Spatial propagation for some nonlocal and non-autonomous reaction-diffusion systems arising in population dynamics

Zhucheng Jin

Normandie Université

THÈSE

Pour obtenir le diplôme de doctorat

Spécialité MATHEMATIQUES

Préparée au sein de l'Université Le Havre Normandie

Spatial propagation for some nonlocal and non-autonomous reaction-diffusion systems arising in population dynamics

Présentée et soutenue par
ZHUCHENG JIN

Thèse soutenue le 12/12/2022 devant le jury composé de		
M. FRANCOIS HAMEL	PROFESSEUR DES UNIVERSITES, AIX-MARSEILLE UNIVERSITE	Rapporteur du jury
M. XING LIANG	PROFESSOR, UNIV OF SCIENCES AND TECHNOLOGIES CHINE	Rapporteur du jury
M. QUENTIN GRIETTE	PROFESSEUR DES UNIVERSITES, Université Le Havre Normandie	Membre du jury
M. PIERRE MAGAL	PROFESSEUR DES UNIVERSITES, UNIVERSITE DE BORDEAUX	Membre du jury
MME DANIELLE HILHORST	DIRECTEUR DE RECHERCHE EMERITE, UNIVERSITE PARIS-SACLAY	Président du jury
M. ARNAUD DUCROT	PROFESSEUR DES UNIVERSITES, Université Le Havre Normandie	Directeur de thèse

Thèse dirigée par ARNAUD DUCROT (Laboratoire de Mathématiques Appliquées du Havre)

UNIVERSITÉ LE HAVRE NORMANDIE

MI∫IΣ

LMAH

Abstract

In this thesis, we study the spatial propagation for some non-autonomous reaction-diffusion equations and systems. Here we mainly consider that coefficients depend on time in a general way. To obtain some sharp results, we assume that the time variations exhibit somehow good averaging properties, including periodicity and almost periodicity as special cases. In the first work, we investigate the so-called generalized travelling wave solutions for a Fisher-KPP equation with nonlocal diffusion. Both the nonlocal kernel and the reaction term depend on time. The existence and nonexistence of such solutions are proved. The second work is concerned with the spreading properties of solutions to non-autonomous Fisher-KPP equations with nonlocal diffusion. Under certain assumptions on the coefficients, a definite spreading speed is obtained. Then, we focus on a two-components non-autonomous prey-predator system with diffusion. We are interested in the case where both the prey and the predator can co-invade the empty environment. Based on the derivation of local pointwise estimates between the densities of the two species, we obtain the spreading speeds for such a system. Lastly, for a class prey-predator systems posed on a lattice with a discrete convolution dispersion, we can also derive similar estimates. By combining the method developed for nonlocal equation in our second work, the large time behaviour of some solutions to such a problem is described.

Keywords. Propagation phenomena; Time heterogeneity; Reaction-diffusion equations; Nonlocal diffusion; Prey-predator systems.

Résumé

Dans cette thèse, on étudie la propagation spatiale pour certaines équations et systèmes d'équations de réaction-diffusion non autonomes. On considère que les coefficients dépendent du temps de façon générale. Cependant, pour obtenir des résultats optimaux, notamment sur les vitesses d'expansion de certaines solutions, nous supposerons que les variations temporelles possèdent des propriétés supplémentaires de moyennisation. Les cas de coefficients périodiques ou presque périodiques seront des cas particulier de notre analyse, pour lesquels nous pouvons obtenir l'existence d'une vitesse d'expansion. Dans un premier travail, nous nous intéressons à l'existence et la non existence de solutions de type ondes progressives généralisées pour une équation de Fisher-KPP avec diffusion non locale. Ici le noyau de diffusion non local ainsi que le terme de réaction dépendent du temps. L'existence et la non existence de telles solutions sont prouvées. Le deuxième travail concerne l'étude de la propriété de d'expansion pour des solutions d'une équations non autonome de Fisher-KPP avec diffusion non locale. Sous certaines hypothèses de moyennisation temporel sur les coefficients, on décrit une vitesse d'expansion pour certaines classes de donnée initiales. On considère ensuite un système proie-prédateur non autonome avec diffusion locale. On s'intéresse tout particulièrement au cas où la proie et le prédateur peuvent tout deux envahir l'environnement, initialement vide des deux espèces. En prouvant des estimations ponctuelles locales entre les densités des deux espèces, nous décrivons les vitesses de propagation des deux espèces. Enfin, nous étudions certaines propriétés d'expansion pour les solutions d'une classe de systèmes de type proie-prédateur posé sur un réseau discret. La dispersion spatiale des deux espèces suit des lois données par des noyaux de convolution. En combinant des estimations ponctuelles similaires au chapitre précédent avec des méthodes développées pour les équations non locales dans notre second travail, nous décrivons le comportement en temps grand de certaines solutions de co-invasion.

Mots clés. Phénomènes de propagation; Hétérogénéité temporelle; Équations de réaction-diffusion; Diffusion non locale; Système de type proie-prédateur.

Résumé étendu

L'objet de cette thèse est l'étude des phénomènes de propagation pour des équations de réaction-diffusion non autonomes de la forme générale suivante :

$$\partial_t \mathbf{u} = \mathcal{A}(t)[\mathbf{u}] + \mathbf{F}(t, \mathbf{u}),$$

où $\mathbf{u} = \mathbf{u}(t, x)$ est une fonction scalaire ou vectorielle dépendant d'une variable temporelle $t \in I$ et d'une variable spatiale $x \in \mathcal{H}$. Ici, on considère $I = \mathbb{R}$ ou $I = (0, \infty)$ et l'environnement $\mathcal{H}$ est l'espace entier $\mathbb{R}$ ou l'espace discret $\mathbb{Z}$. Cette équation décrit la variation instantanée en temps $\partial_t \mathbf{u}$ de la fonction $\mathbf{u} = \mathbf{u}(t, x)$ qui est causé par le *terme de dispersion linéaire* $\mathcal{A}(t)[\mathbf{u}]$ et le terme *non linéaire terme de réaction* $\mathbf{F}(t, \mathbf{u})$. Les équations et systèmes de réaction-diffusion modélisent de nombreux phénomènes en biologie et écologie, en particulier en dynamique des populations. Dans ce travail de thèse, on s'intéresse tout particulièrement à des questions liées à l'invasion spatiale d'espèces.

Une première étude de ce type de question remonte aux années 30, où l'équation dite de Fisher-KPP ou simplement KPP a été étudié. Un exemple typique est donné par l'équation suivante :

$$\partial_t u = \partial_{xx} u + u(1 - u), \quad t > 0, \ x \in \mathbb{R}.$$

En 1937, Fisher a proposé et utilisé cette équation pour étudier la propagation de traits génétiques dans une population donnée. La même année, une analyse mathématique a été donnée par Kolmogorov, Petrovsky et Piskunov. En 1951, Skellam a utilisé cette équation pour étudier l'invasion spatiale d'une espèce biologique, qui est une tentative pour comprendre le rôle de la diffusion en biologie des populations. Il a notamment utilisé l'équation KPP pour étudier l'invasion du rat musqué en Europe de l'Est et a montré que ce modèle donne une description cohérente avec les observations et données précises de terrain. Depuis ces travaux pionniers, les équations et systèmes de réaction-diffusion ont suscité beaucoup d'attention de la part des mathématiciens, biologistes et écologues.

Dans le premier chapitre de ce manuscrit, nous rappelons deux notions importantes: *ondes progressives* et *vitesse de d'expansion*, qui permettent de décrire quantitativement les phénomènes de propagation. Il existe une littérature très vaste consacrée à l'étude des ondes progressives et à la vitesse d'expansion pour des équations de réaction-diffusion. Dans ce chapitre, on se concentre principalement sur les points suivants: terme de réaction de type KPP, diffusion locale, diffusion non locale, hétérogénéité temporelle (coefficients homogènes, périodiques, quasi périodiques...), système de type proie-prédateur et équations posées sur des réseaux. On passe en revue quelques résultats classiques et des développements récents.

On présente également quelques motivations de ce travail, avant de proposer un résumé des résultats principaux obtenus et présenté dans ce manuscrit. Par ailleurs, nous donnons quelques idées de démonstration de ces théorèmes. Finalement, nous discutons quelques problèmes ouverts et perspectives de ce travail. On s'attend en effet à ce que certaines méthodes développées dans ce manuscrit puissent être étendues et utilisée pour étudier d'autres problèmes.

Le Chapitre 2 présente un travail en collaboration avec Arnaud Ducrot. Les résultats présentés font l'objet d'un article publié dans *Annali di Matematica Pura ed Applicata*.

On étudie l'existence et la non-existence de solutions particulières sous forme d'ondes progressives généralisées pour l'équation de diffusion non locale non autonome suivante:

$$\partial_t u(t,x) = \int_{\mathbb{R}} K(t,y)\,[u(t,x-y) - u(t,x)]\,\mathrm{d}y + F\big(t, u(t,x)\big),\ (t,x) \in \mathbb{R} \times \mathbb{R}.$$

La noyau de dispersion $K = K(t,y)$ est une fonction positive ou nulle, dépendante du temps et bornée exponentiellement uniformément pour $t \in \mathbb{R}$ tandis que le terme non linéaire $F = F(t,u)$ est de type Fisher-KPP avec

$$F(t,0) = F(t,1) = 0,\ \forall t \in \mathbb{R}.$$

Cette équation modélise l'évolution spatio-temporelle d'une population dans un environnement. Ici l'individu présente une dispersion à longue distance selon le noyau K. C'est-à-dire, la quantité $K(t, x-y)$ correspond à la probabilité de sauter de y à x au temps t. Et la dynamique de la population locale (processus de naissance et de mort) est décrite par le terme de réaction de type Fisher-KPP qui varie avec le temps. On donne quelques résultats d'existence et non existence de telles solutions. On prouve également quelques estimations pour l'ensemble de vitesse admissible. De plus, sous certaines hypothèses appropriées portant sur les coefficients, on obtient certaines estimations optimales pour l'ensemble des vitesses admissibles.

Le chapitre 3 est un travail en collaboration avec Arnaud Ducrot, qui fait l'objet d'un article actuellement soumis. Il porte sur l'étude d'une équation de Fisher-KPP non autonome avec diffusion non locale, qui s'écrit comme suit:

$$\partial_t u(t,x) = \int_{\mathbb{R}} K(y)\,[u(t,x-y) - u(t,x)]\,\mathrm{d}y + F(t,u),\ \forall t \geq 0, x \in \mathbb{R},$$

et est munie d'une donnée initiale $u(0,\cdot) = u_0(\cdot)$. On suppose que le noyau de dispersion K est à queue fine, c'est-à-dire intégrale contre certaines exponentielles. Ce chapitre étudie les propriétés d'expansion des solutions de cette équation avec des données initiales qui sont respectivement avec une décroissance exponentielle rapide et à décroissance exponentielle plus lente.

De façon générale, on donne des estimations supérieures et inférieures de la vitesse d'expansion. A l'aide de d'hypothèses supplémentaires portant notamment sur des propriétés d'existence de moyenne pour les coefficients dépendant du temps, on prouve que la vitesse d'expansion est bien définie.

Pour l'estimation inférieure de la vitesse d'expansion, on propose une nouvelle approche, basée sur un lemme de persistance pour des solutions uniformément continues. Ce lemme clé assure grosso modo que si une solution uniformément continue $u = u(t,x)$ admet un chemin $t \mapsto X(t)$ le long duquel elle se propage et si elle persiste en $x = 0$, alors la solution persiste sur l'intervalle $[0, kX(t)]$ avec tout k dans $(0,1)$. Autrement dit, u reste uniformément éloigné de 0 sur cet intervalle, quand le temps est grand. En appliquant ce lemme clé, on obtient notre estimation inférieure de la vitesse d'expansion. L'estimation supérieure est quand à elle obtenue comme la vitesse linéaire.

Le Chapitre 4 présente des résultats obtenus dans un travail en collaboration avec Arnaud Ducrot, qui est actuellement soumis pour publication.

Dans ce travail, on étudie la vitesse de propagation pour des systèmes de réaction-diffusion de type proie-prédateur qui s'écrivent sous la forme suivante:

$$\begin{cases} \partial_t u = d(t)\partial_{xx} u + uf\left(t,u,v\right), \\ \partial_t v = \partial_{xx} v + vg\left(t,u,v\right), \end{cases}$$

où $t > 0$ et $x \in \mathbb{R}$. Ce problème est associé à des données initiales convenables avec un support compact pour les deux composantes,

$$u(0,x) = u_0(x) \text{ et } v(0,x) = v_0(x) \text{ pour } x \in \mathbb{R}.$$

Les fonctions $u = u(t,x)$ et $v = v(t,x)$ représente respectivement la densité de la proie et celle du prédateur. Un exemple typique de système est donné comme suit:

$$\begin{cases} \partial_t u = d(t)\partial_{xx} u + r(t)u\left(1-u\right) - p(t)uv, \\ \partial_t v = \partial_{xx} v + q(t)uv - \nu(t)v, \end{cases}$$

où $t > 0$ et $x \in \mathbb{R}$.

Dans ce travail, on considère que la proie et le prédateur sont tous les deux introduits dans un environnement où ces deux espèces sont absentes. On s'intéresse à l'invasion et la co-invasion de ces deux espèces. Pour ce système non autonome, en supposant des propriétés de moyenne temporelle, on a prouvé l'existence de vitesse d'expansion dans deux cas différents. Dans le premier cas, le prédateur envahit le milieu plus lentement que la proie. Dans ce cas, la propagation se produit en deux étapes distinctes impliquant un équilibre intermédiaire (à savoir $u = 1, v = 0$) dans la zone intermédiaire. Pour le deuxième cas, le prédateur envahit l'environnement plus rapidement que la proie. Dans cette situation, on prouve que la proie et le prédateur envahissent l'espace vide simultanément, à $o(t)$ près quand $t \to \infty$.

On fournit ici une nouvelle méthode pour étudier la vitesse de propagation dans le type de système proie-prédateur. En utilisant le principe du maximum fort pour une équation parabolique scalaire, on prouve des estimations ponctuelles entre les densités de proie et de prédateur. Avec ces estimations, on peut comparer les solutions du système proie-prédateur à celles d'équations scalaires de type Fisher-KPP dans un domaine approprié (l'espace entier et des domaines mobiles).

Le dernier chapitre de ce manuscrit expose un travail en collaboration avec Arnaud Ducrot, en cours de finalisation.

Il considère un problème de Cauchy de type proie-prédateur posé sur le réseau infini discret $\mathbb{Z}$:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u(t,i) = \sum_{j\in\mathbb{Z}} J_1(t,j)\left[u(t,i-j) - u(t,i)\right] + u(t,i)f\left(t,u(t,i),v(t,i)\right), \\ \dfrac{\mathrm{d}}{\mathrm{d}t} v(t,i) = \sum_{j\in\mathbb{Z}} J_2(t,j)\left[v(t,i-j) - v(t,i)\right] + v(t,i)g\left(t,u(t,i),v(t,i)\right), \end{cases}$$

où $t > 0$ and $i \in \mathbb{Z}$. Ce système est associé à des données initiales positives (ou nulle) et bornées

$$u(0,i) = u_0(i) \text{ and } v(0,i) = v_0(i).$$

On suppose que les deux ensembles

$$\{i \in \mathbb{Z};\ u_0(i) \neq 0\} \neq \emptyset \text{ et } \{i \in \mathbb{Z};\ v_0(i) \neq 0\} \neq \emptyset$$

ont tous deux un nombre fini d'éléments. Les noyaux de dispersion J_1 et J_2 sont supposés être des fonctions exponentiellement bornées. Comme pour le Chapitre 4, un exemple typique de non linéarité (f, g) est la suivante:

$$\begin{cases} f(t,u,v) = u\,(1-u) - p(t)uv, \\ g(t,u,v) = q(t)uv - \nu(t)v. \end{cases}$$

Pour étudier ce système, on adapte des idées similaires à celles développées au Chapitre 4 afin de pouvoir comparer les solutions du système avec celles d'équations scalaires de type Fisher-KPP dans des domaines spatio-temporels appropriés.

Un résultat principal de ce chapitre décrit la vitesse d'expansion exacte des solutions du système, en supportant là encore l'existence de moyennes temporelles pour les coefficients du problème. Nos résultats sont proches de ceux obtenus au Chapitre 4, avec une diffusion locale. C'est-à-dire, si le prédateur envahit le milieu vide plus lentement que la proie, la proie envahit d'abord l'espace, puis le prédateur suit donnant lieux à la co-exsistence des deux populations. D'autre part, si la proie envahit l'espace vide plus lentement que le prédateur, alors la proie et le prédateur envahissent simultanément l'environnement.

Acknowledgements

First and foremost, I would like to express my greatest gratitude to my supervisor, Professor Arnaud Ducrot. Without him, this thesis would not have been possible. I do not know how to express thanks in here. Please permit me to come back to the beginning of story. A sudden opportunity, I became a PhD student work with Arnaud. Everything is like a dream. He provides a lot of help in every step, from choosing each problem to solving them to organizing my thesis defense. Thanks for introducing me into this interesting topic spatial propagation in population dynamic models. During the past time working for this thesis, he showed me the charms of mathematics and the values of our problems in each discussion. His passion affected me a lot. He often asked me "Is it interesting?", especially, when we find some difficulties but we do not know how to solve them. Then, we spend a lot of time to understand them. Sometimes we can be lucky enough. Of course, sometimes we have to choose "plan B". I often lost in the "uniformity" for some proofs. Then, after his careful reading, he pointed my mistakes and explained many times to me when I trapped. Sometimes I cannot get his ideas instantly, he explained those with utmost patience. He guided me step by step to become a good researcher in mathematics. Perhaps I should stop too much detailed and private expositions. Some memories are my treasures. As someone said, "scientific research is an adventure". It is great fortunate for me that I can adventure with Arnaud. I really appreciate him, my mentor and friend.

I am extremely grateful to the professors who agreed to be the jury members out of their busy schedules for participating my defense: Danielle Hilhorst, François Hamel, Xing Liang, Pierre Magal, and Quentin Griette. Many thanks to them.

Particularly, I would like to express my gratitude to Professor François Hamel and Professor Xing Liang for agreeing to review my thesis during their busy schedules. It is my honor to have my thesis reviewed by them.

I warmly thank to Professor Rong Yuan and Professor Zhihua Liu in Beijing Normal University. They suggested and recommended me to come to France to work with Arnaud. Thanks for their many help before I came to France.

I also want to express many thanks to the members of my "Individual Supervision Committee": Thomas Giletti and Benjamin Ambrosio. They gave me some suggestions about my works and reports during each meeting. Their advice improved my academic presentation skills.

I would like to express a lot of thanks to some friends in the laboratory. I really thank to Ahmadou Sylla for helping me a lot for instance some procedures when I arrive in France. Since I cannot speak french, sometimes I have to bother Ahmadou Sylla when I meet some difficulties in daily life. I am very grateful for his help. I want to say thanks to Alexandre Thorel, Irmand Mikiela and Kamal Khalil. Their hospitality allowed me to adapt quickly to the new environment. I also want to thank to my colleagues Fatima Zahra Lahbiri and Yaheng Cui. In particular, I am very happy to recognize Hao Kang and Liangliang Deng. Every Sunday afternoon, we walk together around this city. We went to the sea, to the forest, to Éterat and to some parks. During the walk, we discuss the mathematics or some other story. It is a nice memory.

I also want to thank the faculty and staff who have helped me in this laboratory for instance Aziz Alaoui, David Manceau and other people.

Many thanks to some friends who are not around. We contact regularly via internet. All of you provide a lot of emotional support during this thesis work. We exchange some interesting things or some difficulties encountered during the PhD. Many thanks to Bingyao Huang, Long Huang, Ran Wang, Zhi Chen, Yajing Li, Min Zhao, Xiao Zhao. Particularly, I would like to thank Min Zhao for reading the first chapter in this manuscript and pointing out some typos.

I would like to take this opportunity to thank the financial support provided by région Normandie.

Lastly, I would like to express my gratitude to my families for their irreplaceable support and love at all times, even though they are not familiar with mathematical research. Especially my mom, I am forever grateful to her.

Contents

Chapter 1

General introduction

This thesis is devoted to the study of propagation phenomena emerging from various non-autonomous reaction-diffusion equations of the following general form:

$$\partial_t \mathbf{u} = \mathcal{A}(t)[\mathbf{u}] + \mathbf{F}(t, \mathbf{u}), \tag{1.0.1}$$

where $\mathbf{u}$ is a scalar or vector-valued function depending on time $t \in I$ and location $x \in \mathcal{H}$. Herein we consider $I = \mathbb{R}$ or $I = (0, \infty)$ and $\mathcal{H} = \mathbb{R}$ or $\mathbb{Z}$. This equation describes the instantaneous time change $\partial_t \mathbf{u}$ of $\mathbf{u}(t, x)$ at time t and location x caused by the *linear dispersal term* $\mathcal{A}(t)[\mathbf{u}]$ and time heterogeneous nonlinear *reaction term* $\mathbf{F}(t, \mathbf{u})$.

The first investigation of the above type equation can be dated back to 1930s. Fisher [70] and Kolmogorov, Petrovsky and Piskunov [97] independently introduced and studied the following equation,

$$\partial_t u = \partial_{xx} u + u(1 - u), \quad x \in \mathbb{R}.$$

This equation is often referred as the Fisher-KPP equation or KPP equation. The original motivation of the Fisher-KPP equation is to model the spread of advantageous genetic traits in space in a given population. In 1951, Skellam [149] used this KPP equation to study biological invasion, which is a systematic attempt to examine the role of diffusion in population biology. He showed that the model yields a good description consistent with observations in precise data. Since these pioneering works, reaction-diffusion equations and systems arise as a basic model in mathematical biology and ecology.

There is a large literature devoted to describing and understanding the propagation phenomena in (1.0.1) from many aspects including reaction term (KPP-type, bistable, ignition...), diffusion mechanism (random diffusion, nonlocal diffusion...), media (homogeneous, periodic coefficients, almost periodic coefficients...), multi-species (predation, competition...) and so on. In order to study propagation phenomena in reaction-diffusion equations, there are two important mathematical notions (which will be shown in the following section): *travelling waves* and *spreading speeds*.

In this chapter, we first give a review of the state of the art as well show some motivations for our work. Then, we present the important mathematical results obtained in this thesis and explain key ideas of the proofs. Lastly, we discuss some open problems for future work.

1.1 Literature review

In this section, we will start from the classical Fisher-KPP equations to introduce two important notions in studying propagation phenomena: *travelling waves* and *spreading speed*. We expose some celebrated results in KPP equations and show the connection of two notions. Then, we review recent works in time heterogeneous KPP equations to understand more complex spatio-temporal behaviours caused by time heterogeneity. Next, we review some works in nonlocal diffusion equations which aim to describe some long distance dispersal processes in population dynamics. Further, the developments of spreading behaviours in prey-predator systems are exposed. The results about the existence of travelling waves and spreading speed in prey-predator systems are elaborated. Lastly, we give an overview of lattice equations which can model species living in patch environments.

1.1.1 Classical Fisher-KPP equations

Let us first recall some well known results for the following classical Fisher-KPP equation

$$\partial_t u = \partial_{xx} u + f(u), \text{ for } x \in \mathbb{R}, \tag{1.1.2}$$

where $f \in C^1([0,1])$ satisfies

$$\begin{cases} f(0) = f(1) = 0, \\ 0 < f(u) \leq f'(0)u, \ \forall u \in (0,1), \end{cases} \quad \text{(KPP conditions)}. \tag{1.1.3}$$

Note that $f(u) = u(1-u)$ is a typical example of above assumption, see Figure 1.1. A particular case of (1.1.2) was introduced by Fisher [70] to investigate the propagation of genetic traits in a given population. A mathematical treatment was given in [97]. This equation plays an important role in population dynamics, we refer to some monographs [30, 121, 122, 147]. In the literature, there are also some strong KPP conditions such as $f(u)/u$ is nonincreasing for $u \in (0,1)$ or $f'(u) < f'(0)$ for $u \in (0,1)$.

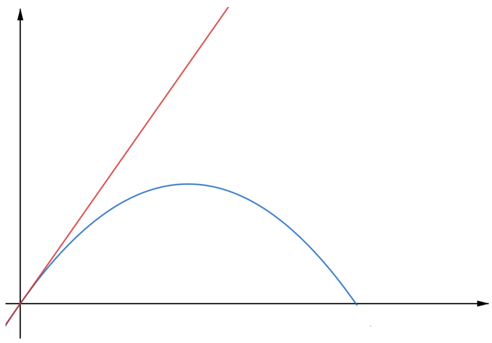


Figure 1.1: A typical example of KPP type function.

Travelling waves

The first important notion to describe propagation phenomena in quantity and mathematically is *travelling waves*.

Definition 1.1.1. *A* ***travelling wave*** *solution with speed $c \in \mathbb{R}$ of (1.1.2) is a solution $u(t,x) = \varphi(x - ct)$ with $\varphi(-\infty) = 1$ and $\varphi(\infty) = 0$.*

The function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ is called wave profile and c is the wave speed. Note that φ satisfies the ODE

$$-\varphi'' - c\varphi' = f(\varphi),$$

with

$$\varphi(-\infty) = 1, \varphi(\infty) = 0 \text{ and } \varphi(z) \geq 0, \forall z \in \mathbb{R}.$$

By the phase plane analysis, [97] and [8] obtained the following theorem.

Theorem 1.1.2 ([8, 97])**.** *There exists a travelling wave solution with speed* c *of the KPP equation (1.1.2) if and only if* $c \geq c^* := 2\sqrt{f'(0)}$*. Moreover, the travelling wave is unique up to translation and* $0 < \varphi < 1$ *is a decreasing function.*

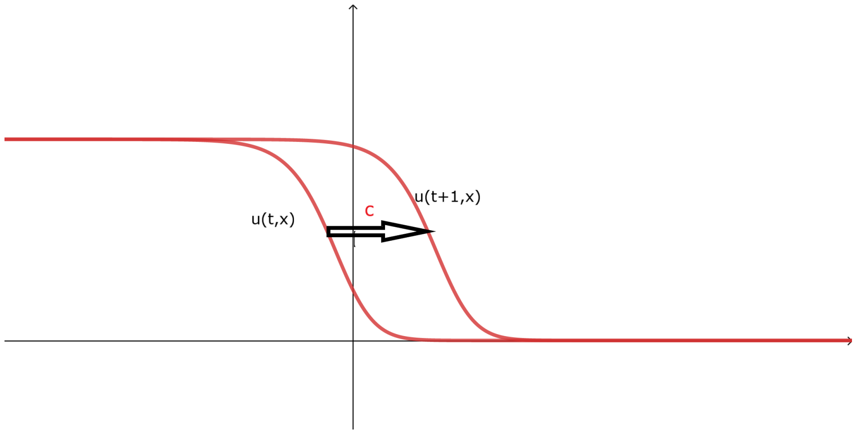


Figure 1.2: A right moving travelling wave solutions in (1.1.2).

The quantity c^* in the above theorem is called the *minimal speed* of wave propagation. Let us observe that the minimal speed of KPP equation is *linearly determined.* Indeed, linearizing the equation satisfied by φ at $\varphi = 0$, one has

$$-\varphi'' - c\varphi' = f'(0)\varphi.$$

Note that the characteristic polynomial is

$$\lambda^2 - c\lambda + f'(0) = 0.$$

Hence, there exists a positive solution for above linear equation if and only if $c \geq 2\sqrt{f'(0)}$.

Here the front (φ, c) with $c \geq c^* = 2\sqrt{f'(0)}$ is also called *pulled front* which was first introduced in [151]. This means that the speed of wave propagation is determined by the leading edge of the population distribution while the front is being pulled by the leading edge. For more information about *pulled front* and the corresponding notion *pushed front*, we refer the reader to [74, 134, 151].

We also point out that for the high dimensional case $x \in \mathbb{R}^N$, the definition of (planar) travelling waves is given by $u(t, x) = \varphi(x \cdot e - ct)$ where $e \in S^{N-1}$ is a given direction. We refer the reader to the monograph of Volpert *et al.* [154] for more information about travelling waves. There are some other types front in high dimensional space such as curved fronts, see [84, 126, 152].

Spreading speed

Another important notion to understand the spatio-temporal dynamic in an unbounded domain is the *asymptotic speed of spread* (in short *spreading speed*).

Let us consider the Cauchy problem of (1.1.2) supplemented with initial data $u(0, x) = u_0(x)$ where $u_0 \geq 0$ and $u_0 \not\equiv 0$, namely

$$\begin{cases} \partial_t u = \partial_{xx} u + f(u), & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

Aronson and Weinberger [7, 8] proved the following theorem.

Theorem 1.1.3. *Let $u = u(t,x)$ be a solution of (1.1.2) equipped with initial data u_0. If u_0 is compactly supported, then there exists a quantity $w_* = 2\sqrt{f'(0)}$ such that the solution u satisfies the following property:*

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{|x|\geq ct} u(t,x) = 0, & \text{for } c > w_*, \\ \lim\limits_{t\to\infty} \sup\limits_{|x|\leq ct} |1-u(t,x)| = 0, & \text{for } c \in [0, w_*). \end{cases}$$

Here the quantity w_* is called ***spreading speed****.*

Remark 1.1.4. *From the above theorem, one can note that there exists the* ***hair trigger effect*** *in the Cauchy problem of (1.1.2), namely, if $u_0 \not\equiv 0$, then the solution $u = u(t,x)$ to (1.1.2) supplemented with u_0 satisfies*

$$\lim_{t\to\infty} u(t,x) = 1 \text{ locally uniformly for } x \in \mathbb{R}.$$

The above theorem means that for the compactly supported initial data, there are full of species u in the area $(-w_*t, w_*t)$ after a long time t. This theorem provides a rigorously mathematically support for the empirical work by Skellam [149] which used the KPP equation to study the invasion of the muskrat in Eastern Europe. With the precise available data, Skellam plotted the square root of the area occupied by the population of muskrats with respect to the observed years and illustrated the propagation of muskrats at a constant speed, see Figure 1.3.

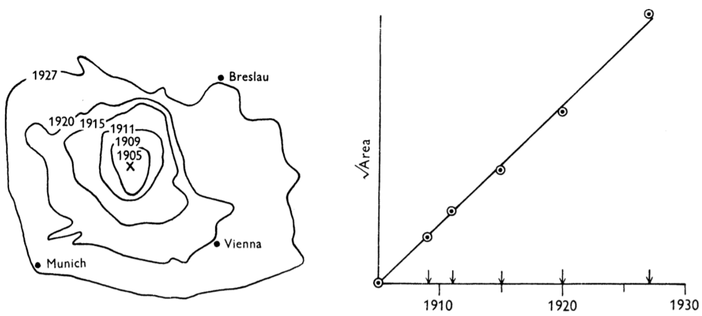


Figure 1.3: The invasion of muskrat in Eastern Europe. This figure is taken from [149].

Convergence results

From the above two theorems, one can note that $w_* = c^* = 2\sqrt{f'(0)}$. This means that the spreading speed of solutions with compactly supported initial data coincides with the minimal speed of travelling waves. In the following theorem, the connection between the two concepts for homogeneous KPP equations can be observed more accurately.

Theorem 1.1.5 ([97])**.** *Let $u = u(t,x)$ be the solution of (1.1.2) with initial data u_0 which is Heaviside function. Let φ_{c^*} be the travelling wave solutions of (1.1.2) with minimal speed c^*. There exists a function $m : \mathbb{R} \to \mathbb{R}$ such that $\lim_{t\to\infty} m(t)/t = 0$ and*

$$\lim_{t\to\infty} |u(t,x) - \varphi_{c^*}(x - c^*t - m(t))| = 0, \quad \text{uniformly for } x \in \mathbb{R}.$$

Then several authors have refined this property. For a large class of Heaviside-like initial data, Uchiyama [153] proved that $m(t)$ might have a nontrivial behaviour as:

$m(t) = -3/(2\lambda^*)\ln t + O(\ln\ln t)$, where λ^* is the root of $\lambda^2 - c^*\lambda + f'(0) = 0$. The following sharpest asymptotic is given by Bramson [26, 27], used probabilistic methods,

$$m(t) = -\frac{3}{2\lambda^*}\ln t + O(1).$$

These results were also proved by Lau [98] using intersection number theory. More recently, the paper by Hamel, Nolen, Roquejoffre and Ryzhik [85] proposed a PDE method to give a short proof for this problem.

According to these celebrated results, we can conclude that although the spreading speed equal to minimal wave speed in (1.1.2), this does not mean that the solution propagates parallel to the travelling wave with speed c^*. There is a backward phase drift of $O(\ln t)$ from the position c^*t in KPP equations.

Influence of initial data

Some works also showed that the spreading speed is affected by the tail of the initial data. Before coming to the precise results, let us recall that if φ_c is a travelling wave solution of (1.1.2) with speed $c > c^*$, then for some $M > 0$ large enough, one has

$$\varphi_c(z) \sim Me^{-\lambda z} \text{ as } z \to +\infty,$$

where λ is the smallest root of

$$\lambda^2 - c\lambda + f'(0) = 0.$$

Thus, we may expect that the estimate of spreading speed is related to the exponential decaying rate of initial data.

Set

$$c(\lambda) := \lambda + \frac{f'(0)}{\lambda}, \lambda > 0.$$

Note that the minimum value of $c(\lambda)$ is $c^* = 2\sqrt{f'(0)}$ and $c^* = c(\lambda^*)$ with $\lambda^* = \sqrt{f'(0)} = c^*/2$.

From the work by Aronson and Weinberger [8] and Uchiyama [153], one can see the following relationship between the spreading speed and the tail of initial data.

Theorem 1.1.6. *Let $u = u(t,x)$ be a solution of (1.1.2) supplemented with nonzero initial data $1 \geq u_0 \geq 0$.*

(i) If $u_0(x) = O(e^{-\lambda x})$ as $x \to +\infty$ for some given $\lambda \geq \lambda^*$, then c^* is the spreading speed to the right, namely, the following property holds

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{x\geq ct} u(t,x) = 0, & \text{for } c > c^*, \\ \lim\limits_{t\to\infty} \sup\limits_{x\in[0,ct]} |1 - u(t,x)| = 0, & \text{for } c \in [0, c^*). \end{cases}$$

(ii) If $u_0(x) \sim e^{-\lambda x}$ as $x \to +\infty$ for some given $\lambda \in (0, \lambda^*)$, then $c(\lambda)$ is the spreading speed to the right, namely,

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{x\geq ct} u(t,x) = 0, & \text{for } c > c(\lambda), \\ \lim\limits_{t\to\infty} \sup\limits_{x\in[0,ct]} |1 - u(t,x)| = 0, & \text{for } c \in [0, c(\lambda)). \end{cases}$$

Observing the second conclusion in the above theorem, intuitively, if we let $\lambda \to 0^+$, then one has $c(\lambda) \to \infty$. We may suspect that the acceleration phenomena appears when the initial data decays more slowly than exponential decaying. The spreading speed may not be a finite number any more.

In [86], the authors considered that the initial data is globally front like and decays more slowly than any exponential decaying function, namely, u_0 satisfies

$$u_0 > 0 \text{ in } \mathbb{R}, \lim_{x\to-\infty} u_0(x) > 0 \text{ and } \lim_{x\to+\infty} u_0(x) = 0,$$

and

$$u_0(x)e^{\varepsilon x} > 0 \text{ as } x \to +\infty, \text{ for all } \varepsilon > 0.$$

In this case, the spreading speed notion used before may not be suitable. The authors in [86] try to describe the location of the level set. By the way, travelling waves and spreading speed can be regarded as a way to show the motion of level sets of solutions. In [86], for $\lambda \in (0,1)$ and $t \geq 0$, they denote the level set $E_\lambda(t)$ by

$$E_\lambda(t) := \{x \in \mathbb{R};\ \ u(t,x) = \lambda\}.$$

They proved that all level sets of solution move infinitely fast as time goes to infinity and displayed the locations of the level sets according to the decay of the initial conditions. Here we only show some examples in [86] instead of the exposition of precise theorem.

Example 1.1.7. *Let $C, \alpha > 0$ and $\beta \in (0,1)$ be given.*
(i) (Spread algebraic fast with t) If $u_0 \sim Ce^{-\alpha x^\beta}$ as $x \to \infty$, then

$$\min E_\lambda(t) \sim \max E_\lambda(t) \sim f'(0)^{1/\beta}\alpha^{-1/\beta}t^{1/\beta}, \ \text{as } t \to \infty,$$

(ii) (Spread exponential fast with t) If $u_0 \sim Cx^{-\alpha}$ as $x \to \infty$, then

$$\ln(\min E_\lambda(t)) \sim \ln(\max E_\lambda(t)) \sim f'(0)\alpha^{-1}t, \ \text{as } t \to \infty,$$

(iii) (Spread doubly-exponential fast with t) If $u_0 \sim C(\ln x)^{-\alpha}$ as $x \to \infty$, then

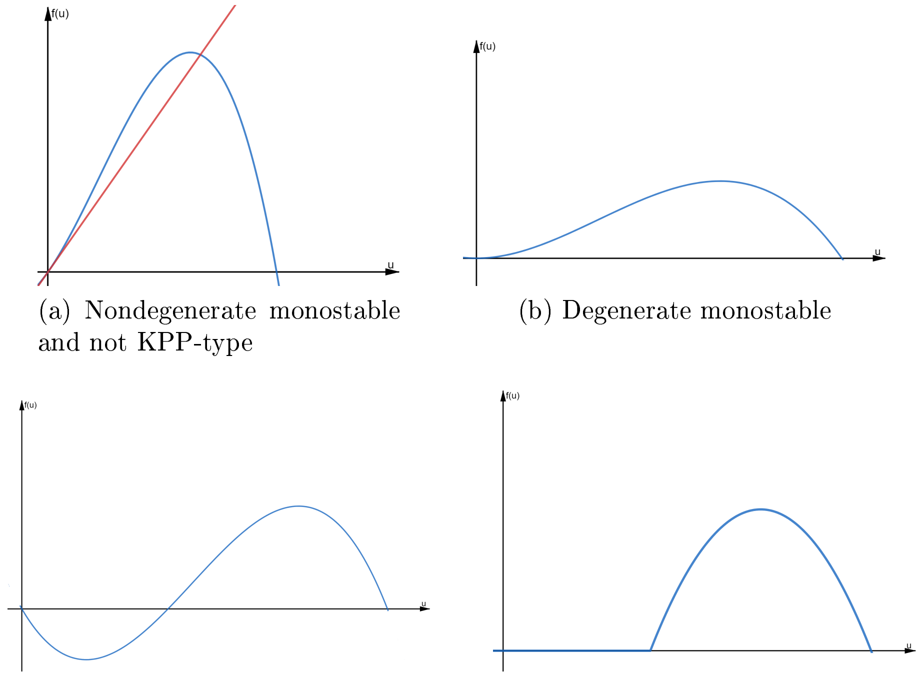
$$\ln\ln(\min E_\lambda(t)) \sim \ln\ln(\max E_\lambda(t)) \sim f'(0)\alpha^{-1}t, \ \text{as } t \to \infty.$$

We also refer the reader to [83] for spreading speed of (1.1.2) with front like and asymptotically oscillating initial data. These works show that the spreading speed is strongly affected by the tail of initial function.

Other nonlinear reaction terms

For the sake of completeness, we also mention some other types of nonlinear reaction term $f = f(u)$. In the following, we give the definition and typical example for nondegenerate monostable, degenerate monostable, bistable and ignition.

(i) **Nondegenerate monostable** if $f \in C^1([0,1])$, $f(0) = f(1) = 0$, $f > 0$ in $(0,1)$ and $f'(0) > 0$. Example: $f(u) = u(1-u)(1+au)$ with $a \geq 0$.
– *KPP type* is a special case of monostable. Example: $f(u) = u(1-u)(1+au)$ for $0 \leq a \leq 1$, see Figure 1.1 above.
– *Non KPP type monostable* if the maximum value of $\frac{f(u)}{u}$ is not reach at $u = 0$. This corresponds to the so-called *weak Allee effect* in population dynamics. Example: $f(u) = u(1-u)(1+au)$ with $a > 1$, see Figure 1.4 (a).



(a) Nondegenerate monostable and not KPP-type

(b) Degenerate monostable

(c) Bistable

(d) Ignition

Figure 1.4: Different reaction terms.

(ii) **Degenerate monostable** if $f \in C^1([0,1])$, $f(0) = f(1) = 0$, $f > 0$ in $(0,1)$ and $f'(0) = 0$. Example: $f(u) = u^p(1-u)$ with $p > 1$, see Figure 1.4 (b).

(iii) **Bistable** if $f \in C^1([0,1])$, $f(0) = f(1) = 0$, and exists $\theta > 0$ such that $f < 0$ in $(0,\theta)$ and $f > 0$ in $(\theta, 1)$. This corresponds to *strong Allee effect*. Example: $f(u) = u(1-u)(u-\theta)$, see Figure 1.4 (c).

(iv) **Ignition** if $f \in C^1([0,1])$, $f(0) = f(1) = 0$, and exists $\theta > 0$ such that $f = 0$ in $(0,\theta)$ and $f > 0$ in $(\theta,1)$, see Figure 1.4 (d). This type appears in combustion problems, θ is known as the ignition temperature.

We point out some phenomena which are different from KPP equations. For instance, in the bistable case, there is a unique speed c of travelling wave and the sign of c is same as $\int_0^1 f(s)\mathrm{d}s$. As well as, the hair trigger effect property does not hold in bistable case.

There is a huge literature studying propagation phenomena with these different reaction terms. It is impossible to exhaust all literature in these topics. We refer the reader to some earlier works [8, 67, 134] and to the monograph [154].

1.1.2 Time heterogeneity

Note that fluctuating environment modeled by time heterogeneities is important in biology and ecology, particularly in population dynamics. Various important factors vary in time seasonally or daily as for instance physical environmental conditions (temperature, rainfall, wind...), species mobility, the availability of food and so on. Therefore, it is important to study wave propagation and spatial spread behaviours in equations with time periodic coefficients as well as more general time dependence such as time almost periodic.

In this subsection, we mainly recall the propagation phenomena in the non-autonomous KPP equation. For statement simplicity and clarity, we consider the following time dependent diffusive logistic equation

$$\partial_t u = \partial_{xx} u + \mu(t)u(1-u), \quad t \in \mathbb{R}, x \in \mathbb{R}, \tag{1.1.4}$$

where $\mu = \mu(t)$ is uniformly continuous and bounded for $t \in \mathbb{R}$ and $\inf_{t\in\mathbb{R}} \mu(t) > 0$. Note that $f(t,u) = \mu(t)u(1-u)$ is a typical example of KPP type nonliearity.

Due to the time heterogeneity, the notion of classical *travelling wave* is not suitable. In order to describe the wave propagation for reaction-diffusion equations in heterogeneous media, some other or generalized notions are introduced such as periodic travelling wave or pulsating wave (for periodic environment) and generalized transition front (for almost periodic and more general heterogeneous environment). Note that different classes of time heterogeneities may affect the dynamical behaviour. Before going to these precise results, let us recall definitions and examples for some important classes of heterogeneities.

Definition 1.1.8. *(i) A function $h : \mathbb{R}^m \to \mathbb{R}$ is called* ***periodic*** *if there exist some positive constants $L_1, \ldots, L_m$ such that $h(z) = h(z+L)$, where $L = (L_1, \cdots, L_m)$.*

(ii) A function $g : \mathbb{R}^m \to \mathbb{R}$ is called ***almost periodic*** *if for any sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{R}^m$ one can extract a subsequence $(z_{n_k})_{k\in\mathbb{N}}$ such that $g(z_{n_k} + z)$ converges uniformly for $z \in \mathbb{R}^m$.*

(iii) A uniformly continuous and bounded function $f : \mathbb{R}^m \to \mathbb{R}$ is called ***uniquely ergodic*** *if there exists a unique invariant probability measure $\mathbb{P}$ on the hull*

$$\mathcal{H}(f) := cl\left\{f(\cdot + \tau),\ \tau \in \mathbb{R}^m\right\},$$

where $\mathcal{H}(f)$ is the closure of the translation set of f with respect to local uniform topology.

Example 1.1.9. *Some examples are given in below:*

(i) Periodic function: $h(z) = \sin z$ for $z \in \mathbb{R}$.

(ii) Almost periodic function: $g(z) = \sin z + \sin(\sqrt{2}z)$ for $z \in \mathbb{R}$.

(iii) Uniquely ergodic function: A trivial example is a bounded continuous function f satisfies $f(z) \to C$ as $|z| \to \infty$ for some constant C. A classical example is a bounded uniformly continuous function on $\mathbb{R}^2$ whose level sets exhibit the Penrose tiling pattern, see Figure 1.5. For more examples and properties, we refer to [118].

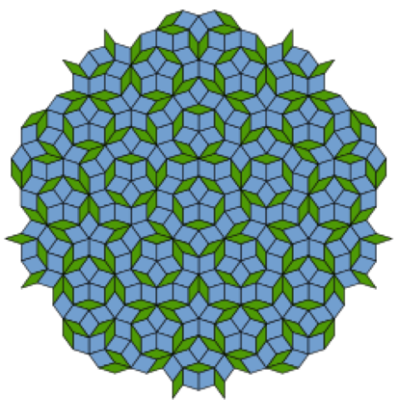


Figure 1.5: Penrose tiling. Source from Wikipedia.

From the above definition, one can observe that periodic functions are almost periodic and both are uniquely ergodic. Due to the equivalent characterization of uniquely ergodic in [118], one has the following property:

Proposition 1.1.10. *Let $\mu : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous and bouned function. If $\mu = \mu(t)$ is uniquely ergodic, then the following limit exists*

$$\langle \mu \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mu(\tau + s)\mathrm{d}s, \text{ uniformorly for } \tau \in \mathbb{R}.$$

The quantity $\langle \mu \rangle$ is called ***mean value****.*

There are also some functions that do not have a mean value. We show the following example which was given in [124].

Example 1.1.11. *Functions without mean value.*
Set $t_1 := 2$ and for $n \in \mathbb{N}$,

$$\sigma_n := t_n + n, \quad \tau_n := \sigma_n + n, \quad t_{n+1} := \tau_n + 2^n.$$

The function μ is defined by

$$\mu(t) := \begin{cases} 3 & \text{if } t_n < t < \sigma_n, n \in \mathbb{N}, \\ 1 & \text{if } \sigma_n < t < \tau_n, n \in \mathbb{N}, \\ 2 & \text{else.} \end{cases}$$

Note that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mu(s)\mathrm{d}s = 2 \text{ and } \lim_{t\to\infty} \inf_{h>0} \frac{1}{t} \int_0^t \mu(h+s)\mathrm{d}s = 1.$$

Therefore, μ does not admit a uniform mean value $\langle \mu \rangle$ over $(0, \infty)$.

Time periodic travelling wave

When μ is a periodic function with period $T > 0$ in (1.1.4), the classical travelling wave no longer exists and the relevant notion is *time periodic travelling wave* which is defined below.

Definition 1.1.12. *Assume that in (1.1.4) there exists $T > 0$ such that $\mu(t) = \mu(t + T)$ for all $t \in \mathbb{R}$. A solution u of (1.1.4) is called* ***time periodic travelling wave*** *with speed c if $u(t, x) = \varphi(x - ct, t)$ and $\varphi = \varphi(z, t)$ satisfies*

$$\begin{cases} \partial_t\varphi - \partial_{zz}\varphi - c\partial_z\varphi = \mu(t)\varphi(1-\varphi), & \forall(z,t) \in \mathbb{R}^2, \\ \varphi(-\infty, t) = 1 \text{ and } \varphi(+\infty, t) = 0, & \forall t \in \mathbb{R}, \\ \varphi(z,t) = \varphi(z, t+T) \text{ and } \varphi \geq 0, & \forall(z,t) \in \mathbb{R}^2. \end{cases}$$

This type of solution was also investigated in [3] for time periodic bistable equation.

Another well-known notion for equation in periodic media is the *pulsating wave*, namely there exist $T > 0$ and $L > 0$ such that

$$u(t + T, x + L) = u(t, x) \quad \forall(t, x) \in \mathbb{R}^2.$$

The number $c := L/T$ can be regarded as the velocity of propagating wave front. This notion was first introduced by Shigesada, Kawasaki and Teramoto [148] for space periodic reaction-diffusion equations. One can observe that the two notions, namely *time periodic travelling wave* and *pulsating wave*, are equivalent for (1.1.4). The following existence result of such solution in (1.1.4) can be yielded from [123] which studied in a more general framework of space-time periodic reaction-diffusion equations.

Theorem 1.1.13 (see [123])**.** *Let $\mu = \mu(t)$ in (1.1.4) be a periodic function with period T. There exists a time periodic travelling wave solution with speed c in (1.1.4) if and only if $c \geq c^* := 2\sqrt{\frac{1}{T}\int_0^T \mu(s)\mathrm{d}s}$.*

Generalized transition wave

The case of time almost periodic and bistable reaction term has been investigated by Shen [136, 137]. The author introduced an appropriate notion of wave which incorporated a time almost periodic speed function $c = c(t)$.

In order to investigate more general heterogeneous (for instance uniquely ergodic coefficients) equations, Berestycki and Hamel [19, 20] proposed the notion of *generalized transition wave*. We also refer to Matano [117] and Shen [138] for related notions in random media. Next, we introduce the definition of generalized transition wave adapted to (1.1.4).

Definition 1.1.14. *A* ***generalized transition wave*** *connecting* 1 *and* 0 *for (1.1.4) is a solution* $u = u(t, x) : \mathbb{R}^2 \to [0, 1]$ *for which there exists some interface function* $X : \mathbb{R} \to \mathbb{R}$ *such that*

$$\begin{cases} \lim_{x\to-\infty} u(t, x + X(t)) = 1, \\ \lim_{x\to\infty} u(t, x + X(t)) = 0, \end{cases} \quad \text{uniformly with respect to } t \in \mathbb{R}.$$

The important point in the above definition is that the transition between 0 and 1 is well localized in space, uniformly in t. That means for a given transition wave u, for all $0 < \alpha \le \beta < 1$, the level set $\{x \in \mathbb{R} : \ \alpha \le u(t, x) \le \beta\}$ has a bounded length uniformly in $t \in \mathbb{R}$.

Next, we recall the definition of *generalized travelling wave* which is used in [124, 125].

Definition 1.1.15. *A function* $u = u(t, x) : \mathbb{R}^2 \to [0, 1]$ *is said to be a* ***generalized travelling wave*** *of (1.1.4) with the wave speed function* $c = c(t) \in L^\infty(\mathbb{R})$ *if* u *is a transition wave of (1.1.4) with the interface function*

$$X(t) = \int_0^t c(s)\mathrm{d}s, \forall t \in \mathbb{R}.$$

In this case, we define its profile $\varphi : \mathbb{R}^2 \to [0, 1]$ *by*

$$\varphi(t, z) = u\left(t, z + \int_0^t c(s)\mathrm{d}s\right), \ \forall (t, z) \in \mathbb{R}^2.$$

The profile function $\varphi : \mathbb{R}^2 \to [0, 1]$ *satisfies the following behaviours at* $z = \pm\infty$*:*

$$\lim_{z\to-\infty} \varphi(t, z) = 1 \text{ and } \lim_{z\to\infty} \varphi(t, z) = 0 \text{ uniformly for } t \in \mathbb{R}.$$

Note that this definition is nothing but a transition wave associated with a globally Lipschitz continuous interface function.

In time uniquely ergodic case, Shen [141] proved the minimal speed of generalized travelling waves in (1.1.4) by a dynamical system approach.

For (1.1.4) with more general time heterogeneities (such as the mean value does not exist), Nadin and Rossi [124] used an average, so-called *least mean*, to provide a sharp estimate of the minimal speed of generalized travelling wave.

Definition 1.1.16. *For any given function* $g \in L^\infty(\mathbb{R})$*, the quantity* $\lfloor g \rfloor$ *is called* ***least mean*** *of function* g *if*

$$\lfloor g \rfloor := \lim_{T\to\infty} \inf_{t\in\mathbb{R}} \frac{1}{T} \int_0^T g(t + s)\mathrm{d}s.$$

One can observe that if g exists mean value, then $\lfloor g \rfloor = \langle g \rangle$.

Theorem 1.1.17 ([124]). *Assume that $\mu \in L^\infty(\mathbb{R})$ and $\inf_{t\in\mathbb{R}} \mu(t) > 0$ in (1.1.4).*

(i) For all $\gamma > 2\sqrt{\lfloor \mu \rfloor}$, there exists a generalized travelling wave u with a speed function c such that $\lfloor c \rfloor = \gamma$.

(ii) There exists no generalized travelling wave with a speed c such that $\lfloor c \rfloor < 2\sqrt{\lfloor \mu \rfloor}$.

By the way, we point out that the wave speed function constructed in [124] has the particular form

$$c(t) = \lambda + \frac{\mu(t)}{\lambda}, \quad \lambda \in (0, \sqrt{\lfloor \mu \rfloor}).$$

However, there might exist some generalized travelling waves with a speed which cannot be written in this form. For instance, we refer the reader to [20] for an interesting example which shows a transition front temporally connecting between two classical travelling waves with two different wave speeds. As well as, [87] obtained the set of admissible asymptotic future and past speeds of generalized transition waves for (1.1.4) with $\mu(t)$ admitting two limits as $t \to \pm\infty$.

Spreading speed results

Now we review the work of spreading property in the non-autonomous KPP equation. Let us consider the Cauchy problem associated with (1.1.4), namely,

$$\begin{cases} \partial_t u = \partial_{xx} u + \mu(t) u(1-u), & t \in (0,\infty), x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1.5}$$

In the time periodic case, the spreading speed in monotone systems was studied by Liang, Yi and Zhao [103] by the abstract dynamical system method. Shen [140] investigated the spreading speed in time almost periodic and space periodic case for KPP equations. Here, we recall precisely that spreading properties in general time heterogeneities was proved by Nadin and Rossi [124].

Theorem 1.1.18 ([124]). *Assume that $\mu \in L^\infty(0,\infty)$ and $\inf_{t\geq 0} \mu(t) > 0$. Let the non-trivial continuous initial function $0 \leq u_0 \leq 1$ be compactly supported. Then the solution u of (1.1.5) satisfies*

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{|x| \geq 2\sqrt{t\int_0^t \mu(s)\mathrm{d}s} + \sigma t} u(t,x) = 0, & \forall \sigma > 0, \\ \lim\limits_{t\to\infty} \inf\limits_{|x|\leq ct} u(t,x) = 0, & \forall c \in [0, 2\sqrt{\lfloor \mu \rfloor_+}), \end{cases}$$

where

$$\lfloor \mu \rfloor_+ = \lim_{T\to\infty} \inf_{t>0} \frac{1}{T} \int_0^T \mu(t+s)\mathrm{d}s.$$

Remark 1.1.19. *Note that if*

$$\frac{1}{t}\int_0^t \mu(s)\mathrm{d}s \to \lfloor \mu \rfloor_+, \ \text{as } t \to \infty,$$

then the above theorem result is optimal. This condition holds for instance if μ exists a mean value. Either the choice of ways of averaging causes that the exact spreading speed *is not obtained, or the time heterogeneity structure leads to no exact spreading speed, see some examples in Section 13 in [23]. For*

$$\lfloor \mu \rfloor_+ < \gamma < \liminf_{t\to\infty} \frac{1}{t}\int_0^t \mu(s)\mathrm{d}s,$$

this limit $\lim_{t\to\infty} u(t,\gamma t)$ *remains unknown.*

We also refer the reader to Berestycki *et al.* [21, 23] for different methods of studying spreading speed in general heterogeneous media. In [23], the authors used homogenization techniques and generalized principal eigenvalues to prove the spreading speed in the framework of space-time heterogeneous equations.

There are also a lot of works considering propagation behaviour of reaction-diffusion equations in spatial varying or time-space heterogeneous environments, as well as in space with obstacles. In this manuscript, we do not plan to go further in this part. We refer the reader to [19, 23] and references cited therein for a nice survey in more general media.

1.1.3 Nonlocal diffusion

Dispersal is a driving factor for species expanding the distribution of its population. In previous, we discuss the *local diffusion* case, described by $\partial_{xx}u$, that is the motion governed by random work. However, some species in nature may disperse at a long distance in a short time. For example, pollen can be blown far away by wind and the spread of seeds can be affected by some animals possess caching behaviours such as some birds and squirrels. We refer the reader to [130] for more biological examples about long distance diffusion. To take into account this long range dispersion, the following intergro-differential equation was introduced,

$$\partial_t u = \int_{\mathbb{R}} J(x-y)\,[u(t,y)-u(t,x)]\,\mathrm{d}y + f(u), \text{ for } x\in\mathbb{R}. \tag{1.1.6}$$

Here the diffusion mechanism is described by convolution operator

$$\phi \mapsto \int_{\mathbb{R}} J(x-y)\,[\phi(y)-\phi(x)]\,\mathrm{d}y,$$

the quantity $J(x-y)$ represents the probability distribution of an individual of the species jumping from position y to position x. Hence the integral $\int_{\mathbb{R}} J(x-y)u(t,y)\mathrm{d}y$ shows the rate of individuals arriving at location x from other places, while the integral $\int_{\mathbb{R}} J(x-y)u(t,x)\mathrm{d}y$ is the rate of individuals leaving from location x to other places.

The propagation phenomena in the above equation have attracted a lot of interests in the last decades. Despite its biological sense, the nonlocal diffusion operator also brings some new mathematical difficulties such as the dynamical system generated by (1.1.6) is noncompact.

Local diffusion vs. Nonlocal diffusion

We first focus on the pure diffusive cases: the linear equation with local diffusion (namely heat equation),

$$\begin{cases} \partial_t u(t,x) = \partial_{xx}u(t,x), & t>0, x\in\mathbb{R},\\ u(0,x)=u_0(x), & x\in\mathbb{R}, \end{cases} \tag{1.1.7}$$

and the linear equation with nonlocal diffusion,

$$\begin{cases} \partial_t u(t,x) = \int_{\mathbb{R}} J(x-y)\,[u(t,y)-u(t,x)]\,\mathrm{d}y, & t>0, x\in\mathbb{R}, \\ u(0,x) = u_0(x), & x\in\mathbb{R}. \end{cases} \tag{1.1.8}$$

Let us compare the two diffusion mechanisms from various aspects.

• **Derivation of diffusion:** For the local diffusion case, there are two ways to introduce the notion of local diffusion, see Murray [121]. One way is according to Fick's law, namely, the flux of material which could be cells, amount of animals and so on, is proportional to the negative gradient of concentrations of the material. The other way is considering the random walk of the dif fusing particles. More details can be found in [121].

For the derivation of nonlocal diffusion model, we refer the reader to [92] where used discretization and approximation. We also refer to [114] for deriving the theoretical forms of dispersal kernels.

• **Fundamental solution:** The function

$$\Phi(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}}, \quad t>0, x\in\mathbb{R},$$

is the fundamental solution of the heat equation. We can employ Φ to fashion a solution to Cauchy problem (1.1.7) as

$$u(t,x) = \int_{\mathbb{R}} \Phi(t,x-y)u_0(y)\mathrm{d}y = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} u_0(y)\mathrm{d}y.$$

The fundamental solution of nonlocal diffusion equation (1.1.8) is

$$\Phi_J(t,x) = e^{-t}\delta_0(x) + \mathcal{K}_t(x),$$

where δ_0 is a Dirac function at 0 and $\mathcal{K}_t = \mathcal{K}(t,x)$ is a smooth function defined in Fourier variables by

$$\hat{\mathcal{K}_t}(\xi) = e^{-t}\left(e^{t\hat{J}(\xi)} - 1\right).$$

Moreover the solution of (1.1.8) can be written as

$$u(t,x) = \int_{\mathbb{R}} \Phi_J(t,x-y)u_0(y)\mathrm{d}y.$$

We refer the reader to monograph [5] for more details.

• **Parabolic regularity:** From the above fundamental solutions, one can observe the following facts instantly. The solution of heat equation (1.1.7) enjoys the parabolic regularizing effect. However, there is no parabolic regularity in nonlocal diffusion equation (1.1.8). As well as, the semi-flow generated by such nonlocal diffusion equation is noncompact. These differences bring new difficulties in analysis nonlocal diffusion equations no matter using PDE arguments or dynamical system methods.

• **Maximum principles:** Both cases enjoy the maximum principles, see [64, 5].

• **Approximation:** The heat equation (1.1.7) can be seen as an approximation of (1.1.8). Indeed if we consider the kernel function J which is compactly supported and symmetric. Let us define J_ε given by

$$J_\varepsilon(x) := \frac{1}{\varepsilon} J(\frac{x}{\varepsilon}), \text{ for } 0<\varepsilon\ll 1.$$

Then formally we have

$$
\begin{aligned}
\int_{\mathbb{R}} J_\varepsilon(x-y)\,[u(t,y)-u(t,x)]\,\mathrm{d}y &= \frac{1}{\varepsilon}\int_{\mathbb{R}} J(\frac{x-y}{\varepsilon})\,[u(t,y)-u(t,x)]\,\mathrm{d}y \\
&= \int_{\mathbb{R}} J(z)\,[u(t,x+\varepsilon z)-u(t,x)]\,\mathrm{d}z \\
&= \varepsilon\partial_x u(t,x)\int_{\mathbb{R}} J(z)z\mathrm{d}z + \frac{\varepsilon^2}{2}\partial_{xx}u(t,x)\int_{\mathbb{R}} J(z)z^2\mathrm{d}z + o(\varepsilon^2) \\
&= C\varepsilon^2\partial_{xx}u(t,x) + o(\varepsilon^2).
\end{aligned}
$$

In the above last equality, we use the fact that $\int_{\mathbb{R}} J(z)z\mathrm{d}z = 0$ and $C = \frac{1}{2}\int_{\mathbb{R}} J(z)z^2\mathrm{d}z < \infty$. This is due to J is assumed to be symmetric and compactly supported.

• **Asymptotic behaviour for nonlocal diffusion equation:** In [34], the authors proved that the long time behavior of the solutions to (1.1.8) is determined by the behaviour of J at infinity. They showed that if the kernel function J is symmetric and decays sufficiently fast at infinity (such as J is compactly supported or $J = e^{-x^2}$), then the asymptotic behaviour is the same as the one for the heat equation, that is the solution $u(t,x)$ to (1.1.8) satisfies:

$$\lim_{t\to\infty} t^{\frac{1}{2}} \max_{x\in\mathbb{R}} |u(t,x) - v(t,x)| = 0,$$

where $v(t,x)$ is the solution of heat equation with initial data $u(0,x)$. As well as, for some kernel function J decays slowly, [34] obtained that the asymptotic behaviour is given by the nonlocal fractional Laplacian parabolic equation. The fractional power of the Laplacian is that the kernel function satisfies $J(x-y) \sim |x-y|^{-1-2s}$ for $s \in (0,1)$.

Note that the tail of the dispersal kernel has an important effect on the dynamical behaviour of nonlocal diffusion equation even for the pure diffusive equations. The next definition gives the classification of kernel function according to its behaviour at infinity.

Definition 1.1.20. *The kernel function $J \in L^1(\mathbb{R})$ is called* ***thin-tailed kernel*** *(or exponentially bounded) if there exists some constant $\lambda_0 > 0$ such that*

$$\int_{\mathbb{R}} J(y)e^{\lambda_0|y|}\mathrm{d}y < \infty.$$

Otherwise, if $\int_{\mathbb{R}} J(y)e^{\lambda|y|}\mathrm{d}y = \infty$ for any $\lambda > 0$, then J is called ***fat-tailed kernel****.*

Some examples of such kernel functions are shown in Figure 1.6. In this manuscript, we mainly focus on the thin-tailed kernel.

Travelling wave results

Now we review the results about propagation phenomena in Fisher-KPP equations with nonlocal diffusion

$$\partial_t u = \int_{\mathbb{R}} J(x-y)\,[u(t,y)-u(t,x)]\,\mathrm{d}y + f(u), \text{ for } x \in \mathbb{R}, \tag{1.1.9}$$

where f satisfies KPP assumption (1.1.3).

The first work about the existence of travelling waves in (1.1.9) is by Schumacher [135]. Then Carr and Chmaj [32] completed this work, which extended the uniqueness of travelling wave to minimal wave speed. We refer the reader to Coville *et al.* [44, 41] for monostable nonlinearity, to Bates *et al.* [15] for bistable case and to Liang and Zhao [104, 105] and Yagisita [168] for abstract dynamical system methodology.

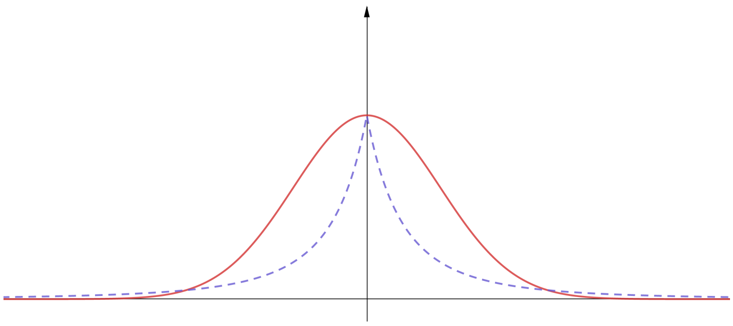


Figure 1.6: The red solid line is an example of the thin-tailed kernel function: $J(x) = e^{-|x|^2}$, while the blue dashed line is an example of the fat-tailed kernel function: $J(x) = (1+|x|)^{-3}$.

Theorem 1.1.21 (see [32, 41, 135]). *Assume that J is a thin-tailed kernel. There exists $c^* \in \mathbb{R}$ such that for all $c \geq c^*$, equation (1.1.9) exists a travelling wave solution $u(t,x) = \varphi(x-ct)$ with speed c and the wave profile φ satisfying*

$$\begin{cases} -c\varphi'(z) = \displaystyle\int_{\mathbb{R}} J(y)\left[\varphi(z-y) - \varphi(z)\right] \mathrm{d}y + f(\varphi(z)), & \text{for } z \in \mathbb{R}, \\ \varphi(-\infty) = 1 \text{ and } \varphi(\infty) = 0. \end{cases}$$

And such solution is unique up to translation. While there is no such solutions if $c < c^*$. Moreover, the minimal speed c^* is characterized by

$$c^* := \inf_{\lambda>0} \frac{\int_{\mathbb{R}} J(y)\left[e^{\lambda y} - 1\right] \mathrm{d}y + f'(0)}{\lambda}.$$

Note that for the nonlocal diffusion KPP equation, the minimal wave speed is also linearly determined. The quantity c^* can be derived similarly as the case of classical KPP equation.

Spreading speed results

Let us consider the Cauchy problem of (1.1.9) supplemented with compactly supported initial data u_0. To describe the large time behaviour of solutions, the asymptotic speed of spread is studied in [114, 167]. The result reads as follows:

Theorem 1.1.22 (see [114, 167]). *Assume that function J is a thin-tailed kernel. Let $u(t,x)$ be the solution of (1.1.9) equipped with initial data u_0. If u_0 is compactly supported, then there exist two constants c_l^* and c_r^* such that*

$$\begin{cases} \displaystyle\lim_{t\to\infty} \sup_{x\geq c_2 t,\ x\leq c_1 t} u(t,x) = 0, & \text{for } c_1 < c_l^* \text{ or } c_2 > c_r^*, \\ \displaystyle\lim_{t\to\infty} \sup_{c_1 t\leq x\leq c_2 t} |u(t,x) - 1| = 0, & \text{for } c_l^* < c_1 < c_2 < c_r^*, \end{cases}$$

where c_l^* and c_r^* are defined by

$$c_r^* := \inf_{\lambda>0} \lambda^{-1}\left(\int_{\mathbb{R}} J(y)\left[e^{\lambda y} - 1\right] \mathrm{d}y + f'(0)\right),$$
$$c_l^* := \sup_{\lambda<0} \lambda^{-1}\left(\int_{\mathbb{R}} J(y)\left[e^{\lambda y} - 1\right] \mathrm{d}y + f'(0)\right).$$

Remark 1.1.23. *Since here the kernel function is not assumed to be symmetric, the speed propagating to the left and to the right can be different. Also, the speed may not be positive. These are different from the local diffusion KPP equation.*

Note that the minimal speed of wave (propagating to right) coincides with the right spreading speed which is finite for (1.1.9) with thin-tailed nonlocal dispersal kernel.

In the case of the fat-tailed dispersal kernel in (1.1.9), the acceleration phenomena may appear and the spreading speed is infinite. The first rigorous mathematical results are due to Garnier [73]. Herein, instead of stating precise results in [73], we only recall the simulation in [73] to show the acceleration phenomena, see Figure 1.7.

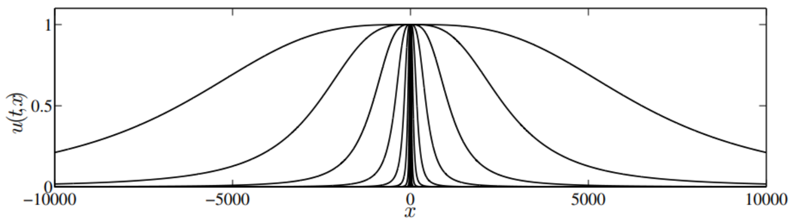


Figure 1.7: The solution $u(t,x)$ of (1.1.9) with $f(u) = u(1-u)$, $u_0(x) = \max\{0, (1 - (x/10)^2)\}$ and fat-tailed kernel $J(x) = (1+|x|)^{-3}$, see [73].

We also refer the reader to some recent results about Bramson correction for nonlocal diffusion KPP equations, see [78, 132].

Time varying environment

As far as the propagation phenomena in time heterogeneous nonlocal diffusion equations are concerned, we refer the reader to [93, 94] in time periodic environment, to [143, 144, 145] in general time heterogeneities and references cited therein.

Again, we recall the example of long range dispersion that the pollen can be blown far away by wind. Note that the wind velocity is varying with time. Thus, in Chapter 2 of this manuscript, we consider the nonlocal diffusion KPP equations with both dispersal kernel and reaction term are dependent on time,

$$\partial_t u = \int_{\mathbb{R}} J(t, x-y)\,[u(t,y) - u(t,x)]\,\mathrm{d}y + f(t,u), \ \text{for } x \in \mathbb{R}.$$

We also refer the reader to [42, 107, 146] for nonlocal diffusion equation with spatial heterogeneous reaction term, to [108, 131] for spatial heterogeneous kernel function and references cited therein for a nice review about propagation phenomena in spatial (and time) heterogeneous nonlocal diffusion equations.

1.1.4 Prey-predator systems

The population dynamic of species can be affected by other interacting species. The systems of equations involving two or more species are considered in mathematical biology and ecology. There are three main types of interaction: (i) The *prey-predator* type means that the growth rate of one population is decreased while the other one is increased; (ii) The *competition* type means that the growth rate of each species is decreased due to this type interaction; (iii) If the growth rate of each species is increased then it is called *mutualism*. In this manuscript, we only focus on the prey-predator situation.

Let us first recall the classical Lotka-Volterra prey-predator system

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = ru - puv, \\ \frac{\mathrm{d}v}{\mathrm{d}t} = quv - \nu v, \end{cases} \tag{1.1.10}$$

where $u = u(t)$ and $v = v(t)$ denote the density of the prey and the predator at time t, respectively. The parameters r, p, q and ν are real and positive numbers. In the first component u-equation, the parameter r is the growth rate of the prey while the predation term puv describes the rate of predation upon the prey in the form of proportional to the rate at which the predator and the prey meet. In the predator equation, the term quv represents the growth of the predator species contributed by the prey. Note that it is proportional to the available prey as well to the size of the predator population. The parameter ν is the death rate of the predator. The term $-\nu v$ in (1.1.10) leads to exponentially decay in the absence of any prey.

In the 1920s, Volterra used the simple prey-predator model (1.1.10) to explain the oscillatory levels of certain fish catches in the ocean. This model was also derived by Lotka in the theory of chemical reaction. Later, the model was extended to some more general form as:

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = uh(u) - \Pi(u)v, \\ \frac{\mathrm{d}v}{\mathrm{d}t} = \mu\Pi(u)v - \nu v. \end{cases}$$

Herein, the constants μ and ν describe the conversion rate of biomass and death rate of the predator respectively. The function $h : [0, \infty) \to \mathbb{R}$ represents the intrinsic growth rate of the prey, for example the logistic growth $h(u) = r(1 - u)$. The function $\Pi : [0, \infty) \to \mathbb{R}$ is the functional response to predator which varies with the prey density, some typical examples as:

$$\Pi(u) = qu, \ \Pi(u) = \frac{mu^n}{b + u^n} \text{ with } n \geq 1, \text{ and } \Pi(u) = m(1 - e^{-u}),$$

where q, m and b are positive numbers. The form of the functional response was developed by Holling [89] who showed some saturation effect. We also refer the reader to [39, 129, 133] for more examples of functions h and Π.

Since the prey and the predator are spatially distributed, then the systems of reaction-diffusion equations have attracted a lot of attention in the last decades. In order to understand the propagation behaviours in diffusive prey-predator systems, the two important mathematical notions, namely travelling wave and spreading speed, are used to study such systems also. Next, we recall some well known results about propagation phenomena in reaction-diffusion systems of prey-predator type.

Travelling wave results

The pioneering work by Dunbar [62, 63] considered following diffusive Lotka-Volterra system with logistic growth of the prey,

$$\begin{cases} \partial_t u = d\partial_{xx} u + ru\,(1 - u) - puv, \\ \partial_t v = \partial_{xx} v + quv - \nu v, \end{cases} \tag{1.1.11}$$

where $d \in [0, 1]$ is the diffusion rate of the prey, the positive parameters r, p, q and ν represent the growth rate of the prey, the predation rate, the conversion rate and the death rate of the predator, respectively. As well as assume that $q > \nu$. Note that there are three steady states in the above system: $(0,0)$, $(1,0)$ and

$$(u^*, v^*) = \left(\frac{\nu}{q}, \frac{r(q-\nu)}{pq}\right).$$

One can observe that $(0,0)$ and $(1,0)$ are unstable while the coexistence equilibrium (u^*, v^*) is stable. [62, 63] used the shooting method and LaSalle's invariance principle to study the existence of travelling wave in (1.1.11). The precise theorem reads as follows:

Theorem 1.1.24. *There exists a travelling wave solution* $(u,v)(t,x) = (U,V)(x-ct)$ *satisfying* $(U,V)(-\infty) = (u^*, v^*)$ *and* $(U,V)(+\infty) = (1,0)$ *in (1.1.11) if and only if* $c \geq c^* := 2\sqrt{q-\nu}$.

We remark that the travelling wave in system (1.1.11) indicates the existence of a transition zone from a boundary equilibrium to a coexistence steady state. The wave is analogous to the travelling wave in KPP-type scalar equations. However, here the travelling wave in the prey-predator system may be non-monotone, see Figure 1.8.

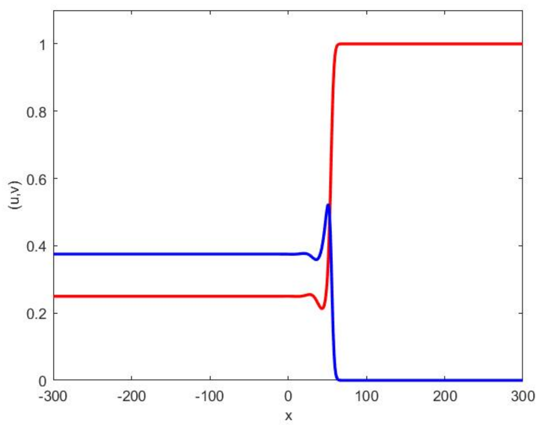


Figure 1.8: A solution (u,v) of (1.1.11) at $t = 50$ and with parameter $d = 1$, $r = 1$, $p = 2$, $q = 1.5$ and $\nu = 1$.

We also refer the reader to the pioneering work [72] via connection index by Gardner and to some recent works [90, 91, 61], as well as the survey paper [100] and the reference cited therein for the existence of travelling wave solutions in the prey-predator system with a more general functional response and for a nice review of this topic.

Spreading speed results

Let us consider the Cauchy problem of (1.1.11), namely

$$\begin{cases} \partial_t u = d\partial_{xx} u + ru(1-u) - puv, \\ \partial_t v = \partial_{xx} v + quv - \nu v, \end{cases}$$

equipped with initial data

$$u(0,x) = u_0(x), v(0,x) = v_0(x). \tag{1.1.12}$$

In this part, we only assume that all parameters in (1.1.11) are positive numbers and $q > \nu$.

One may expect that the large time behaviour of solutions to the Cauchy problem (1.1.11)-(1.1.12) is already determined by such travelling wave solutions, however, travelling wave is only a special class solution. The connection between wave solutions and the asymptotic behaviour of the Cauchy problem (1.1.11)-(1.1.12) has been rarely studied. We refer the reader to [71] which investigated the local stability of wave solutions.

Before the recent work [55] by Ducrot, Giletti and Matano, little has been known about the spreading speed of prey-predator systems (including (1.1.11)-(1.1.12) considered here), largely extent because of the lack of the comparison principle for such system. In [55], the authors obtained exact spreading speeds for a large class reaction-diffusion systems of prey-predator type by using some ideas from uniform persistence theory in dynamical systems. For the persistence theory, we refer the reader for instance to Hale and Waltman [82], to Magal and Zhao [116] and to the monograph [150] by Smith and Thieme.

Next, we introduce some notations and recall the precise spreading speed results for (1.1.11)-(1.1.12) which is obtained in [55]. Let us define quantities c_u^* and c_v^* by

$$c_u^* := 2\sqrt{dr} \text{ and } c_v^* := 2\sqrt{q-\nu}.$$

One can note that c_u^* is the spreading speed of the prey u in the absence of predator. This is due to when $v \equiv 0$, the u-equation in (1.1.11) becomes a KPP equation

$$\partial_t u = d\partial_{xx} u + ru\,(1-u)\,.$$

On the other hand, when the prey is abundant, namely $u \equiv 1$, the function v satisfies

$$\partial_t v = \partial_{xx} v + (q-\nu)\,v.$$

Then one can use the same argument as in [8] to show that c_v^* is the spreading speed of the above equation equipped with compactly supported initial data. Here the only difference is that the solution may not converge to a stationary state after propagation but grow unbounded.

In [55], the authors considered the prey and the predator can co-invade an empty space. Their first theorem showed that if the predator invades the empty environment slower than the prey, then the propagation occurs in two separate steps involving an intermediate equilibrium (namely $u = 1, v = 0$) in the middle zone, see Figure 1.9.

Theorem 1.1.25 ([55]). *Let u_0 and v_0 be two given bounded and continuous functions in $\mathbb{R}$ with compact support, and $0 \not\equiv\leq u_0 \leq 1$, $0 \not\equiv\leq v_0$. Let $(u,v) = (u(t,x), v(t,x))$ be the solution of (1.1.11) with initial data (u_0, v_0). If $c_u^* > c_v^*$, then the function pair (u,v) satisfies:*

(i) for all $c > c_u^*$, one has $\lim_{t\to\infty} \sup_{|x|\geq ct} u(t,x) = 0$;

(ii) for all $c_v^* < c_1 < c_2 < c_u^*$ and for all $c > c_v^*$ one has:

$$\lim_{t\to\infty} \sup_{c_1 t\leq |x|\leq c_2 t} |1-u(t,x)| + \lim_{t\to\infty} \sup_{|x|\geq ct} v(t,x) = 0;$$

(iii) for all $c \in [0, c_v^*)$ one has:

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} v(t,x) > 0,$$

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} u(t,x) > 0 \text{ and } \limsup_{t\to\infty} \sup_{|x|\leq ct} u(t,x) < 1.$$

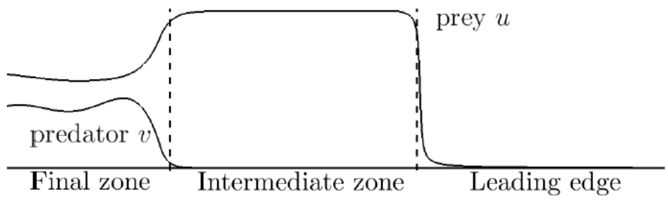


Figure 1.9: Slow predator case. This figure is taken from [55].

The following second main result in [55] showed that if the predator invades the empty environment faster than the prey, then the predator's population could grow fast enough to overtake the prey. One can note that the system spreading speed is c_u^*, which means that the prey and the predator invade the empty space almost simultaneously, see Figure 1.10.

Theorem 1.1.26 ([55]). *Let u_0 and v_0 be two given bounded and continuous functions in $\mathbb{R}$ with compact support, and $0 \not\equiv\leq u_0 \leq 1$, $0 \not\equiv\leq v_0$. Let $(u, v) = (u(t, x), v(t, x))$ be the solution of (1.1.11) with initial data (u_0, v_0). If $c_u^* \leq c_v^*$, then the function pair (u, v) satisfies:*

(i) for all $c > c_u^*$, one has $\lim_{t\to\infty} \sup_{|x|\geq ct} [u(t, x) + v(t, x)] = 0$;

(ii) for all $c \in [0, c_u^*)$ one has:

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} v(t, x) > 0,$$

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} u(t, x) > 0 \text{ and } \limsup_{t\to\infty} \sup_{|x|\leq ct} u(t, x) < 1.$$

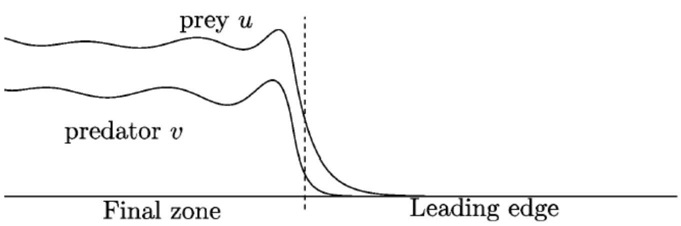


Figure 1.10: Fast predator. This figure is taken from [55].

The similar idea in [55] was also extended to study the spreading speed for prey-predator systems in a shift environment in [40] and to investigate the propagation behaviour arising in the interaction between two predators and one prey, see [53]. We also refer the reader to [37, 49, 51, 110] for studying large time behaviour of solutions in other types prey-predator systems.

Time heterogeneous systems

As shown before, there are many biotic and abiotic factors that vary with time. These affect the species a lot. In the case of interactive species, it is also necessary to include time variations in modeling such as the predation rate and conversion rate of biomass may depend on time, see [28, 45, 75] for the non-autonomous prey-predator system of ordinary differential equations. We refer the reader to [25, 158, 170, 171] for travelling wave in time-periodic reaction-diffusion systems and to recent paper [4] for generalized travelling wave in non-autonomous prey-predator systems.

For the spreading speed results in non-autonomous monotone systems, we refer to [66, 103] in periodic media and to [12] for the almost periodic case. It seems that there are only partial results about spreading speed for time periodic prey-predator systems, see [157] for some estimates about spreading speed.

To the best of our knowledge, the spreading behaviours of prey-predator systems with time heterogeneity might remain unknown at least theoretically. Thus, we study the spreading speed for reaction-diffusion systems of prey-predator type with general time heterogeneities in Chapter 4. As well as, we provide a different method compared with [55].

1.1.5 Discrete equations

In the above, we have recalled propagation results for reaction-diffusion equations in continuous time and space variables which show spatio-temporal dynamic behaviour of solutions in an unbounded domain. There are also large classes of models in which the time and space variables are allowed to be discrete.

The *difference equations* (that is discrete in time variable) rose to fame since in 1975, May [119] discovered that the complex and chaotic dynamic behavior could be generated by simple density-dependent growth functions. The difference equations sometimes are easier to formulate and simulate. The celebrated work which investigates asymptotic properties for discrete-generation population dynamic models with dispersal in continuous space or discrete space was given by Weinberger [162]. We refer to monograph [113] for studying *integrodifference equations* (where discrete in time variable and continuous in space variable) and its application in ecology. For dynamical models where time and space are discrete, known as *coupled map lattices*, we refer the reader to [46] and references cited therein.

In this manuscript, we mainly focus on the *lattice differential equations*, sometimes known as *patch models*, which is with continuous time variable and discrete space variable. On one hand, the lattice differential equations arise in several different contexts, for instance modeling species grow over patchy environment, we refer the reader to [16, 96] and to [47] for a list of ecological scenarios with patchy environments. As well as, such lattice equations can be used to describe phase transition, see [14]. On the other hand, lattice equations are the discretization of the differential equations in which the spatial variable are continuous.

Let us first recall some propagation results for the following Fisher-KPP equation on lattice,

$$\frac{\mathrm{d}u(t,i)}{\mathrm{d}t} = u(t,i+1) - 2u(t,i) + u(t,i-1) + f(u(t,i)),\ i \in \mathbb{Z}, \tag{1.1.13}$$

where $f \in C^1([0,1])$ satisfies KPP conditions. In fact, it is an infinite system of ordinary differential equation indexed by points in a lattice $\mathbb{Z}$. As well as, it is a discrete version of (1.1.2). The propagation phenomena in lattice single equations and systems have attracted a lot of interest. In [174], the authors proved the following theorem.

Theorem 1.1.27. *There exists a travelling wave solution $u(t,i) = U(i - ct)$ with speed c in (1.1.13) if and only if $c \geq c_*$, where*

$$c_* := \min_{\lambda>0} \frac{e^{\lambda} - 2 + e^{-\lambda} + f'(0)}{\lambda}. \tag{1.1.14}$$

The next theorem about spreading speed for (1.1.13) was obtained in [162].

Theorem 1.1.28. *If (1.1.13) is equipped with initial data u_0 where $1 \geq u_0 \geq 0, u_0 \not\equiv 0$ and $u_0(j) = 0$ for all $|j| \geq k$ with $k \in \mathbb{Z}$. Then c_* defined in (1.1.14) is spreading speed of (1.1.13) supplemented with u_0.*

Recently, the Bramson correction for lattice equation (1.1.13) has been proved by the paper [24].

The spatial motion of individuals may have different form with (1.1.13). Some general diffusion is modeled by a discrete convolution operator. There are some propagation results in [14, 32] for lattice differential equations with nonlocal diffusion as follows:

$$\frac{\mathrm{d}u(t,i)}{\mathrm{d}t} = \sum_{j=-\infty}^{\infty} J(i-j)[u(t,j) - u(t,i)] + f(u(t,i)),\ i \in \mathbb{Z}.$$

We also refer to [35, 36, 65, 115] for existence, nonexistence and uniqueness of travelling waves and spreading speed in other forms lattice differential equations.

As we discussed in the previous subsections for space continuous variable, time heterogeneity is an important factor in biological modeling and mathematical structure. The generalized transition fronts and spreading speeds for lattice KPP equations in time varying environments are investigated in [139, 31, 155, 156]. We also mention some works devoted to understanding spreading phenomena in spatially heterogeneous lattice equations, see [79, 106] and references cited therein.

It is also necessary to consider the system of lattice equations since species living in patchy environment may interact. For the existence of travelling fronts in lattice systems, we refer to [80, 81] in monotone systems and to [38] for an endemic model. However, it seems that there are few results about the spreading speeds in lattice systems, especially of the prey-predator type, not to mention with general time heterogeneities. Thus, in Chapter 5, we will investigate spreading speeds of the non-autonomous prey-predator system in a lattice where the diffusion is described by a discrete convolution operator with time dependent kernel.

1.2 Our results

In this section, we state some important results obtained in this thesis. We focus on the propagation phenomena in non-autonomous equations and systems. General time heterogeneity is a common feature of my works. Before stating the precise results, let us introduce some notions related to time averaging. These notions, so called *least mean*, *upper mean* and *mean value*, will be used often throughout this thesis. Some of them have already been introduced in the previous section. Note that these notions are successfully used to study propagation phenomena in non-autonomous reaction-diffusion equations, see [124, 125, 140], (also refer to [4] for systems).

Definition 1.2.1. *The* ***least mean*** *(resp. the* ***upper mean****) of a function $g \in L^\infty(\mathbb{R})$ is defined as follows*

$$\lfloor g \rfloor := \sup_{T>0} \inf_{t\in\mathbb{R}} \frac{1}{T}\int_0^T g(t+s)\mathrm{d}s, \quad \left(resp.\ \lceil g \rceil := \inf_{T>0} \sup_{t\in\mathbb{R}} \frac{1}{T}\int_0^T g(t+s)\mathrm{d}s \right).$$

The least mean and upper mean value enjoy the following property.

Proposition 1.2.2 (see [124]). *For each function $g \in L^\infty(\mathbb{R})$, the least mean $\lfloor g \rfloor$ satisfies*

$$\lfloor g \rfloor = \lim_{T\to+\infty} \inf_{t\in\mathbb{R}} \frac{1}{T}\int_0^T g(t+s)\mathrm{d}s = \sup_{A\in W^{1,\infty}(\mathbb{R})} \inf_{t\in\mathbb{R}} (A'+g)(t),$$

while the upper mean $\lceil g \rceil$ satisfies

$$\lceil g \rceil = \lim_{T\to+\infty} \sup_{t\in\mathbb{R}} \frac{1}{T}\int_0^T g(t+s)\mathrm{d}s = \inf_{A\in W^{1,\infty}(\mathbb{R})} \sup_{t\in\mathbb{R}} (A'+g)(t).$$

The next notion is about mean value.

Definition 1.2.3. *A function $g \in L^\infty(\mathbb{R})$ is said to have a **mean value** if the least mean and the upper mean coincide, namely*

$$\lfloor g \rfloor = \lceil g \rceil.$$

This means that there exists some constant $\langle g \rangle \in \mathbb{R}$ such that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g(t+s)\mathrm{d}s = \langle g \rangle \text{ exists uniformorly for } t \in \mathbb{R}.$$

Remark 1.2.4. *For function $h \in L^\infty(0,\infty)$, we can define the least mean (resp. the upper mean) of h as (for notation simplicity, without any confusion, we still use the same notation)*

$$\lfloor h \rfloor := \sup_{T>0} \inf_{t\geq 0} \frac{1}{T}\int_0^T h(t+s)\mathrm{d}s, \quad \left(resp.\ \lceil h \rceil := \inf_{T>0} \sup_{t\geq 0} \frac{1}{T}\int_0^T h(t+s)\mathrm{d}s\right).$$

Similarly, we can define the mean value for $h \in L^\infty(0,\infty)$ by

$$\langle h \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T h(t+s)\mathrm{d}s, \text{ uniformly for } t \geq 0.$$

The similar property as Proposition 1.2.2 also holds true for $\lfloor h \rfloor$ and $\lceil h \rceil$.

As noticed in Proposition 1.1.10, mean value exists for a large class functions such as periodic, almost periodic and uniquely ergodic functions.

In our works, nonlocal diffusion is also a key factor. The focus of this manuscript is on exponentially bounded kernel functions. We give the definition of abscissa of convergence below.

Definition 1.2.5. *Let $(X, \|\cdot\|_X)$ be a Banach space. For $f \in L^1(\mathbb{R}; X)$ and $g \in l^1(\mathbb{Z}; X)$, we define quantities $\sigma(f)$ and $abs(g)$ which are called the abscissa of convergence of f and g respectively, as follows*

$$\sigma(f) = \sup\left\{\lambda \geq 0 : \text{ the improper integral } \int_{-\infty}^{\infty} e^{\lambda s} f(s)\mathrm{d}s \text{ converges in } X\right\},$$

and respectively

$$abs(g) = \sup\left\{\lambda \geq 0 : \text{ the series } \sum_{j=-\infty}^{\infty} e^{\lambda j} f(j) \text{ converges in } X\right\}.$$

1.2.1 Summary of Chapter 2: Generalized travelling fronts for nonautonomous Fisher-KPP equations with nonlocal diffusion

This work in collaboration with Arnaud Ducrot is published in *Annali di Matematica Pura ed Applicata* [58].

Problem

We investigate the existence and nonexistence of the generalized travelling wave solutions for the following non-autonomous nonlocal diffusion equation

$$\partial_t u(t,x) = \int_{\mathbb{R}} K(t,y)\,[u(t,x-y) - u(t,x)]\,\mathrm{d}y + F\big(t, u(t,x)\big),\ (t,x) \in \mathbb{R}\times\mathbb{R}. \quad (1.2.15)$$

Here $K = K(t,y)$ denotes a nonnegative time dependent and exponentially bounded dispersal kernel function while the nonlinear term $F = F(t,u)$ is of Fisher-KPP type with

$$F(t,0) = F(t,1) = 0,\ \forall t \in \mathbb{R}.$$

This equation typically models the spatio-temporal evolution of an invading population into some empty environment. Here the individual exhibits a long distance dispersion according to the kernel K, in other words, the quantity $K(t, x-y)$ corresponds to the probability of jumping from y to x at time t; while the local population dynamics (birth and death processes) is described by the time varying Fisher-KPP nonlinearity F.

Assumptions

Assumption 1.2.6 (Kernel $K = K(t,y)$)**.** *The kernel $K : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ satisfies the following set of assumptions:*

(i) The function K is measurable, nonnegative and $K(\cdot,y) \in L^\infty_+(\mathbb{R})$ for almost every $y \in \mathbb{R}$;

(ii) The map $\tilde{K} : y \mapsto K(\cdot,y)$ satisfies $\tilde{K} \in L^1(\mathbb{R}; L^\infty(\mathbb{R}))$;

(iii) The abscissa of convergence satisfies

$$\sigma(\tilde{K}) > 0.$$

In the following, for notational simplicity, we use $\sigma(K)$ instead of $\sigma\left(\tilde{K}\right)$.

For instance $K(t,y) = \exp\{-y^2/(1+t^2)\}$ satisfies Assumption 1.2.6. Next, we turn to our KPP assumptions for the nonlinear function $F = F(t,u)$.

Assumption 1.2.7 (KPP nonlinearity)**.** *We assume that the function F takes the form $F(t,u) = uf(t,u)$ where the function $f : \mathbb{R}\times[0,1] \to \mathbb{R}$ satisfies the following set of hypotheses:*

(f1) Assume that $f(\cdot,u) \in L^\infty(\mathbb{R})$ for all $u \in [0,1]$, and f is Lipschitz continuous with respect to $u \in [0,1]$, uniformly with respect to $t \in \mathbb{R}$;

(f2) Let $f(t,0) = 1, f(t,1) = 0$ for a.e. $t \in \mathbb{R}$ and

$$h(u) := \inf_{t\in\mathbb{R}} f(t,u) > 0 \text{ for all } u \in [0,1);$$

(f3) For almost every $t \in \mathbb{R}$, the function $u \mapsto f(t,u)$ is nonincreasing on $[0,1]$.

Remark 1.2.8. *In the above set of hypotheses, a typical example is $F(t,u) = u(1-u)$. We have assumed, for simplicity, that $f(t,0) \equiv 1$. This assumption can be relaxed by using a change of variable in time to take into account more general KPP nonlinearity function $F(t,u) = uf(t,u)$ such that $f(t,1) \equiv 0$ and $f(\cdot,0) = \mu \in L^\infty(\mathbb{R})$ with $\inf_{t\in\mathbb{R}} \mu(t) > 0$. Indeed, if $u = u(t,x)$ is a solution of (1.2.15) then by setting*

$$t \mapsto \tau(t) = \int_0^t \mu(s)\mathrm{d}s \text{ and } \hat{u}(\tau(t),x) = u(t,x),$$

the function $\hat{u}$ becomes a solution of the following equation

$$\partial_\tau \hat{u}(\tau,x) = \int_{\mathbb{R}} \hat{K}(\tau,y)\left[\hat{u}(\tau,x-y) - \hat{u}(\tau,x)\right]\mathrm{d}y + \hat{u}(\tau,x)\hat{f}(\tau,\hat{u}(\tau,x)),$$

wherein we have set

$$\hat{K}(\tau,y) = \frac{K(t,y)}{\mu(t)}, \quad \hat{f}(\tau,\hat{u}) = \frac{f(t,\hat{u})}{\mu(t)}.$$

Hence $\hat{F}(\tau,u) = u\hat{f}(\tau,u)$ becomes a KPP nonlinearity with $\hat{f}(\tau,0) \equiv 1$, while $\hat{K}$ satisfies Assumption 1.2.6 with $\sigma(K) = \sigma(\hat{K})$.

Now, for the reader convenience, we recall again the definition of generalized travelling wave which was used in [124, 125].

Definition 1.2.9. *A continuous function $u = u(t,x) : \mathbb{R}^2 \to [0,1]$ is said to be a **generalized travelling wave** of (1.2.15) with the wave speed function $c = c(t) \in L^\infty(\mathbb{R})$ if $u(t,x)$, solution to (1.2.15), can rewrite as*

$$u(t,x) = \phi\left(t, x - \int_0^t c(s)\mathrm{d}s\right), \ \forall (t,x) \in \mathbb{R}^2,$$

and the profile function $\phi : \mathbb{R}^2 \to [0,1]$ satisfies the following behaviours at $z = \pm\infty$:

$$\lim_{z\to-\infty} \phi(t,z) = 1 \text{ and } \lim_{z\to+\infty} \phi(t,z) = 0 \text{ uniformly for } t \in \mathbb{R}.$$

Recall that generalized travelling waves are nothing but *generalized transition waves* (see Definition 1.1.14) associated to a globally Lipschitz continuous interface function $X(t) = \int_0^t c(s)\mathrm{d}s$. Let us also notice that when the profile ϕ of a generalized travelling wave $u = u(t,x)$ with a speed function $c = c(t)$ is rather smooth in space and time, say locally Lipschitz continuous, then it satisfies the following equation for almost every $(t,z) \in \mathbb{R}^2$:

$$\partial_t \phi(t,z) = c(t)\partial_z \phi(t,z) + \int_{\mathbb{R}} K(t,y)\left[\phi(t,z-y) - \phi(t,z)\right]\mathrm{d}y + F(t,\phi(t,z)), \tag{1.2.16}$$

together with the limit behaviours

$$\lim_{z\to-\infty} \phi(t,z) = 1 \text{ and } \lim_{z\to+\infty} \phi(t,z) = 0 \text{ uniformly for } t \in \mathbb{R}. \tag{1.2.17}$$

Linear speed

Linearizing (1.2.16) at $\phi = 0$, one has

$$\partial_t \varphi(t,z) = c(t)\partial_z \varphi(t,z) + \int_{\mathbb{R}} K(t,y)\left[\varphi(t,z-y) - \varphi(t,z)\right] \mathrm{d}y + \varphi(t,z). \tag{1.2.18}$$

For some $a \in W^{1,\infty}(\mathbb{R})$, ansatz $\varphi(t,z) = e^{-\lambda(z+a(t))}$ into above linear equation, we obtain that

$$c(t) = \lambda^{-1}\left(\int_{\mathbb{R}} K(t,y)[e^{\lambda y} - 1]\mathrm{d}y + 1\right) + a'(t).$$

Now for each $\lambda \in (0, \sigma(K))$ and $a \in W^{1,\infty}(\mathbb{R})$, for $t \in \mathbb{R}$, let us introduce

$$c(\lambda)(t) := \lambda^{-1}\left(\int_{\mathbb{R}} K(t,y)\left[\mathrm{e}^{\lambda y} - 1\right] \mathrm{d}y + 1\right), \tag{1.2.19}$$

and define $c_{\lambda,a} \in L^\infty(\mathbb{R})$ given by

$$c_{\lambda,a}(t) = c(\lambda)(t) + a'(t). \tag{1.2.20}$$

From the property of least mean (see Proposition 1.2.2), one can observe that

$$\lfloor c_{\lambda,a}(\cdot) \rfloor = \lfloor c(\lambda)(\cdot) \rfloor.$$

In order to define the critical speed, we consider the set

$$\Lambda = \left\{\lambda \in (0, \sigma(K)) : \exists \lambda' \in (\lambda, \sigma(K)),\ \forall k \in (\lambda, \lambda'],\ \lfloor c(\lambda) - c(k) \rfloor > 0\right\}.$$

We can show the following property of speed function.

Proposition 1.2.10. *There exists $\lambda^* \in (0, \sigma(K)]$ such that $\Lambda = (0, \lambda^*)$ and $\lambda \mapsto \lfloor c(\lambda) \rfloor$ is decreasing on Λ. Moreover, one has $c(\lambda)$ is of class C^1 from $(0, \sigma(K))$ to $L^\infty(\mathbb{R})$ and*

$$\left\lfloor -\frac{\mathrm{d}c(\lambda)}{\mathrm{d}\lambda} \right\rfloor > 0,\ \forall \lambda \in (0, \lambda^*) \text{ and } \left\lfloor -\frac{\mathrm{d}c(\lambda^*)}{\mathrm{d}\lambda} \right\rfloor = 0 \text{ if } \lambda^* < \sigma(K).$$

Next we introduce the admissible speed function set.

Definition 1.2.11. *The set $\mathscr{C} \subset L^\infty(\mathbb{R})$ is called to be* ***admissible speed function set****, if the function $c \in \mathscr{C}$, then there exists a generalized travelling wave with the speed function c in (1.2.15).*

In this work, we give some estimates for the admissible speed function set $\mathscr{C}$. In addition, under suitable assumptions on time varying coefficients, we can derive a sharp estimate for the addmissible speed set.

Existence of generalized travelling waves

Using above notations, our first theorem ensures the existence of generalized travelling waves for problem (1.2.15) with the speed function $c_{\lambda,a}$, for each $\lambda \in (0, \lambda^*)$ and $a \in W^{1,\infty}(\mathbb{R})$.

Theorem 1.2.12 (Existence)**.** *Let Assumption 1.2.6 and 1.2.7 be satisfied. Recalling that λ^* is defined in Proposition 1.2.10, for each $\lambda \in (0, \lambda^*)$ and each $a \in W^{1,\infty}(\mathbb{R})$, problem (1.2.15) possesses a generalized travelling wave with the speed function $c_{\lambda,a} \in L^\infty(\mathbb{R})$, defined in (1.2.20). Furthermore, these travelling wave profiles are globally Lipschitz continuous on $\mathbb{R}^2$.*

In other words, the above theorem ensures that

$$\left\{t \mapsto c_{\lambda,a}(t), \quad \lambda \in (0,\lambda^*) \text{ and } a \in W^{1,\infty}(\mathbb{R})\right\} \subset \mathscr{C}.$$

Recalling the definition of $c(\lambda)$ in (1.2.19) and Proposition 1.2.10, one also obtains that

$$\left(\lim_{\lambda\to\lambda^*} \lfloor c(\lambda)\rfloor, \infty\right) \subset \lfloor \mathscr{C} \rfloor := \{\lfloor c\rfloor \,,\; c \in \mathscr{C}\}. \tag{1.2.21}$$

Sketch the proof of Theorem 1.2.12

Now we give the sketch of proof for above Theorem 1.2.12. It is a standard argument. Generally speaking, we first construct proper super-solutions and sub-solutions, then we consider the Cauchy problem with a suitable initial data at time $t = -n$, for some integer $n \geq 1$. By a limiting argument, letting $n \to \infty$, we obtain a generalized travelling wave solution. It is unlike the classical diffusion case that one can use parabolic regularity results and Arzelà-Ascoli theorem to obtain the limit function. Due to the nonlocal diffusion operator, such regularity results are not available, as discussed in the previous section. This is the main technical difficulty in nonlocal diffusion problem. Here we will provide Lipschitz regularity estimates for the solution of Cauchy problem at $t = -n$.

Let us first construct super-solution and sub-solution of (1.2.16) with the speed function $c(t) = c_{\lambda,a}(t)$ where $\lambda \in (0,\lambda^*)$ and $a \in W^{1,\infty}(\mathbb{R})$ are given. Set for $(t,z) \in \mathbb{R}^2$,

$$\overline{\phi}(t,z) = \min\left\{1, e^{-\lambda(z+a(t))}\right\}.$$

By direct computation and the assumption $F(t,\phi) \leq \phi$, one has $\overline{\phi}$ is a super-solution of (1.2.16). For the same fixed λ and a, for some $b \in W^{1,\infty}(\mathbb{R})$ and $k > 0$ sufficiently small, for $(t,z) \in \mathbb{R}^2$, we define

$$\underline{\phi}(t,z) := \max\{0, \varphi(t,z)\} \text{ with } \varphi(t,z) = e^{-\lambda(z+a(t))} - e^{-\lambda a(t)+b(t)}e^{-(\lambda+k)z}.$$

From Proposition 1.2.10, one can choose some $k > 0$ small enough such that

$$\lfloor c_{\lambda,a} - c_{\lambda+k,a} \rfloor > 0.$$

Combined with the above inequality and the property of least mean, we can verify that $\underline{\phi}$ is a sub-solution of (1.2.16).

Next, we consider the following initial value problem, posed in $t \geq -n$ and $z \in \mathbb{R}$,

$$\begin{cases} \partial_t \phi = c_{\lambda,a}(t)\partial_z\phi(t,z) + \displaystyle\int_{\mathbb{R}} K(t,y)\left[\phi(t,z-y) - \phi(t,z)\right]\mathrm{d}y + F(t,\phi), \\ \phi(-n,z) = \overline{\phi}(-n,z). \end{cases} \tag{1.2.22}$$

We denote $\phi^n = \phi^n(t,z)$ to be the solution of the above equation and define the function $u^n = u^n(t,z)$ by

$$u^n(t,z) = \phi^n\left(t, z - \int_0^t c_{\lambda,a}(s)\mathrm{d}s\right).$$

One may observe that the function $u^n(t,z)$ satisfies the following equation without the drift term $c_{\lambda,a}(t)\partial_z$,

$$\begin{cases} \partial_t u(t,z) = \displaystyle\int_{\mathbb{R}} K(t,y)\left[u(t,z-y) - u(t,z)\right]\mathrm{d}y + F(t,u), \; t \geq -n, \; z \in \mathbb{R}, \\ u(-n,z) = \overline{\phi}\left(-n, z - \displaystyle\int_0^{-n} c_{\lambda,a}(s)\mathrm{d}s\right), \; z \in \mathbb{R}. \end{cases} \tag{1.2.23}$$

Applying the comparison principle (which has been constructed in this work), one obtains that

$$\underline{\phi}\left(t, z - \int_0^t c_{\lambda,a}(s)\mathrm{d}s\right) \le u^n(t,z) \le \overline{\phi}\left(t, z - \int_0^t c_{\lambda,a}(s)\mathrm{d}s\right).$$

Moreover since the function $z \mapsto \overline{\phi}(-n, z - \int_0^{-n} c_{\lambda,a}(s)\mathrm{d}s)$ is nonincreasing in $\mathbb{R}$, then the function $z \mapsto u^n(t,z)$ is also nonincreasing with respect to $z \in \mathbb{R}$ for each given $t \ge -n$.

In order to pass to the limit $n \to \infty$, let us observe that u^n is a Lipschitz continuous function for $(t,z) \in [-n,\infty) \times \mathbb{R}$. Due to (1.2.23) and $0 \le u^n \le 1$, one can note that

$$\|\partial_t u^n\|_\infty \le 2\int_{\mathbb{R}} \|K(\cdot,y)\|_\infty \mathrm{d}y + 1, \ \forall n \ge 1.$$

Then let us show that

$$|u^n(t,z+h) - u^n(t,z)| \le \min\left\{1, e^{m|h|} - 1\right\}, \ \forall t \ge -n, \ \forall z \in \mathbb{R}, \forall n \ge 1. \qquad (1.2.24)$$

Indeed, for $h > 0$, there exists some $m > \lambda$ such that

$$e^{-mh} \le \frac{u^n(-n, z+h)}{u^n(-n,z)} \le 1, \ \forall n \ge 1.$$

We can verify that $v^h(t,z) := e^{mh}u^n(t,z+h)$ is the super-solution of (1.2.23). The comparison principle applies and ensures that

$$u^n(t,z) \le e^{mh}u^n(t,z+h), \ \forall (t,z) \in [-n,\infty) \times \mathbb{R}.$$

Due to $z \mapsto u^n(t,z)$ is nonincreasing for all $t \ge -n$, one observes that

$$|u^n(t,z+h) - u^n(t,z)| = u^n(t,z) - u^n(t,z+h) \le e^{mh} - 1.$$

The case of $h < 0$ can be proved similarly. Hence, we obtain the estimate (1.2.24).

Next, Arzelà-Ascoli theorem ensures that there exists a subsequence of $\{u^n\}$, still denoted with the same indexes, and a globally Lipschitz continuous function $u = u(t,z) : \mathbb{R}^2 \to \mathbb{R}$ such that

$$u^n(t,z) \to u(t,z) \text{ as } n \to \infty,$$

locally uniformly for $(t,z) \in \mathbb{R}^2$. This also allows us to define the Lipschitz continuous function $\phi = \phi(t,z)$ by

$$\phi(t,z) = u\left(t, z + \int_0^t c_{\lambda,a}(s)\mathrm{d}s\right), \ \forall (t,z) \in \mathbb{R}^2.$$

Lastly, we show that

$$\lim_{z\to-\infty} \phi(t,z) = 1 \text{ and } \lim_{z\to\infty} \phi(t,z) = 0 \text{ uniformly for } t \in \mathbb{R}.$$

The limit at $z = \infty$ can be obtained from the super-solution $e^{-\lambda(z+a(t))}$. The behaviour of ϕ as $z \to -\infty$ can be shown by a contradiction argument. We assume that there exists a sequence $(t_n, z_n)_n$ such that

$$\lim_{n\to\infty} u\left(t_n, z_n + \int_0^{t_n} c_{\lambda,a}(s)\mathrm{d}s\right) = \Theta, \text{ with } 0 < \Theta < 1.$$

Let us consider the time-space shift function

$$u_n(t,z) := u\left(t+t_n, z+z_n+\int_0^{t_n} c_{\lambda,a}(s)\mathrm{d}s\right).$$

One can observe that

$$u_n(t,z) \to u_\infty(t,z) \text{ as } n\to\infty \text{ locally uniformly for } (t,z)\in\mathbb{R}^2,$$

and $u_\infty(0,0)=\Theta$.

Next we derive the equation satisfied by u_∞. Then for this equation, one can construct a suitable sub-solution to obtain that $\Theta = 1$. This is a contradiction. So the limit behaviour is obtained.

Nonexistence of generalized travelling wave

Our next result provides further properties for the admissible speed set $\mathscr{C}$. This result reads as follows.

Theorem 1.2.13 (Wave speed lower estimate)**.** *Let Assumption 1.2.6 and 1.2.7 be satisfied. Define for $\lambda\in(0,\sigma(K))$ the function $t\mapsto \underline{c}(\lambda)(t)\in L^\infty(\mathbb{R})$ given by*

$$\underline{c}(\lambda)(t) := \int_{-\infty}^{\infty} zK(t,z)\mathrm{e}^{\lambda z}\mathrm{d}z,$$

Then for any $c\in\mathscr{C}$ the following estimate holds

$$\lceil \underline{c}(\lambda)(\cdot) - c(\cdot)\rceil \le 0,\ \forall\lambda\in(0,\lambda^*)\,. \tag{1.2.25}$$

As a consequence one also has

$$\sup_{\lambda\in(0,\lambda^*)} \lfloor \underline{c}(\lambda)\rfloor \le \inf\lfloor\mathscr{C}\rfloor\,.$$

As a corollary of the above theorem, we can derive some conditions ensuring that the estimate of $\lfloor\mathscr{C}\rfloor$ provided in (1.2.21) is sharp. This is somehow an extension of the well known results for the travelling waves of the Fisher-KPP equation either local or nonlocal diffusion, for which we refer to [70, 97] and [44, 135].

Corollary 1.2.14. *Under the same assumptions as in Theorem 1.2.13, assume that $\lambda^* < \sigma(K)$ and that*

$$\lceil c(\lambda^*)(\cdot) - \underline{c}(\lambda^*)(\cdot)\rceil \le 0. \tag{1.2.26}$$

Then $\lfloor\mathscr{C}\rfloor$ is an unbounded interval with

$$\inf\lfloor\mathscr{C}\rfloor = \lfloor c(\lambda^*)(\cdot)\rfloor\,.$$

Within the framework of the above corollary and due to (1.2.21), one obtains that the set $\lfloor\mathscr{C}\rfloor$ is given by

$$\text{either } \left(\lfloor c(\lambda^*)(\cdot)\rfloor, \infty\right) \text{ or } \left[\lfloor c(\lambda^*)(\cdot)\rfloor, \infty\right).$$

By analogy with the usual Fisher-KPP equation, we suspect that $\lfloor\mathscr{C}\rfloor$ coincides with the closed interval. However we are not able to prove it for the moment. In other words, we cannot prove that $c_{\lambda^*,a}$ is an admissible wave speed function, for some $a\in W^{1,\infty}(\mathbb{R})$.

Let us comment on the additional conditions $\lambda^* < \sigma(K)$ and (1.2.26).

Remark 1.2.15. *The first condition* $\lambda^* < \sigma(K)$ *holds if we assume that*

$$\limsup_{\lambda \to \sigma(K)^-} \frac{1}{\lambda} \lfloor L(\lambda) \rfloor = \infty.$$

By combining the decreasing property of the map $\lambda \mapsto \lfloor c(\lambda) \rfloor$ *on* $(0, \lambda^*)$ *and* $\lfloor c(\lambda)(\cdot) \rfloor \to \infty$ *as* $\lambda \to 0^+$*, one can observe that* $\lambda^* < \sigma(K)$*.*

For the condition (1.2.26), let us observe that

$$-\lambda \frac{\mathrm{d}c(\lambda)}{\mathrm{d}\lambda} = c(\lambda) - \underline{c}(\lambda), \ \forall \lambda \in (0, \sigma(K)).$$

Recalling the property that

$$\left\lfloor -\frac{\mathrm{d}c(\lambda^*)}{\mathrm{d}\lambda} \right\rfloor = 0 \text{ if } \lambda^* < \sigma(K),$$

one can observe that condition (1.2.26) is equivalent to the function $c(\lambda^*)(\cdot) - \underline{c}(\lambda^*)(\cdot)$ exists a mean value, that is

$$\lfloor c(\lambda^*)(\cdot) - \underline{c}(\lambda^*)(\cdot) \rfloor = \lceil c(\lambda^*)(\cdot) - \underline{c}(\lambda^*)(\cdot) \rceil = 0.$$

Condition (1.2.26) can be satisfied for instance we assume that the function

$$t \mapsto \int_{-\infty}^{\infty} K(t, y)[e^{\lambda y} - 1] \mathrm{d}y$$

is uniquely ergodic for all λ closed λ^*.

Sketch the proof of Theorem 1.2.13

Now we explain our ideas of proving Theorem 1.2.13. Roughly speaking, we first construct a nonnegative sub-solution with compact support. Using the comparison principle in a spatial moving domain and the limit behaviour of wave profile, we derive the lower estimate of admissible speed function.

Let $\gamma \in (0, \lambda^*)$ be given. For $B > 0$ and $R > 0$, for some $a \in W^{\infty}(\mathbb{R})$, we define

$$c_{R,B}(\gamma)(t) := \frac{2R}{\pi} \int_{-B}^{B} K(t, z) e^{\gamma z} \sin(\frac{\pi z}{2R}) \mathrm{d}z,$$

and set

$$u_{R,B}(t, x) = \begin{cases} e^{a(t)} e^{-\gamma x} \cos(\frac{\pi x}{2R}) & \text{for } t \in \mathbb{R} \text{ and } x \in [-R, R], \\ 0 & \text{else.} \end{cases}$$

Then for $R > 0$ and $B > 0$ large enough, for some $\theta > 0$ sufficiently small, we show that $u_{R,B}(t, x)$ satisfies following equation for all $x \in [-R, R]$ and $t \in \mathbb{R}$,

$$(\partial_t - c_{R,B}(\gamma)(t)\partial_x)\, u_{R,B}(t, x) \leq \int_{\mathbb{R}} K(t, x-y)[u_{R,B}(t, y) - u_{R,B}(t, x)] \mathrm{d}y + (1 - \theta)\, u_{R,B}(t, x).$$

Recall that $u = u(t, x)$ denotes a generalized travelling wave of (1.2.15) with speed function $c = c(t) \in \mathscr{C}$ while $\phi = \phi(t, z)$ denotes its wave profile. Next, we introduce the parameter $\tau \in \mathbb{R}$ and define

$$u(t, x; \tau) := \phi\left(t - \tau, x - \int_0^t c(l - \tau) \mathrm{d}l\right), \ \forall t \in \mathbb{R}, x \in \mathbb{R}, \tau \in \mathbb{R}.$$

It satisfies the equation

$$\partial_t u(t,x;\tau) = \int_{\mathbb{R}} K(t-\tau,y)[u(t,x-y;\tau)-u(t,x;\tau)]\mathrm{d}y + F(t-\tau,u).$$

For some $\eta > 0$ small enough, the comparison principle in a moving domain applies and ensures that

$$\eta u_{R,B}\left(t-\tau, x - \int_0^t c_{R,B}(\gamma)(s-\tau)\mathrm{d}s\right) \le u(t,x;\tau),$$

for all $t \ge 0$, $\tau \in \mathbb{R}$ and $x \in \mathbb{R}$. This rewrites as

$$0 < \eta u_{R,B}\,(t-\tau,0) \le \phi\left(t-\tau, \int_0^t \left[c_{R,B}(\gamma)(l-\tau) - c(l-\tau)\right]\mathrm{d}l\right),\ \forall t \ge 0, \tau \in \mathbb{R}.$$

Lastly, recalling that the limit behaviour

$$\lim_{z\to\infty} \phi(t,z) = 0, \text{ uniformly for } t \in \mathbb{R},$$

we can derive that

$$\lceil c_{R,B}(\gamma)(\cdot) - c(\cdot)\rceil \le 0.$$

Letting $R, B \to \infty$ in above inequality, we obtain the estimate (1.2.25). From the definition of least mean and upper mean, we can complete the proof of Theorem 1.2.13.

1.2.2 Summary of Chapter 3: Spreading properties for nonautonomous Fisher-KPP equations with nonlocal diffusion

This joint work with Arnaud Ducrot has been submitted, see [59].

Problem

We consider the following non-autonomous Fisher-KPP equation with nonlocal diffusion

$$\partial_t u(t,x) = \int_{\mathbb{R}} K(y)\left[u(t,x-y)-u(t,x)\right]\mathrm{d}y + F(t,u),\ \forall t \ge 0, x \in \mathbb{R}, \tag{1.2.27}$$

which is equipped with initial data u_0. Here the function K is a thin-tailed kernel. We investigate spreading properties for solutions of (1.2.27) equipped with fast exponential decaying and slow exponential decaying initial data respectively.

For a better exposition, let us first use the following non-autonomous Logistic equation to illustrate our ideas. We consider

$$\partial_t u(t,x) = \int_{\mathbb{R}} K(y)\left[u(t,x-y)-u(t,x)\right]\mathrm{d}y + \mu(t)u(1-u), \tag{1.2.28}$$

posed for time $t \ge 0$ and $x \in \mathbb{R}$. This evolution problem is supplemented with initial data $u(0,x) = u_0(x)$.

Assumptions

Now we present the main assumptions that shall be used in this work.

Assumption 1.2.16. *We assume that the kernel $K : \mathbb{R} \to [0,\infty)$ satisfies the following set of assumptions:*

(i) *The function* $y \mapsto K(y)$ *is non-negative, continuous and integrable;*

(ii) *The abscissa of convergence of* K *enjoys* $\sigma(K) > 0$*;*

(iii) *Assume that* $K(0) > 0$.

We assume that μ satisfies following assumption.

Assumption 1.2.17. *The bounded and uniformly continuous function* $t \mapsto \mu(t)$ *satisfies* $\inf_{t\geq 0} \mu(t) > 0$ *and the least mean* $\lfloor \mu \rfloor$ *enjoys*

$$\lfloor \mu \rfloor > \overline{K} := \int_{\mathbb{R}} K(y)\mathrm{d}y.$$

Remark 1.2.18. *The above inequality is imposed for some technical reasons. It will be used to prove the hair trigger effect property in our problem (1.2.27).*

Linear speed

Ansatz the function $\exp\left\{-\lambda\left(x - \int_0^t c(\lambda)(s)\mathrm{d}s\right)\right\}$ into the linearized equation of (1.2.28) at $u = 0$, one obtains that

$$\lambda c(\lambda)(t) = \int_{\mathbb{R}} K(y)[e^{\lambda y} - 1]\mathrm{d}y + \mu(t), \ \forall t \geq 0.$$

For $\lambda \in (0, \sigma(K))$, $a \in W^{1,\infty}(0,\infty)$ and $t \geq 0$, we set

$$c(\lambda)(t) := \lambda^{-1}\left(\int_{\mathbb{R}} K(y)[e^{\lambda y} - 1]\mathrm{d}y + \mu(t)\right),$$

and

$$c_{\lambda,a}(t) := \lambda^{-1}\left(\int_{\mathbb{R}} K(y)[e^{\lambda y} - 1]\mathrm{d}y + \mu(t)\right) + a'(t). \tag{1.2.29}$$

Similar to Proposition 1.2.10 in previous, we also have the following properties about $c(\lambda)$.

Proposition 1.2.19. *Let Assumption 1.2.16 and 1.2.17 be satisfied. Then the following properties hold:*

(i) *The map* $\lambda \mapsto \lfloor c(\lambda)(\cdot) \rfloor$ *from* $(0, \sigma(K))$ *to* $\mathbb{R}$ *is of class* C^1.

(ii) *Set* $c_r^* := \inf_{\lambda \in (0,\sigma(K))} \lfloor c(\lambda)(\cdot) \rfloor$. *There exists* $\lambda_r^* \in (0, \sigma(K)]$ *such that*

$$\lim_{\lambda \to (\lambda_r^*)^-} \lfloor c(\lambda)(\cdot) \rfloor = c_r^*.$$

The map $\lambda \mapsto \lfloor c(\lambda)(\cdot) \rfloor$ *is decreasing on* $(0, \lambda_r^*)$.

(iii) *One has* $c_r^* > 0$.

(iv) *Assume that* $\lambda_r^* < \sigma(K)$. *One has*

$$c_r^* = \int_{\mathbb{R}} K(y) e^{\lambda_r^* y} y \mathrm{d}y. \tag{1.2.30}$$

Let us first observe that $c_r^* > 0$. Indeed, from assumption $\lfloor \mu \rfloor > \overline{K}$ and Proposition 1.2.2, one can choose some function $a \in W^{1,\infty}(0,\infty)$ such that $\mu(t) - \overline{K} + a'(t) \geq 0$ for all $t \geq 0$. Recall that for $\lambda \in (0, \sigma(K))$,

$$\lambda c(\lambda)(t) = \int_{\mathbb{R}} K(y) e^{\lambda y} \mathrm{d}y + \mu(t) - \overline{K}, \ \forall t \geq 0.$$

Since $K(0) > 0$, then for all $\lambda \in (0, \sigma(K))$ and $t \geq 0$, one has

$$\lambda c(\lambda)(t) + a'(t) = \int_{\mathbb{R}} K(y) e^{\lambda y} \mathrm{d}y + \mu(t) - \overline{K} + a'(t) \geq \int_{\mathbb{R}} K(y) e^{\lambda y} \mathrm{d}y > 0,$$

Thus we obtain that $c_r^* > 0$.

Since

$$\lfloor c(\lambda) \rfloor = \lambda^{-1} \left(\int_{\mathbb{R}} K(y)[e^{\lambda y} - 1] \mathrm{d}y + \lfloor \mu \rfloor \right),$$

and the function $\lambda \mapsto \lambda \lfloor c(\lambda) \rfloor$ is convex, then one can prove the other results in the above proposition.

As we discussed in Remark 1.2.15, one can assume that λ_r^* is different from the convergence abscissa of K.

Assumption 1.2.20. *Assume that* $\lambda_r^* < \sigma(K)$.

Upper bound for the propagating set to the right

Theorem 1.2.21. *Let Assumption 1.2.16, 1.2.17 and 1.2.20 be satisfied. Let* $u = u(t,x)$ *denote the solution of (1.2.28) equipped with a continuous initial data* u_0*, with* $0 \leq u_0(\cdot) \leq 1$ *and* $u_0(\cdot) \not\equiv 0$*. The following upper bound for the propagating set holds: if* $u_0(x) = O(e^{-\lambda x})$ *as* $x \to \infty$ *for some* $\lambda > 0$*, then one has*

$$\lim_{t \to \infty} \sup_{x \geq \int_0^t c^+(\lambda)(s) \mathrm{d}s + \eta t} u(t,x) = 0, \ \forall \eta > 0,$$

where the function $c^+(\lambda)(\cdot)$ *is defined by*

$$c^+(\lambda)(\cdot) := \begin{cases} c(\lambda_r^*)(\cdot) & \text{if } \lambda \geq \lambda_r^*, \\ c(\lambda)(\cdot) & \text{if } \lambda \in (0, \lambda_r^*). \end{cases}$$

In order to prove this theorem, it is sufficiently to construct suitable super-solutions and apply the comparison principle. Note that the decay rate of initial data has influences on the spreading speed. If $u_0(x) = O(e^{-\lambda x})$ as $x \to \infty$ for some $\lambda \geq \lambda_r^*$, we construct the super-solution as

$$\overline{u}_1(t,x) := A e^{-\lambda_r^* \left(x - \int_0^t c(\lambda_r^*)(s) \mathrm{d}s \right)}.$$

If $u_0(x) = O(e^{-\lambda x})$ as $x \to \infty$ for some $\lambda \in (0, \lambda_r^*)$, we define

$$\overline{u}_2(t,x) := A e^{-\lambda \left(x - \int_0^t c(\lambda)(s) \mathrm{d}s \right)}.$$

Let $A > 0$ be given large enough such that $\overline{u}_1(0,\cdot) \geq u_0(\cdot)$ and $\overline{u}_2(0,\cdot) \geq u_0(\cdot)$. Applying comparison principle, we can prove the above theorem.

Lower bound for the propagating set to the right

Theorem 1.2.22. *Let Assumption 1.2.16, 1.2.17 and 1.2.20 be satisfied. Let $u = u(t,x)$ denote the solution of (1.2.28) equipped with a continuous initial data u_0, where $0 \leq u_0(\cdot) \leq 1$ and $u_0(\cdot) \not\equiv 0$. Then the following propagation occurs:*

(i) **(Fast exponential decay case)** *If $u_0(x) = O(e^{-\lambda x})$ as $x \to \infty$ for some $\lambda \geq \lambda_r^*$, then one has*
$$\lim_{t\to\infty} \sup_{x\in[0,ct]} |1 - u(t,x)| = 0, \ \forall c \in (0, c_r^*);$$

(ii) **(Slow exponential decay case)** *If $\liminf_{x\to\infty} e^{\lambda x} u_0(x) > 0$ for some $\lambda \in (0, \lambda_r^*)$, then it holds that*
$$\lim_{t\to\infty} \sup_{x\in[0,ct]} |1 - u(t,x)| = 0, \ \forall c \in (0, \lfloor c(\lambda) \rfloor).$$

Remark 1.2.23. *If $\lim_{t\to\infty} \frac{1}{t} \int_0^t c^+(\lambda)(s)\mathrm{d}s = \lfloor c^+(\lambda) \rfloor$, then Theorem 1.2.21 and Theorem 1.2.22 provide the exact spreading speed $\lfloor c^+(\lambda) \rfloor$. This condition holds for instance if $\mu(\cdot)$ has a mean value.*

If one has $\lfloor c^+(\lambda) \rfloor < \liminf_{t\to\infty} \frac{1}{t} \int_0^t c^+(\lambda)(s)\mathrm{d}s$, then behaviour of $u(t, \beta t)$ for $t \gg 1$ is unknown when β satisfies
$$\lfloor c^+(\lambda) \rfloor < \beta < \liminf_{t\to\infty} \frac{1}{t} \int_0^t c^+(\lambda)(s)\mathrm{d}s,$$
This open problem is similar to the Fisher-KPP equation with local diffusion [124].

Remark 1.2.24. *In the above results, we only consider the propagation to the right-hand side of the real line. This is intentional, for the sake of brevity and clarity. To study the propagation of the left-hand side, it is sufficiently to change x to $-x$. As a consequence, one can obtain the spreading speed for equation supplemented with the initial data which has different tails on the left and right-hand side.*

Note also that the kernel is not assumed to be symmetric, so that the minimal spreading speeds on the right and the left can be different even if the initial data with the same decay rate on the left and right-hand sides.

Next we state the scheme of proof Theorem 1.2.22. In this work, we provide a new point of view to study the spreading speed of nonlocal diffusion problems. It is different from the well developed monotone semi-flow method, refer to [162, 94, 103, 104, 105]. Roughly speaking, we first prove a persistence lemma for uniformly continuous solutions. This key lemma ensures that if the uniformly continuous solution $u = u(t,x)$ admits a propagating path $t \mapsto X(t)$, then $[0, kX(t)]$ with any $k \in (0,1)$ is a propagating interval, that is u stays uniformly far from 0 on this interval, in the large time. By applying this key lemma, we obtain a lower estimate of spreading speed.

We also apply this idea to obtain the spreading speed for non-autonomous KPP equations with nonlocal diffusion in a lattice, see Chapter 5 for some details.

As mentioned previously, it is not easy to obtain that the uniform continuity of solution to nonlocal diffusion equations. Note that in [101], the authors showed that when the nonlinear term satisfies $F_u(u) < \overline{K}$ for any $u \geq 0$, where $\overline{K} = \int_{\mathbb{R}} K(y)\mathrm{d}y$, then the solutions of the homogeneous problem inherit the Lipschitz continuity property from their initial data. Here we require $\lfloor \mu \rfloor > \overline{K}$. The above condition fails. We show the regularity of solutions to Logistic equation (1.2.28) supplemented with suitable initial data.

A key persistence lemma

To show the persistence lemma, let us introduce some notations.

Definition 1.2.25 (Limit orbits set). *Let $u = u(t,x)$ be a uniformly continuous function on $[0,\infty) \times \mathbb{R}$ into $[0,1]$, which is a solution to (1.2.28). We define $\omega(u)$,* **the set of the limit orbits***, as: the function $\tilde{u} \in \omega(u)$ if there exist sequences $(x_n)_n \subset \mathbb{R}$ and $(t_n)_n \subset [0,\infty)$ such that $t_n \to \infty$ as $n \to \infty$ and*

$$u(t+t_n, x+x_n) \to \tilde{u}(t,x), \text{ as } n \to \infty, \text{ locally unifomrly for } (t,x) \in \mathbb{R}^2.$$

Observe that if u is bounded and uniformly continuous on $[0,\infty) \times \mathbb{R}$, then Arzelà-Ascoli theorem ensures that $\omega(u)$ is not empty. Indeed, for each sequence $(t_n)_n$ with $t_n \to \infty$ and $(x_n)_n \subset \mathbb{R}$, the sequence of function $(t,x) \mapsto u(t+t_n, x+x_n)$ is equi-continuous and thus has a converging subsequence with respect to the local uniform topology.

From the strong maximum principle, we can claim that the set $\omega(u)$ enjoys the following property:

Claim 1.2.26. *Let $\tilde{u} \in \omega(u)$ be given, then one has:*

$$\text{Either } \tilde{u}(t,x) > 0 \text{ for all } (t,x) \in \mathbb{R}^2 \text{ or } \tilde{u}(t,x) \equiv 0 \text{ on } \mathbb{R}^2.$$

With the above notations, now we state our persistence lemma.

Lemma 1.2.27 (Uniform persistence lemma). *Let Assumption 1.2.16 (i) and (iii), Assumption 1.2.17 be satisfied. Let $u = u(t,x) : [0,\infty) \times \mathbb{R} \to [0,1]$ be a uniformly continuous solution of (1.2.28). Let $t \mapsto X(t)$ from $[0,\infty)$ to $[0,\infty)$ be a given continuous function. Assume that the following set of hypothesis holds,*

(H1) Assume that $\liminf_{t\to\infty} u(t,0) > 0$;

(H2) There exists $\tilde{\varepsilon}_0 > 0$ such that

$$\liminf_{t\to\infty} \tilde{u}(t,0) > \tilde{\varepsilon}_0, \quad \forall \tilde{u} \in \omega(u) \setminus \{0\};$$

(H3) The map $t \mapsto X(t)$ is a propagating path for u, in the sense that

$$\liminf_{t\to\infty} u(t, X(t)) > 0.$$

Then for any $k \in (0,1)$, one has

$$\liminf_{t\to\infty} \inf_{0 \le x \le kX(t)} u(t,x) > 0.$$

Remark 1.2.28. *The above result holds without assuming that the convolution kernel is exponentially bounded. We expect that this key lemma may also be useful to study the spatial propagation for Fisher-KPP equation with fat-tailed dispersion kernel, which may accelerate, see [29, 69, 73].*

The above lemma is proved by some ideas coming from uniform persistence theory, somehow close to those developed in [53, 55].

We first state the idea of proving the regularity of solution, which is very technical. Then it remains to choose proper $X(t)$ and verify conditions $(H1)$-$(H3)$ in above lemma are satisfied.

Regularity of solution

Now we explain the idea of showing the regularity of solution. We consider two cases: an initial data with compact support in the right half line and an initial data with prescribed exponential decay for $x \gg 1$. Note that

$$\|\partial_t u\|_\infty \le 2\overline{K} + \|\mu\|_\infty.$$

Hence, the solution $u(t,x)$ is Lipschitz continuous for $t \in [0,\infty)$, uniformly with respect to $x \in \mathbb{R}$.

Next we investigate the regularity with respect to the spatial variable $x \in \mathbb{R}$. For the case in Section 1.2.1, we can find some $m > 0$ such that

$$e^{-mh} \le \frac{u(0,x+h)}{u(0,x)} \le 1, \ \forall x \in \mathbb{R}.$$

However, here the initial data is not monotone and u_0 may vanish at some point. We make a slight modification. For all $h > 0$ sufficiently small, we show that there exists some $0 < \sigma(h) < 1$ such that $\sigma(h) \to 1$ as $h \to 0$ and

$$u(\sqrt{h}, x) \ge \sigma(h) u_0(x-h), \ \forall x \in \mathbb{R}.$$

However, due to the shift in time, we can not apply the comparison principle directly. We define function $b_h(t)$ as follows,

$$b_h(t) = b_h(0) \exp\left\{ \int_0^t \left[\mu(s+\sqrt{h}) - \mu(s) \right] \mathrm{d}s \right\}, \quad \text{for all } t \ge 0.$$

And $b_h(0)$ is some constant depending on h and satisfies the following three conditions:

(i) $0 < b_h(0) \le \sigma(h) < 1$,
(ii) $b_h(0) \to 1$ as $h \to 0$,
(iii) for all $h > 0$ small enough,

$$b_h(0) \le \inf_{t\ge 0} \frac{\mu(t)}{\mu(t+\sqrt{h})} \exp\left\{ \int_0^t \left[\mu(s) - \mu(s+\sqrt{h}) \right] \mathrm{d}s \right\}.$$

Then we derive the equation satisfied by $u(t+\sqrt{h},x) - b_h(t)u(t,x-h)$ and apply the maximum principle to show that

$$u(t+\sqrt{h},x) - b_h(t)u(t,x-h) \ge 0, \forall t \ge 0.$$

For $h < 0$, we can analysis similarly. Combined with the Lipschitz continuity with respect to variable $t \in [0,\infty)$, we can obtain that u is uniformly continuous with respect to spatial variable $x \in \mathbb{R}$ uniformly for $t \in [0,\infty)$.

Remark 1.2.29. *The construction of $b_h(t)$ is in order to eliminate some "bad" term which appears in the equation satisfied by $u(t+\sqrt{h},x) - b_h(t)u(t,x-h)$. We should point out that we only show the uniform continuity of solutions to Logistic equation. For the moment, we still do not know how to construct proper $b_h(t)$ for general KPP-type equation.*

For the case of slow exponential decaying initial data, we use the similar idea to prove the uniform continuity. The main difference appears in showing the existence of proper $\sigma(h)$ such that $u(\sqrt{h},x) \ge \sigma(h)u_0(x-h)$ for all $x \in \mathbb{R}$. To do this, we need to show such solutions decay at the same rate as the initial data, at least in a short time.

Sketch the proof of Theorem 1.2.22 (i)

We focus on the case of initial data u_0 is fast exponential decaying, that is $u_0 = O(e^{-\lambda x})$ as $x \to \infty$ with $\lambda \geq \lambda_r^*$. We show that the Lispschitz continuous solution $u(t,x)$ satisfies conditions $(H1)$-$(H3)$ in Lemma 1.2.27. In some extent, $(H1)$ and $(H2)$ can be regarded as hair trigger effect.

Let us first show that $(H1)$ is satisfied. Recall that the kernel function K is continuous and $K(0) > 0$. Hence, there exist $\delta > 0$ and a continuous function $k : \mathbb{R} \to [0,\infty)$ which is even and compactly supported such that

$$\begin{aligned} &\operatorname{supp} k = [-\delta,\delta],\ k(y) > 0,\ \forall y \in (-\delta,\delta),\\ &k(y) \leq K(y) \text{ and } k(y) = k(-y),\ \forall y \in \mathbb{R}. \end{aligned}$$

Then, one can observe that $u = u(t,x)$, the solution of (1.2.28) with suitable initial data u_0, satisfies

$$\partial_t u(t,x) \geq \int_{\mathbb{R}} k(y)u(t,x-y)\mathrm{d}y - \overline{K}u(t,x) + \mu(t)u(t,x)\left(1 - u(t,x)\right).$$

Due to $\lfloor \mu \rfloor > \overline{K}$, one can choose some $a \in W^{1,\infty}(0,\infty)$ such that $\mu(t) - \overline{K} + a'(t) \geq 0$ for all $t \geq 0$. Set $w(t,x) := e^{a(t)}u(t,x)$. Note that w satisfies

$$\partial_t w(t,x) \geq \int_{\mathbb{R}} k(y)w(t,x-y)\mathrm{d}y - \bar{k}w(t,x) + w(t,x)\left(m - e^{\|a\|_\infty}\|\mu\|_\infty w(t,x)\right),$$

where $\bar{k} = \int_{\mathbb{R}} k(y)\mathrm{d}y$ and $m := \inf_{t\geq 0}\left(\bar{k} + \mu(t) - \overline{K} + a'(t)\right) \geq \bar{k} > 0$. Let $\underline{w} = \underline{w}(t,x)$ be the solution of following equation

$$\partial_t \underline{w}(t,x) = k * \underline{w}(t,x) - \bar{k}\underline{w}(t,x) + \underline{w}(t,x)\left(m - e^{\|a\|_\infty}\|\mu\|_\infty \underline{w}(t,x)\right). \tag{1.2.31}$$

supplemented with the initial data $\underline{w}(0,x) = e^{-\|a\|_\infty}u_0(x)$.

Recall the spreading speed results for the above autonomous Fisher-KPP equation with nonlocal dispersal, see [114, 167]. Applying comparison principle, one obtains

$$\liminf_{t\to\infty} u(t,0) \geq \lim_{t\to\infty} e^{-\|a\|_\infty}\underline{w}(t,0) = \frac{m}{\|\mu\|_\infty e^{2\|a\|_\infty}} > 0.$$

The condition $(H1)$ is fulfilled.

Next, for all $\tilde{u} \in \omega(u) \setminus \{0\}$, one can derive that $\tilde{u}$ satisfies

$$\partial_t \tilde{u}(t,x) \geq \int_{\mathbb{R}} k(y)\tilde{u}(t,x-y)\mathrm{d}y - \overline{K}\tilde{u}(t,x) + \tilde{\mu}(t)\tilde{u}(t,x)\left(1 - \tilde{u}(t,x)\right),\ (t,x) \in \mathbb{R}^2,$$

where $\tilde{\mu}$ is the limit of $\mu(\cdot + t_n)$ in local uniform topology for $t \in \mathbb{R}$. By the similar analysis in proving $(H1)$, one can show that the condition $(H2)$ is satisfied.

Now let us choose proper $X(t)$. For all $B, R > 0$, $\gamma \in \mathbb{R}$, we define the quantity $c_{R,B}(\gamma)$ by

$$c_{R,B}(\gamma) := \frac{2R}{\pi}\int_{-B}^{B} K(z)\mathrm{e}^{\gamma z}\sin\left(\frac{\pi z}{2R}\right)\mathrm{d}z. \tag{1.2.32}$$

Note that $\gamma \mapsto c_{R,B}(\gamma)$ is continuous and recall that $c_r^* = \int_{\mathbb{R}} K(y)e^{\lambda_r^* y}y\mathrm{d}y$. One has

$$\lim_{\gamma\to\lambda_r^*}\lim_{\substack{R\to\infty\\ B\to\infty}} c_{R,B}(\gamma) = c_r^*.$$

So for each given $c \in [0, c_r^*)$ and $c' \in (c, c_r^*)$, one can choose proper $\gamma = \hat{\gamma}$ close to λ_r^* such that for $R, B > 0$ large enough,

$$c' \leq c_{R,B}(\hat{\gamma}).$$

Set $X(t) := c_{R,B}(\hat{\gamma})t$. Observe that for all $\frac{c}{c'} < k < 1$, one has

$$ct \leq kc't \leq kX(t), \ \forall t > 0.$$

We construct $\underline{u}_1$ to be the sub-solution of (1.2.28) as follows. For all $R, B > 0$ large enough, for some suitable $a \in W^{1,\infty}(\mathbb{R})$, for $\eta > 0$ small enough, we define

$$u_{R,B}(t,x) = \begin{cases} \eta e^{a(t)} e^{-\hat{\gamma}x} \cos(\frac{\pi x}{2R}) & \text{if } t \geq 0 \text{ and } x \in [-R, R], \\ 0 & \text{else.} \end{cases}$$

One can verify that

$$\underline{u}_1(t,x) := u_{R,B}(t, x - X(t)), \text{ with } X(t) = c_{R,B}(\hat{\gamma})t,$$

is the sub-solution of (1.2.28). The comparison principle applies and ensures that

$$\liminf_{t\to\infty} u(t, X(t)) \geq \liminf_{t\to\infty} \underline{u}_1(t, X(t)) = \liminf_{t\to\infty} u_{R,B}(t, 0) > 0,$$

which implies that $(H3)$ is satisfied. The key Lemma 1.2.27 ensures that

$$\liminf_{t\to\infty} \inf_{0\leq x\leq kX(t)} u(t,x) > 0.$$

Recalling that $ct \leq kX(t)$ for $t > 0$, one has

$$\liminf_{t\to\infty} \inf_{0\leq x\leq ct} u(t,x) > 0, \ \forall c \in [0, c_r^*). \tag{1.2.33}$$

Moreover, we can show that

$$\liminf_{t\to\infty} \inf_{0\leq x\leq ct} u(t,x) = 1, \ \forall c \in [0, c_r^*).$$

Sketch the proof of Theorem 1.2.22 (ii)

For the case of slow exponential decay initial data u_0, the proof is similar to the first case. To prove $(H1)$, it is also sufficiently to consider a sub-solution like $\underline{w}$ with compactly supported initial data $\underline{w}_0$ satisfying $\underline{w}_0 \leq u_0$. The condition $(H2)$ can be proved similarly as $(H1)$.

Next, let us introduce some functions. For each $\lambda \in (0, \lambda_r^*)$, for the given $c \in [0, \lfloor c(\lambda) \rfloor)$, due to the property of least mean, one can choose some $a \in W^{1,\infty}(0,\infty)$ such that $c < c_{\lambda,a}(t)$ for all $t \geq 0$, where $c_{\lambda,a}$ is defined in (1.2.29). For some $B \in W^{1,\infty}(0,\infty)$ and $\varepsilon > 0$ small enough, we define that

$$\varphi(t,x) = e^{-\lambda(x+a(t))} - e^{-\lambda a(t)+B(t)} e^{-(\lambda+\varepsilon)x}, \ \ t \geq 0, x \in \mathbb{R}, \tag{1.2.34}$$

With suitable parameters $B \in W^{1,\infty}(0,\infty)$ and $\varepsilon > 0$, one can verify that $\underline{\phi}$ defined as follows is the sub-solution of (1.2.28),

$$\underline{\phi}(t,x) := \max\left\{0, \ \varphi\left(t, x - \int_0^t c_{\lambda,a}(s)\mathrm{d}s\right)\right\}.$$

Note that $\underline{\phi}$ is positive when $x > \|B\|_\infty/\varepsilon$.

Let us choose proper $X(t)$ for each given $\lambda \in (0, \lambda_r^*)$ and $c \in [0, \lfloor c(\lambda) \rfloor)$. We define

$$X(t) := \int_0^t c_{\lambda,a}(s)\mathrm{d}s + P,$$

where $P > \frac{\|B\|_\infty}{\varepsilon} > 0$. As well as, for some $k \in (0,1)$, one has $kX(t) \geq ct$ for all $t \geq 0$. This is due to $c_{\lambda,a}(t) \geq c$ for all $t \geq 0$. By comparison principle, one has

$$\liminf_{t\to\infty} u(t, X(t)) \geq \liminf_{t\to\infty} \underline{\phi}(t, X(t)) = \liminf_{t\to\infty} \varphi(t, P) > 0.$$

Next one can apply persistence Lemma 1.2.27 to obtain that

$$\liminf_{t\to\infty} \inf_{x\in[0,ct]} u(t,x) > 0, \ \forall c \in [0, \lfloor c(\lambda) \rfloor).$$

Moreover, we can show that the above limit equals to 1.

General KPP-type

As a corollary, we can also show the spreading speed for general KPP-type equation (1.2.27). However, due to lack of the regularity results for this general situation, we only show the solution is persistence on a spreading interval without obtaining the results of the convergence to steady state.

Let us state the assumption of F.

Assumption 1.2.30 (KPP nonlinearity). *We assume that the function $F : [0,\infty) \times [0,1] \to \mathbb{R}$ takes the form $F(t,u) = uf(t,u)$ where the function $f : [0,\infty) \times [0,1] \to \mathbb{R}$ satisfies the following set of hypotheses:*

- *(f1) For all $u \in [0,1]$, function $f(\cdot,u) \in L^\infty(0,\infty;\mathbb{R})$, and f is Lipschitz continuous with respect to $u \in [0,1]$, uniformly with respect to $t \geq 0$;*
- *(f2) Let $f(t,1) \equiv 0$ and $\mu(t) := f(t,0)$. Assume $\mu(\cdot)$ is a bounded and uniformly continuous function. Also, we assume that*

$$h(u) := \inf_{t\geq 0} f(t,u) > 0 \text{ for all } u \in [0,1);$$

- *(f3) For almost every $t \geq 0$, the function $u \mapsto f(t,u)$ is nonincreasing on $[0,1]$;*
- *(f4) We assume that $\lfloor \mu \rfloor > \overline{K}$.*

From the above assumption, one can find some constant $C > 0$ such that

$$\mu(t)u(1 - Cu) \leq F(t,u) \leq \mu(t)u, \ \forall t \geq 0.$$

Hence, the solution of a Logistic-type equation can be treated as the sub-solution of (1.2.27). From above inequality and Theorem 1.2.22, we obtain the following spreading properties for (1.2.27).

Corollary 1.2.31. *Let Assumption 1.2.16, 1.2.20 and 1.2.30 be satisfied. Let $u = u(t,x)$ denote the solution of (1.2.27) supplemented with a continuous initial data u_0, with $0 \leq u_0(\cdot) \leq 1$ and $u_0(\cdot) \not\equiv 0$. Then the following propagation result holds true:*

(i) **(Fast exponential decay case)** *If $u_0(x) = O(e^{-\lambda x})$ as $x \to \infty$ for some $\lambda \geq \lambda_r^*$, then one has*

$$\begin{cases} \lim_{t\to\infty} \sup_{x \geq \int_0^t c(\lambda_r^*)(s)\mathrm{d}s + \eta t} u(t,x) = 0, & \forall \eta > 0, \\ \lim_{t\to\infty} \inf_{x\in[0,ct]} u(t,x) > 0, & \forall c \in (0, c_r^*); \end{cases}$$

(ii) **(Slow exponential decay case)** *If $u_0(x) \sim e^{-\lambda x}$ as $x \to \infty$ for some $\lambda \in (0, \lambda_r^*)$, then one has*

$$\begin{cases} \lim_{t\to\infty} \sup_{x \geq \int_0^t c(\lambda)(s)\mathrm{d}s + \eta t} u(t,x) = 0, & \forall \eta > 0, \\ \lim_{t\to\infty} \inf_{x\in[0,ct]} u(t,x) > 0, & \forall c \in (0, \lfloor c(\lambda) \rfloor). \end{cases}$$

1.2.3 Summary of Chapter 4: Spreading speeds for time heterogeneous prey-predator systems with diffusion

This is a joint work with Arnaud Ducrot. It has been submitted, see [60].

Problem

We study spreading speed for the following reaction-diffusion systems of prey-predator type,

$$\begin{cases} \partial_t u = d(t)\partial_{xx} u + uf(t,u,v), \\ \partial_t v = \partial_{xx} v + vg(t,u,v), \end{cases} \tag{1.2.35}$$

posed in $t > 0$ and $x \in \mathbb{R}$. This problem is supplemented with suitable compactly supported initial data

$$u(0,x) = u_0(x) \text{ and } v(0,x) = v_0(x) \text{ for } x \in \mathbb{R}. \tag{1.2.36}$$

Here $u = u(t,x)$ and $v = v(t,x)$ denote the density of the prey and the predator, respectively. Also, the prey and the predator are able to co-invade the empty space. Without loss of generality, here we assume that the diffusion rate of predator equals to one. Similarly to Remark 1.2.8, when the diffusion rate of v is $d_v(t)$, we can achieve that by a suitable time transformation as $\tau(t) = \int_0^t d_v(s)\mathrm{d}s$.

As mentioned in the previous section, the spreading speed for homogeneous prey-predator systems has been obtained in [55]. In this work we provide a new method that allows us to study non-autonomous prey-predator systems and to give a shorter proof for the homogeneous problem as in [55]. Our analysis is based on the derivation of some local pointwise estimates so that we can compare the solutions of the prey-predator problem with those of a KPP scalar equation on suitable spatio-temporal domains.

Rather similar pointwise estimates have been obtained and used by Wu in [165] to study the invasion of a single predator with two abundant preys in the case where the two prey species have the same diffusion coefficient. The analysis in [165] is based on the equation formed by the total density of the two preys coupled with refined estimates of the heat kernel.

Here the situation is different since we study the co-invasion of the two species, the prey and the predator. We extend the analysis to handle time heterogeneities and propose a new methodology based on suitable applications of the strong comparison principle for scalar parabolic equations. This methodology is rather general and can be extended to other problems. Indeed, it can be extended to handle predator-prey systems on discrete lattices (see Chapter 5 in this manuscript).

Assumptions and biological explanation

Assumption 1.2.32. *We assume that* $d : [0, \infty) \to \mathbb{R}$ *is a bounded and uniformly continuous function with a mean value* $\langle d \rangle$ *and* $\inf_{t \geq 0} d(t) > 0$.

Assumption 1.2.33. *The function* $f : [0, \infty)^3 \to \mathbb{R}$ *satisfies:*

(f1) For each given $u, v \geq 0$*, the function* $t \mapsto f(t, u, v)$ *is bounded and uniformly continuous from* $[0, \infty)$ *to* $\mathbb{R}$*, and* $t \mapsto f(t, u, v)$ *has a mean value* $\langle f(\cdot, u, v) \rangle$*. The function* $(u, v) \mapsto f(t, u, v)$ *is Lipschitz continuous with respect to* $u, v \geq 0$*, uniformly for* $t \geq 0$*;*

(f2) For all $t \geq 0$ *and* $u > 0$*, the map* $v \mapsto f(t, u, v)$ *is strictly decreasing;*

(f3) Assume $f(t, 1, 0) = 0$ *for all* $t \geq 0$ *and*

$$h(u) := \inf_{t \geq 0} f(t, u, 0) > 0, \ \forall u \in [0, 1);$$

(f4) For all $t \geq 0$ *and* $v \geq 0$*, the map* $u \mapsto f(t, u, v)$ *is nonincreasing;*

(f5) For all $v > 0$*, the function* f *further satisfies* $\sup_{t \geq 0} f(t, 1, v) < 0$*.*

Assumption 1.2.34. *The function* $g : [0, \infty)^3 \to \mathbb{R}$ *satisfies:*

(g1) For each given $u, v \geq 0$*, the function* $t \mapsto g(t, u, v)$ *is bounded and uniformly continuous from* $[0, \infty)$ *to* $\mathbb{R}$*, and* $t \mapsto g(t, u, v)$ *has a mean value* $\langle g(\cdot, u, v) \rangle$*, while the function* $(u, v) \mapsto g(t, u, v)$ *is Lipschitz continuous with respect to* $u, v \geq 0$*, uniformly with respect to* $t \geq 0$*;*

(g2) For all $t \geq 0$ *and* $v \geq 0$*, the map* $u \mapsto g(t, u, v)$ *is nondecreasing;*

(g3) It satisfies $\inf_{t \geq 0} g(t, 1, 0) > 0$*;*

(g4) For all $t \geq 0$ *and* $u \geq 0$*, the map* $v \mapsto g(t, u, v)$ *is nonincreasing;*

(g5) Let the mean value of function $t \mapsto g(t, 0, 0)$ *satisfy*

$$\langle g(\cdot, 0, 0) \rangle < 0.$$

Now we explain Assumption 1.2.33 and 1.2.34 in the biological context.

- The species usually live in a time varying environment. Thus we assume that f and g both depend on time. We require that these variations in time exhibit an averaging property.

- Assumptions $(f2)$ and $(g2)$ describe predatory behaviour. Condition $(f2)$ means that more predators reduce the prey density while $(g2)$ implies that more prey leads to an increase in the predator population. Due to this asymmetry, the comparison principle does not apply to (1.2.35).

- When there is no predator, $(f3)$ ensures that $u \equiv 1$ is the maximal environmental carrying capacity of the prey. $(g3)$ means that the predator density will increase when the prey is abundant.

- $(f4)$ and $(g4)$ imply that the growth rate of each species is maximal at low density. By analogy with the Fisher-KPP equation, this indicates that the propagation of two species is driven by the leading edge of the invasion.
- $(f5)$ is a technical assumption. Note also that $(f2)$ and $f(t,1,0)\equiv 0$ already ensure that $f(t,1,v)<0$ for all $t\geq 0$ and $v>0$. $(f5)$ implies that the prey cannot reach the environmental carrying capacity 1 as long as there exists the predator. $(g5)$ means that the predator cannot survive without the prey. The prey population is the only resource for the growth of the predator.

Let us recall the classical Lotka-Volterra prey-predator system,

$$\begin{cases} \partial_t u = d(t)\partial_{xx}u + r(t)u\,(1-u) - p(t)uv, \\ \partial_t v = \partial_{xx}v + q(t)uv - \nu(t)v. \end{cases} \tag{1.2.37}$$

Note that it corresponds to (1.2.35) with

$$\begin{aligned} f(t,u,v) &= r(t)\,(1-u) - p(t)v, \\ g(t,u,v) &= q(t)u - \nu(t). \end{aligned}$$

With additional smoothness and sign conditions for the coefficients, it satisfies Assumption 1.2.33 and 1.2.34.

From now on and for writing convenience, we set

$$r_1(t) := f(t,0,0) \text{ and } r_2(t) := g(t,1,0). \tag{1.2.38}$$

Due to the monotonicity and regularity assumptions of f and g, there exists some constant $L>0$ such that for all $t\geq 0$, $u\in[0,1]$ and $v\geq 0$,

$$\begin{aligned} r_1(t)\,(1-Lu-Lv) &\leq f(t,u,v) \leq r_1(t), \\ r_2(t)\big(1-L(1-u)-Lv\big) &\leq g(t,u,v) \leq r_2(t). \end{aligned} \tag{1.2.39}$$

Linear speed

To state our main results, we define two speed functions $\lambda\mapsto c_u(\lambda)$ and $\gamma\mapsto c_v(\gamma)$ from $(0,\infty)$ to $L^\infty(0,\infty)$ given by

$$c_u(\lambda)(t) := d(t)\lambda + \frac{r_1(t)}{\lambda} \quad\text{and}\quad c_v(\gamma)(t) := \gamma + \frac{r_2(t)}{\gamma}, \tag{1.2.40}$$

for all $t\geq 0$, where r_1 and r_2 are defined in (1.2.38). These two functions corresponds to linear speeds for u and v respectively, around the stationary state $(0,0)$ (no species) and $(1,0)$ (predator free equilibrium) for solution with exponential decay rate λ and γ. We also introduce the quantities c_u^* and c_v^* given by

$$c_u^* := \inf_{\lambda>0}\langle c_u(\lambda)\rangle \quad\text{and}\quad c_v^* := \inf_{\gamma>0}\langle c_v(\gamma)\rangle.$$

Setting

$$\lambda^* := \sqrt{\frac{\langle r_1\rangle}{\langle d\rangle}} \text{ and } \gamma^* := \sqrt{\langle r_2\rangle}, \tag{1.2.41}$$

one has

$$c_u^* = \langle c_u(\lambda^*)\rangle = 2\sqrt{\langle d\rangle\langle r_1\rangle} \quad\text{and}\quad c_v^* = \langle c_v(\gamma^*)\rangle = 2\sqrt{\langle r_2\rangle}. \tag{1.2.42}$$

Due to $(f1)$, $(f3)$ and $(f4)$, one can observe that for $v \equiv 0$, the system (1.2.35) degenerates to following Fisher-KPP type equation satisfied by u,

$$\partial_t u(t,x) = d(t)\partial_{xx}u(t,x) + u(t,x)f\left(t, u(t,x), 0\right).$$

The quantity c_u^* is the spreading speed of above equation equipped with compactly supported initial data, we refer the reader to [21, 23, 124].

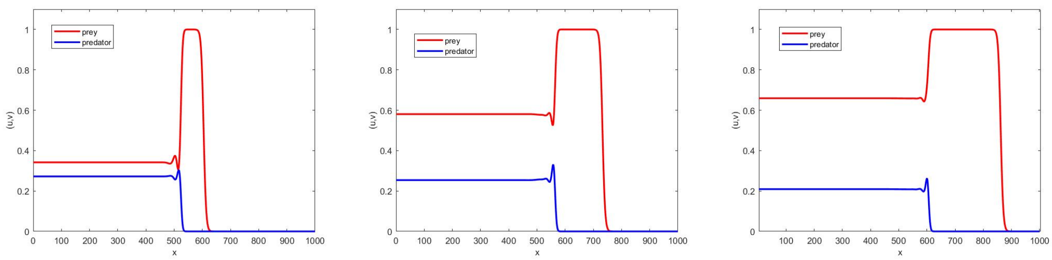
On the other hand, for $u \equiv 1$, the solution v of (1.2.35) satisfies following equation

$$\partial_t v(t,x) = \partial_{xx}v(t,x) + v(t,x)g\left(t, 1, v(t,x)\right).$$

Note that we do not assume the existence of nontrivial stationary state solution in the above equation. It is not a standard KPP-type equation. However, by the similar argument in [21, 124], one can show that c_v^* is the spreading speed of above equation equipped with compactly supported initial data. The main difference is that v may not converge to a stationary state but grow and become unbounded in the large time.

Spreading speed results

With the above notations and assumptions, we state our main results. For the case of the predator invading the empty environment slower than the prey, the first theorem implies that the propagation occurs in two separate steps involving an intermediate equilibrium (namely $u = 1, v = 0$) in the middle zone. One can see the simulation in Figure 1.11.



(a) $t = 30$ (b) $t = 60$ (c) $t = 90$

Figure 1.11: The solution (u, v) at different given times of (1.2.37) associated with compact support initial data where parameters satisfy $d \equiv 5$, $\nu = r \equiv 1$, $q \equiv 2$ and $p(t) = 2 + \sin t$.

Theorem 1.2.35 (Slow predator case)**.** *Let Assumption 1.2.32, 1.2.33 and 1.2.34 be satisfied. We assume that the predator is slower than the prey, in the sense that*

$$c_u^* > c_v^*.$$

Let u_0 and v_0 be two given bounded and continuous functions in $\mathbb{R}$ with compact support, and $0 \not\equiv\leq u_0 \leq 1$, $0 \not\equiv\leq v_0$. Let $(u, v) = (u(t,x), v(t,x))$ be the solution of (1.2.35) with initial data (u_0, v_0). Assume that (u, v) is bounded.
Then the function pair (u, v) satisfies:

(i) for all $c > c_u^*$, one has $\lim_{t\to\infty} \sup_{|x|\geq ct} u(t,x) = 0$;

(ii) for all $c_v^* < c_1 < c_2 < c_u^*$ and for all $c > c_v^*$, one has:

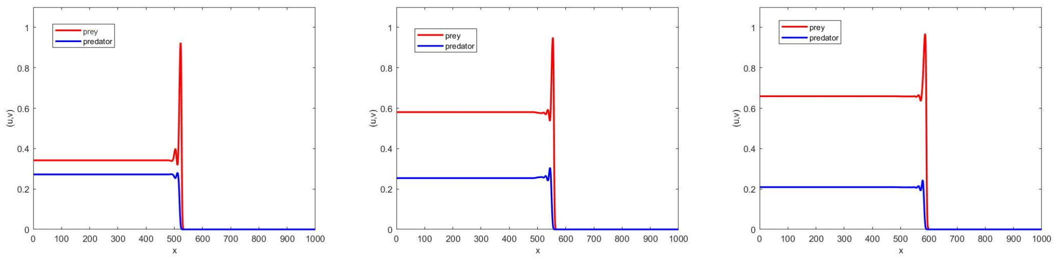
$$\lim_{t\to\infty} \sup_{c_1 t\leq|x|\leq c_2 t} |1 - u(t,x)| = 0 \text{ and } \lim_{t\to\infty} \sup_{|x|\geq ct} v(t,x) = 0;$$

(iii) for all $c \in [0, c_v^*)$ one has:

$$\liminf_{t\to\infty} \inf_{|x|\le ct} v(t,x) > 0,$$

$$\liminf_{t\to\infty} \inf_{|x|\le ct} u(t,x) > 0 \text{ and } \limsup_{t\to\infty} \sup_{|x|\le ct} u(t,x) < 1.$$

In the case of the predator invading the empty environment faster than the prey, the population of the predator could grow fast enough to overtake the prey. From the assumption of g (see Assumption 1.2.34), one can note that the predator cannot survive in the absence the prey at large time. We may expect that the prey and the predator will invade the empty space at the same time. In the next theorem, we show that the spreading speed of the system is c_u^*, which means that the prey and the predator invade the empty space almost simultaneously. One can see the simulation in Figure 1.12.



(a) $t = 30$ (b) $t = 60$ (c) $t = 90$

Figure 1.12: The solution (u, v) at different given times of (1.2.37) associated with compact support initial data where parameters satisfy $d \equiv 0.3$, $\nu = r \equiv 1$, $q \equiv 2$ and $p(t) = 2+\sin t$.

Theorem 1.2.36 (Fast predator case)**.** *Let Assumption 1.2.32, 1.2.33 and 1.2.34 be satisfied and assume that the predator is faster than the prey, in the sense that*

$$c_u^* \le c_v^*.$$

Let u_0 and v_0 be two given bounded and continuous functions in $\mathbb{R}$ with compact support, and $0 \not\equiv\le u_0 \le 1$, $0 \not\equiv\le v_0$. Let $(u, v) = (u(t, x), v(t, x))$ be the solution of (1.2.35) with initial data (u_0, v_0). Assume that (u, v) is bounded.
Then the function pair (u, v) satisfies:

(i) for all $c > c_u^*$, one has $\lim_{t\to\infty} \sup_{|x|\ge ct} [u(t,x) + v(t,x)] = 0$;

(ii) for all $c \in [0, c_u^*)$ one has:

$$\liminf_{t\to\infty} \inf_{|x|\le ct} v(t,x) > 0,$$

$$\liminf_{t\to\infty} \inf_{|x|\le ct} u(t,x) > 0 \text{ and } \limsup_{t\to\infty} \sup_{|x|\le ct} u(t,x) < 1.$$

Remark 1.2.37. *In the situation of all coefficients in (1.2.35) are independent of t, that is $d(t) \equiv d > 0$, $f(t, u, v) \equiv f(u, v)$ and $g(t, u, v) \equiv g(u, v)$, the above two theorems have been proved in Theorem 2.1 and 2.2 in [55]. In this work, we provide a new method that allows 1) to recover this result in the homogeneous case, 2) to extend them for non-autonomous prey-predator systems, 3) to provide a shorter proof as in [55] for the homogeneous problem.*

Sketch the proof of Theorem 1.2.35

Step 1: Upper estimates on spreading speed

To obtain the upper estimates for the spreading speed, we only need to construct suitable super-solutions.

Recall that the definition of c_u^* in (1.2.42) and the property of mean value. For $c > c' > c_u^*$, there exists a function $a \in W^{1,\infty}(0,\infty)$ such that for all $t > 0$,

$$c' \geq d(t)\lambda^* + \frac{r_1(t)}{\lambda^*} + a'(t).$$

For $A > 0$, we define $\overline{u}$ given by

$$\overline{u}(t,x) := Ae^{-\lambda^* a(t)}e^{-\lambda^*(x-c't)}.$$

One can verify that $\overline{u}$ satisfies

$$\partial_t \overline{u}(t,x) - d(t)\overline{u}_{xx}(t,x) - r_1(t)\overline{u}(t,x) \geq 0.$$

From (1.2.39) and comparison principle, we can obtain that

$$\lim_{t\to\infty} \sup_{x\geq ct} u(t,x) = 0, \ \forall c > c_u^*.$$

By a similar symmetric argument, we can obtain the results for $x \leq 0$. Theorem 1.2.35 (i) is proved.

Similarly, for all $c > \tilde{c} > c_v^*$, there exists $\tilde{a} \in W^{1,\infty}(0,\infty)$ such that for all $t > 0$

$$\tilde{c} \geq \gamma^* + \frac{r_2(t)}{\gamma^*} + \tilde{a}'(t).$$

Then the function

$$\overline{v}_1(t,x) := Ae^{-\gamma^* \tilde{a}(t)}e^{-\gamma^*(x-\tilde{c}t)}$$

satisfies the following differential inequality

$$\partial_t \overline{v}_1(t,x) - \partial_{xx}\overline{v}_1(t,x) - r_2(t)\overline{v}_1(t,x) \geq 0.$$

By (1.2.39) and comparison principle, one can obtain the half of statement (ii) in Theorem 1.2.35.

Step 2: Local pointwise estimates

We construct two important lemmas which play a key role in proving Theorem 1.2.35 and 1.2.36. From Assumption 1.2.33 and 1.2.34, we observe two important facts: the predator cannot survive without the prey and the prey asymptotically reach its carrying capacity without the predator.

For simplicity and clarity, in this step, let us use (1.2.37), which is a typical example of (1.2.35), to explain the ideas of deriving the local pointwise estimates between $u(t,x)$ and $v(t,x)$. We recall the classical Lotka-Voterra prey-predator system (1.2.37) below which satisfies our assumptions,

$$\begin{cases} \partial_t u = d(t)\partial_{xx}u + r(t)u\,(1-u) - p(t)uv, \\ \partial_t v = \partial_{xx}v + q(t)uv - \nu(t)v. \end{cases}$$

The first fact: the predator will starve without the prey. Hence if the prey is in the absence, namely $u \equiv 0$, then v becomes a solution of

$$\partial_t v = d_v(t)\partial_{xx} v - \nu(t)v,$$

and v decays exponentially to 0 due to $\inf_{t\geq 0} \nu(t) > 0$. This observation yields our first key lemma.

Lemma 1.2.38. *For all $\delta > 0$, there exist $M_\delta > 0$ and $T_\delta > 0$ such that the following estimate holds true*

$$v(t,x) \leq \delta + M_\delta u(t,x), \ \forall t \geq T_\delta, x \in \mathbb{R}.$$

The proof of above lemma is based on strong maximum principle in parabolic equations. Let us sketch the proof of this key lemma. By a contradiction argument, assume that there exist $\delta_0 > 0$ and sequences $(t_n)_n$ and $(x_n)_n$ such that $t_n \to \infty$ and

$$v(t_n, x_n) > \delta_0 + nu(t_n, x_n), \forall n \geq 1. \tag{1.2.43}$$

Let us consider the time and space shift functions $u_n(t,x) := u(t+t_n, x+x_n)$ and $v_n(t,x) := v(t+t_n, x+x_n)$. The parabolic regularity ensures that there exists function (u_∞, v_∞) such that

$$(u_n, v_n)(t,x) \to (u_\infty, v_\infty)(t,x) \text{ as } n \to \infty \text{ locally uniformly for } (t,x) \in \mathbb{R}^2.$$

As well as, the function (u_∞, v_∞) satisfies

$$\begin{cases} \partial_t u_\infty = \tilde{d}(t)\partial_{xx} u_\infty + \tilde{r}(t)u_\infty (1 - u_\infty) - \tilde{p}(t)u_\infty v_\infty, \\ \partial_t v_\infty = \partial_{xx} v_\infty + \tilde{q}(t)u_\infty v_\infty - \tilde{\nu}(t)v_\infty. \end{cases} \tag{1.2.44}$$

Set $\sigma = \{d, r, p, q, \nu\}$. Herein $\tilde{\sigma}(\cdot)$ is the limit function of $\sigma(\cdot + t_n)$ as $n \to \infty$ in local uniform topology. Due to v is assumed to be bounded, (1.2.43) implies that $u_\infty(0,0) = 0$. The strong maximum principle ensures that $u_\infty \equiv 0$. Then by constructing a proper super-solution for following equation

$$\partial_t v_\infty = \partial_{xx} v_\infty - \tilde{\nu}(t)v_\infty,$$

one can show that $v_\infty(0,0) = 0$. This contradicts (1.2.43). The key lemma is obtained.

Next, we state another key lemma which is due to the following observation: if there is no predator, namely $v \equiv 0$, then the density of the prey satisfies the Fisher-KPP equation

$$\partial_t u = d(t)\partial_{xx} u + r(t)u(1 - u).$$

The prey will spread with the speed $c_u^* = 2\sqrt{\langle d \rangle \langle r \rangle}$. With the help of this observation and strong maximum principle, similarly, we can prove the following key lemma.

Lemma 1.2.39. *Fix $c \in [0, c_u^*)$. For each $\alpha > 0$, there exist $M_\alpha > 0$ and $T_\alpha > 0$ such that the following estimate holds true*

$$1 - u(t,x) \leq \alpha + M_\alpha v(t,x), \ \forall t \geq T_\alpha, |x| \leq ct.$$

Step 3: Middle zone

Now we start to prove Theorem 1.2.35 (*ii*). Recalling (1.2.39) and applying the first key Lemma 1.2.38, one can observe that the solution $u(t,x)$ of (1.2.35) satisfies the following differential inequality

$$\partial_t u(t,x) \geq d(t)\partial_{xx}u(t,x)+r_1(t)u(t,x)\Big(1-Lu(t,x)-L\big(\delta+M_\delta u(t,x)\big)\Big),\ \forall t\geq T_\delta,\ \forall x\in\mathbb{R}.$$

From comparison principle and the spreading speed for Fisher-KPP equation (see [21, 23, 124]), one can obtain that

$$\liminf_{t\to\infty}\inf_{|x|\leq\tilde{c}t} u(t,x)>0,\ \forall\tilde{c}\in[0,c_u^*(\delta)),$$

where the quantity $c_u^*(\delta)$ is given by

$$c_u^*(\delta):=2\sqrt{\langle d\rangle\langle r_1\rangle(1-L\delta)}.$$

From the arbitrariness of $\delta>0$ and $c_u^*(0)=c_u^*$, one obtains that

$$\liminf_{t\to\infty}\inf_{|x|\leq ct} u(t,x)>0,\ \forall c\in[0,c_u^*). \tag{1.2.45}$$

Combining the following limits which has been proved in the first step,

$$\lim_{t\to\infty}\sup_{|x|\geq c_1t} v(t,x)=0,\ \forall c_1>c_v^*,$$

we can further prove that

$$\liminf_{t\to\infty}\inf_{c_1\leq|x|\leq c_2t} u(t,x)=1,\ \forall c_v^*<c_1<c_2<c_u^*.$$

Step 4: Final zone

Since $c_v^*<c_u^*$ and (1.2.45) is obtained, then it remains to show that

$$\liminf_{t\to\infty}\inf_{|x|\leq ct} v(t,x)>0 \text{ and } \limsup_{t\to\infty}\sup_{|x|\leq ct} u(t,x)<1,\quad \forall c\in[0,c_v^*).$$

The proof of these results shall make use of our key Lemma 1.2.39. Recalling (1.2.39), we can derive a differential inequality satisfied by v as follows

$$\partial_t v(t,x)\geq\partial_{xx}v(t,x)+r_2(t)v(t,x)\big(1-L\alpha-L(1+M_\alpha)v(t,x)\big),\ \forall t\geq T_\alpha, x\in[-c't,c't].$$

Next we can construct a nonnegative sub-solution which is compactly supported, to show that

$$\liminf_{t\to\infty} v(t,\pm ct)>0,\ \forall c\in[0,c_v^*).$$

Finally we make use of a positive constant number as a sub-solution on a moving domain to obtain that

$$\liminf_{t\to\infty}\inf_{|x|\leq ct} v(t,x)>0,\ \forall c\in[0,c_v^*).$$

Combining the assumption $\sup_{t\geq0} f(t,1,v)<0$ for all $v>0$, we can show that

$$\limsup_{t\to\infty}\sup_{|x|\leq ct} u(t,x)<1,\ \forall c\in[0,c_v^*).$$

Sketch the proof of Theorem 1.2.36

The proof of Theorem 1.2.36 is similar to above discussion. There is only one difference appears in the part of upper estimate. We explain how to show that v cannot spread faster than c_u^*, namely

$$\lim_{t\to\infty}\sup_{|x|\geq ct} v(t,x)=0,\ \forall c>c_u^*.$$

Same as **Step 1** in the above discussion, we can show that

$$\lim_{t\to\infty}\sup_{|x|\geq ct} u(t,x)=0,\ \forall c>c_u^*.$$

Thus, fixing any $c>c_u^*$ and $\varepsilon>0$ small enough, there exists $T>0$ such that

$$\sup_{t\geq T}\sup_{|x|\geq ct} u(t,x)\leq \varepsilon.$$

From Assumption 1.2.34 ($g1$) and ($g5$), for sufficiently small $\varepsilon>0$, one can choose $b\in W^{1,\infty}(0,\infty)$ such that

$$\sup_{t>0}\{g(t,\varepsilon,0)+b'(t)\}<0.$$

For $B>0$ and for some $\gamma'>0$ small enough, for $c>c''>c_u^*$, we define

$$\overline{v}_2(t,x):=Be^{-\gamma'(x-c''t)}e^{-b(t)}.$$

Since $g(t,u,v)\leq g(t,\varepsilon,0)$ for all $t\geq 0$, $v\geq 0$ and $0\leq u\leq\varepsilon$, then one can verify that $\overline{v}_2(t,x)$ is a super-solution of v-equation in (1.2.35) for all $t\geq T$ and $x\geq c''t$ with $c''>c_u^*$. By comparison principle in the domain $\{(t,x): t\geq T, x\geq c''t\}$, one has

$$\lim_{t\to\infty}\sup_{|x|\geq ct} v(t,x)=0,\ \forall c>c_u^*.$$

Boundedness

Note that in the above two theorems, we require that the solution (u,v) is bounded. For the sake of completeness, we show that the solution can be bounded with some additional assumptions.

We emphasize that the boundedness assumption is satisfied for a large classes of systems. Recall that the comparison principle does not hold for system (1.2.35). However we can apply partial comparison principle to each component equation. Note that $0\leq u(t,x)\leq 1$ and $v(t,x)\geq 0$ for all $t\geq 0$ and $x\in\mathbb{R}$, if the initial data satisfies $0\leq u_0\leq 1$ and $v_0\geq 0$. The boundedness of the solutions can be obtained if we assume that $\limsup_{t\to\infty} g(t,1,\infty)<0$, which is satisfied for the predator with intraspecific competition, for example, $g(t,u,v):=q(t)u-v-\nu(t)$. When this condition is not satisfied, the situation is more complicated. With some additional conditions, we can also show that v is bounded in the next proposition.

Proposition 1.2.40. *Let Assumption 1.2.32, 1.2.33 and 1.2.34 be satisfied. Assume that* $\inf_{t\geq 0} g(t,0,\infty)>-\infty$ *and there exists* $M_0>0$ *such that the mean value* $\langle f(\cdot,0,M)\rangle<0$ *for all* $M\geq M_0$. *Let* $(u,v)=(u,v)(t,x)$ *be the solution of (1.2.35) supplemented with nonnegative and uniformly continuous initial function* (u_0,v_0). *If* $0\leq u_0\leq 1$ *and* $v_0\geq 0$ *is bounded, then the function* $(u,v)=(u,v)(t,x)$ *is bounded on* $[0,\infty)\times\mathbb{R}$.

The proof of the above proposition is close to the idea developed in [4, 55]. We try to give some essential point. Roughly speaking, if v is unbounded, then the decay rate in the component u-equation in (1.2.35) would become very large. This yields that $u\sim 0$. While the v-equation in (1.2.35) with $u\sim 0$, implies that $v\sim 0$. There is a contradiction.

1.2.4 Summary of Chapter 5: Spreading speeds for time heterogeneous prey-predator systems with nonlocal diffusion on lattice

This joint work with Arnaud Ducrot is in preparation.

Problem

We investigate the large time behaviour of solutions for the following Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t,i) = \sum_{j\in\mathbb{Z}} J_1(t,j)\left[u(t,i-j)-u(t,i)\right] + u(t,i)f\left(t,u(t,i),v(t,i)\right), \\ \frac{\mathrm{d}}{\mathrm{d}t}v(t,i) = \sum_{j\in\mathbb{Z}} J_2(t,j)\left[v(t,i-j)-v(t,i)\right] + v(t,i)g\left(t,u(t,i),v(t,i)\right), \end{cases} \tag{1.2.46}$$

posed in $t>0$ and $i\in\mathbb{Z}$. This problem is supplemented with bounded initial data

$$u(0,i)=u_0(i) \text{ and } v(0,i)=v_0(i).$$

Herein the two sets $\{i\in\mathbb{Z};\ u_0(i)\neq 0\}\neq\emptyset$ and $\{i\in\mathbb{Z};\ v_0(i)\neq 0\}\neq\emptyset$ have finite elements.

In the previous literature review, we have known that the spreading speed for non-autonomous diffusive prey-predator system in lattice $\mathbb{Z}$, remains at least theoretically unknown neither time varying in periodicity nor almost periodicity. General time heterogeneity has its meaning in biological modeling and influences the spreading behaviour. Due to the discrete nonlocal operator depending on time, to the best of our knowledge, even the spreading speeds for scalar KPP equations with such dispersion is unknown before this work. We apply the similar idea developed in the prey-predator system with local diffusion, that is using some local pointwise estimates to compare solutions of systems to those of scalar KPP type equation in suitable spatio-temporal domains (see Chapter 4). But the analysis is different from local diffusion case when we study spreading behaviours for scalar equation in a moving domain. We will use some ideas which are developed in studying spreading speed for nonlocal diffusion equation with continuous space variable (see Chapter 3).

Assumptions

Here the nonlinear reaction terms f and g in (1.2.46) satisfies same mathematical structure as Assumption 1.2.33 and 1.2.34. For convenience, we set $f(t,0,0)=1$ and $r(t):=g(t,1,0)$. Instead of repeating Assumption 1.2.33 and 1.2.34, here we only recall a typical example of f and g as

$$\begin{aligned} f(t,u,v) &= 1-u-p(t)v, \\ g(t,u,v) &= q(t)u-\nu(t), \end{aligned} \tag{1.2.47}$$

where the time dependent functions p, q and ν represent the predation rate, the conversion rate and the death rate of the predator, respectively.

Next, we state assumptions for the nonlocal diffusion kernel functions.

Assumption 1.2.41 (Kernel $J_k=J_k(t,i)$)**.** *The kernel function* $J_k:[0,\infty)\times\mathbb{Z}\to[0,\infty)$ *(for* $k=1,2$*) satisfies the following set of assumptions:*

(J1) The function J_k is nonnegative and $J_k(\cdot, i) \in L^\infty(0,\infty)$ has a mean value for each $i \in \mathbb{Z}$;

(J2) The function $\hat{J}_k : i \mapsto J_k(\cdot, i)$ from $\mathbb{Z}$ to $L^\infty(0,\infty)$ whose series is absolutely convergent, that is $\hat{J}_k \in l^1(\mathbb{Z}, L^\infty(0,\infty))$. And we assume that its abscissa of convergence satisfies

$$\mathrm{abs}(\hat{J}_k) > 0.$$

In the following, for notation simplicity, we use $\mathrm{abs}(J_k)$ instead of $\mathrm{abs}(\hat{J}_k)$;

(J3) Assume that $J_k(\cdot, i) = J_k(\cdot, -i)$ for all $i \in \mathbb{Z}$ (symmetric);

(J4) The function J_k satisfies $\inf_{t\geq 0} J_k(t, \pm 1) > 0$;

(J5) Let the following limits hold true

$$\limsup_{\lambda \to \mathrm{abs}(J_1)^-} \lambda^{-1}\Big(\sum_{j\in\mathbb{Z}} \langle J_1(\cdot, j)\rangle\, e^{\lambda j}\Big) = \limsup_{\gamma \to \mathrm{abs}(J_2)^-} \gamma^{-1}\Big(\sum_{j\in\mathbb{Z}} \langle J_2(\cdot, j)\rangle\, e^{\gamma j}\Big) = \infty,$$

where $\langle J_k(\cdot, j)\rangle$ (for $k = 1, 2$) is the mean value of function $t \mapsto J_k(t, j)$ (for $k = 1, 2$) for each $j \in \mathbb{Z}$.

Due to some technical reasons in studying the hair trigger effect for non-autonomous KPP equations with nonlocal diffusion, we impose following assumption.

Assumption 1.2.42. *Set $\overline{J}_k(t) = \sum_{j\in\mathbb{Z}} J_k(t, j)$ for $k = 1, 2$. Assume that*

$$\langle f(t, 0, 0)\rangle > \langle \overline{J}_1(t)\rangle \text{ and } \langle g(t, 1, 0)\rangle > \langle \overline{J}_2(t)\rangle.$$

Spreading speed for scalar KPP equations in a lattice

We investigate the spreading speed for following KPP type equation,

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t, i) = \sum_{j\in\mathbb{Z}} J(t, j)[w(t, i-j) - w(t, i)] + m(t)w(t, i)\big(1 - lw(t, i)\big),\ t \geq 0, i \in \mathbb{Z}, \quad (1.2.48)$$

where the constant $l > 0$. Set $\overline{J}(t) = \sum_{j\in\mathbb{Z}} J(t, j)$.

Let us first show the hair trigger effect property for (1.2.48).

Lemma 1.2.43 (Hair trigger effect). *Assume that kernel function $J = J(t, i)$ is nonnegative and $\inf_{t\geq 0} J(t, \pm 1) > 0$. Let $i \mapsto J(\cdot, i) \in l^1(\mathbb{Z}, L^\infty(0,\infty))$ be satisfied. Assume that the function $m : [0,\infty) \to \mathbb{R}$ is bounded and uniformly continuous with $\inf_{t\geq 0} m(t) > 0$. Assume that m and $\overline{J}$ have mean value, denoted by $\langle m\rangle$ and $\langle \overline{J}\rangle$ respectively, which are satisfying $\langle m\rangle > \langle \overline{J}\rangle$. Let $w(t, i)$ be the solution of (1.2.48) equipped with initial data w_0. If $w_0 \geq 0$ and $w_0 \not\equiv 0$, then there exists a constant $\tilde{\varepsilon}_0 > 0$ which is independent of w_0, such that*

$$\liminf_{t\to\infty} w(t, 0) \geq \tilde{\varepsilon}_0.$$

Remark 1.2.44. *Due to the technical reason, we assume that $\langle m\rangle > \langle \overline{J}\rangle$. The idea of proof this lemma is similar to stated in the proof of Theorem 1.2.22 in Subsection 1.2.2. We will try to prove the hair trigger effect property without this technical condition in the forthcoming work.*

Next, in order to state the spreading speed result in scalar equation, more conditions on J should be given. We assume that J and m satisfy the following assumptions.

Assumption 1.2.45. *The kernel function J satisfies Assumption 1.2.41. Assume that the function $m : [0, \infty) \to \mathbb{R}$ is bounded and uniformly continuous with $\inf_{t\geq 0} m(t) > 0$. Assume that m has a mean value, denoted by $\langle m \rangle$, which satisfies $\langle m \rangle > \langle \overline{J} \rangle$.*

As Subsection 1.2.2, we can introduce the speed function $\mu \mapsto c_w(\mu)$ defined in $(0, \mathrm{abs}(J))$ given by

$$c_w(\mu)(\cdot) := \mu^{-1}\Big(\sum_{j\in\mathbb{Z}} J(\cdot, j)[e^{\mu j} - 1] + m(\cdot)\Big).$$

As well as, define c_w^* by

$$c_w^* := \inf_{0<\mu<\mathrm{abs}(J)} \langle c_w(\mu) \rangle = \inf_{0<\mu<\mathrm{abs}(J)} \mu^{-1}\Big(\sum_{j\in\mathbb{Z}} \langle J(\cdot, j)\rangle \left[e^{\mu j} - 1\right] + \langle m(\cdot)\rangle\Big). \tag{1.2.49}$$

With the above notations, we state the following proposition.

Proposition 1.2.46. *Let Assumption 1.2.45 be satisfied. Let initial data $0 \leq w_0 \leq \frac{1}{l}$ be given. Assume that the set $\{i \in \mathbb{Z} : w_0(i) \neq 0\} \neq \emptyset$ has finite elements. Then the solution $w = w(t, i)$ of (1.2.48) supplemented with initial data w_0 satisfies:*

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{|i|\geq ct} w(t, i) = 0, & \forall c > c_w^*, \\ \lim\limits_{t\to\infty} \inf\limits_{|i|\leq ct} w(t, i) = \frac{1}{l}, & \forall c \in [0, c_w^*), \end{cases}$$

where c_w^* is defined in (1.2.49).

The idea of proving the above proposition is similar to the case of nonlocal diffusion KPP equations with continuous spatial variable (see Chapter 3). We first construct suitable super-solutions to obtain the upper estimate of spreading speed. Then we develop the persistence lemma for lattice equation similar to Lemma 1.2.27. Lastly, by the hair trigger effect property and constructing proper sub-solutions, we can apply the persistence lemma to derive the lower estimate of speed which coincides with the upper estimate. We can obtain that the exact spreading speed of solutions to (1.2.48) is c_w^*.

Note that here we only consider the Logistic equation. In fact, the results can be extended to general KPP type $F(t, u)$ with $\langle F_u'(t, 0)\rangle > \langle \overline{J} \rangle$. The spreading speed result for (1.2.48) is sufficiently to derive our main results in systems.

Spreading speed results for systems

To state our main results, Let us introduce some notations. Define two functions $c_u : (0, \mathrm{abs}(J_1)) \to L^\infty(0, \infty)$ and $c_v : (0, \mathrm{abs}(J_2)) \to L^\infty(0, \infty)$ by

$$\begin{aligned} c_u(\lambda)(\cdot) &:= \lambda^{-1}\left(\sum_{j\in\mathbb{Z}} J_1(\cdot, j)[e^{\lambda j} - 1] + 1\right), \ \ \forall \lambda \in (0, \mathrm{abs}(J_1)), \\ c_v(\gamma)(\cdot) &:= \gamma^{-1}\left(\sum_{j\in\mathbb{Z}} J_2(\cdot, j)[e^{\gamma j} - 1] + r(\cdot)\right), \ \ \forall \gamma \in (0, \mathrm{abs}(J_2)). \end{aligned} \tag{1.2.50}$$

Herein J_1 and J_2 satisfy Assumption 1.2.41 and $r(t) = g(t, 1, 0)$. Set

$$c_u^* := \inf_{\lambda \in (0, \mathrm{abs}(J_1))} \langle c_u(\lambda)(\cdot)\rangle \text{ and } c_v^* := \inf_{\gamma \in (0, \mathrm{abs}(J_2))} \langle c_v(\gamma)(\cdot)\rangle . \tag{1.2.51}$$

From Proposition 1.2.46, one can observe that c_u^* is the spreading speed of solutions to the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t,i) = \sum_{j\in\mathbb{Z}} J_1(t,j)\left[u(t,i-j) - u(t,i)\right] + uf\left(t,u,0\right),$$

equipped with initial data u_0. Similarly, one can show that the quantity c_v^* is the spreading speed of equation

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t,i) = \sum_{j\in\mathbb{Z}} J_2(t,j)\left[v(t,i-j) - v(t,i)\right] + vg\left(t,1,v\right),$$

equipped with initial data v_0. While the difference is that the solution v may no longer converge to some steady state after propagation but may grow and become unbounded.

The first theorem is in the case of the prey invading the empty environment faster than the predator. We show that there are two separate steps involving an intermediate equilibrium (namely $u = 1, v = 0$) in the middle zone in the propagation.

Theorem 1.2.47 (Slow predator). *Let Assumption 1.2.33, 1.2.34, 1.2.41 and 1.2.42 be satisfied. Assume that the predator is slower than the prey, in the sense that*

$$c_v^* < c_u^*.$$

Let $1 \geq u_0 \geq 0$ and $v_0 \geq 0$ be two given bounded functions in $\mathbb{Z}$. Assume that two sets $\{i \in \mathbb{Z}; u_0(i) \neq 0\} \neq \emptyset$ and $\{i \in \mathbb{Z}; v_0(i) \neq 0\} \neq \emptyset$ have finite elements. Let $(u, v) = (u(t, i), v(t, i))$ be the solution of (1.2.46) equipped with initial data (u_0, v_0). Assume that (u, v) is bounded. Then the function pair (u, v) satisfies:

(i) for all $c > c_u^*$, one has $\lim_{t\to\infty} \sup_{|i|\geq ct} u(t,i) = 0$;

(ii) for all $c_v^* < c_1 < c_2 < c_u^*$ and for all $c > c_v^*$, one has

$$\lim_{t\to\infty} \sup_{c_1 t\leq |i|\leq c_2 t} |1 - u(t,i)| = 0 \text{ and } \lim_{t\to\infty} \sup_{|i|\geq ct} v(t,i) = 0,$$

(iii) for all $c \in [0, c_v^*)$, one has

$$\liminf_{t\to\infty} \inf_{|i|\leq ct} v(t,i) > 0,$$

$$\liminf_{t\to\infty} \inf_{|i|\leq ct} u(t,i) > 0 \text{ and } \limsup_{t\to\infty} \sup_{|i|\leq ct} u(t,i) < 1.$$

In the next theorem, we consider the case of the predator invading the empty environment faster than the prey. The population of the predator could grow fast enough to overtake the prey. Then the prey and the predator invade the empty space almost simultaneously.

Theorem 1.2.48 (Fast predator). *Let Assumption 1.2.33, 1.2.34, 1.2.41 and 1.2.42 be satisfied. Assume that the predator is faster than the prey, in the sense that*

$$c_v^* \geq c_u^*.$$

Let $1 \geq u_0 \geq 0$ and $v_0 \geq 0$ be two given bounded functions. Assume that two sets $\{i \in \mathbb{Z}; u_0(i) \neq 0\} \neq \emptyset$ and $\{i \in \mathbb{Z}; v_0(i) \neq 0\} \neq \emptyset$ have finite elements. Let $(u, v) = (u(t, i), v(t, i))$ be the solution of (1.2.46) equipped with initial data (u_0, v_0). Assume that (u, v) is bounded. Then the function pair (u, v) satisfies:

(i) for all $c > c_u^*$, one has $\lim_{t\to\infty} \sup_{|i|\geq ct} [u(t,i) + v(t,i)] = 0$*;*

(ii) for all $c \in [0, c_u^*)$, one has

$$\liminf_{t\to\infty} \inf_{|i|\leq ct} v(t,i) > 0,$$

$$\liminf_{t\to\infty} \inf_{|i|\leq ct} u(t,i) > 0 \text{ and } \limsup_{t\to\infty} \sup_{|i|\leq ct} u(t,i) < 1.$$

We combine the new methods developed in Chapter 3 and Chapter 4 to prove the above two theorems. Roughly speaking, we derive some pointwise estimates between $u(t,i)$ and $v(t,i)$, which are solutions to (1.2.46). These estimates are similar to Lemma 1.2.38 and 1.2.39. According to these estimates, we can compare solutions of (1.2.46) with those of scalar KPP equations with nonlocal diffusion in suitable moving domains.

However, due to the nonlocal diffusion operator, we cannot find a positive constant as a sub-solution in the moving domain to obtain the uniform persistence of solutions. In order to overcome this difficulty, we apply a similar idea in Chapter 3 where considered the scalar nonlocal diffusion equation.

Boundedness

In above two theorems, we require that the solution (u, v) is bounded. This assumption can be satisfied for some systems under certain additional conditions.

Assumption 1.2.49. *Assume that there exist some constants $\varepsilon > 0$, $\eta > 0$ and $\mathcal{M} > 0$ such that for all $0 \leq u \leq 1$, $v \geq 0$ and $t \geq 0$,*

$$uf(t,u,v) + \varepsilon vg(t,u,v) \leq \mathcal{M} - \eta v.$$

Remark 1.2.50. *Let us show that the typical example (1.2.47) satisfies Assumption 1.2.49. Let us choose $0 < \varepsilon < \inf_{t\geq 0} p(t)/\sup_{t\geq 0} q(t)$. Assume that $\inf_{t\geq 0} \nu(t) > 0$. Note that for all $0 \leq u \leq 1$, $v \geq 0$ and $t \geq 0$,*

$$\begin{aligned} uf(t,u,v) + \varepsilon vg(t,u,v) &= u(1-u) - p(t)uv + \varepsilon q(t)uv - \varepsilon\nu(t)v \\ &\leq 1 - \varepsilon \inf_{t\geq 0} \nu(t)v. \end{aligned}$$

Hence (1.2.47) satisfies Assumption 1.2.49 with some given $0 < \varepsilon < \inf_{t\geq 0} p(t)/\sup_{t\geq 0} q(t)$, $\mathcal{M} = 1$ and $\eta = \varepsilon \inf_{t\geq 0} \nu(t)$.

Let $\varepsilon > 0$, $\eta > 0$ and $\mathcal{M} > 0$ be given in Assumption 1.2.49. Set

$$\overline{J}_k(\cdot) = \sum_{j\in\mathbb{Z}} J_k(\cdot, j) \in L^\infty(0,\infty), (k = 1, 2).$$

Proposition 1.2.51. *Let Assumption 1.2.41, 1.2.33, 1.2.34 and 1.2.49 be satisfied. Let $(u,v) = (u,v)(t,i)$ be the solution of (1.2.46) supplemented with initial data (u_0, v_0). If $0 \leq u_0 \leq 1$ and $v_0 \geq 0$ is bounded, then the solution (u, v) is bounded.*

Let us consider $W := u + \varepsilon v$. Due to Assumption 1.2.49 and $0 \leq u \leq 1$, one can observe that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} W(t,i) &\leq \sum_{j\in\mathbb{Z}} J_2(t,j)[W(t,i-j) - W(t,i)] + \|\overline{J}_1\|_\infty + \|\overline{J}_2\|_\infty + \mathcal{M} - \eta \frac{W(t,i) - u(t,i)}{\varepsilon}, \\ &\leq \sum_{j\in\mathbb{Z}} J_2(t,j)[W(t,i-j) - W(t,i)] + \|\overline{J}_1\|_\infty + \|\overline{J}_2\|_\infty + \mathcal{M} + \frac{\eta}{\varepsilon} - \frac{\eta}{\varepsilon} W(t,i), \end{aligned}$$

where η and $\mathcal{M}$ are given in Assumption 1.2.49. Then we can construct a bounded super-solution for the above equation. The comparison principle implies that $u + \varepsilon v$ is bounded. Thus the Proposition 1.2.51 holds.

1.3 Perspectives

In this section, we will discuss some interesting problems for future works.

How time heterogeneities affect generalized travelling waves

In Chapter 2, we show the existence and nonexistence of generalized travelling wave solutions. Under certain conditions such as time uniquely ergodic, we show a sharp estimates of minimal wave speed. However, we do not show whether the time structure (almost periodic, uniquely ergodic and so on) would transmit to generalized travelling waves. We expect that if the coefficients are time almost periodic for nonlocal diffusion equations considered in Chapter 2, then the wave profile is time almost periodic and the speed function is almost periodic. The similar results for non-autonomous reaction-diffusion equations were proved by Shen [141] using dynamical systems theory. We also refer the reader to [102] which showed the almost periodic traveling fronts share the same recurrence property as the structure of the media for KPP lattice equations.

As we noticed in the local diffusion case [87], there may exist some generalized travelling wave solutions with other form speed functions which are different from the following speed function derived from the linear equation as

$$c(t) = \lambda + \frac{f'_u(t,0)}{\lambda}, \ \forall \lambda > 0.$$

It would be interesting to give a more exhaustive description for generalized travelling waves in nonautonomous nonlocal diffusion equation, for instance generalized travelling waves with some other type speed functions not merely as

$$c(t) = \lambda^{-1}\left(\int_{\mathbb{R}} K(t,y)[e^{\lambda y} - 1]\mathrm{d}y + 1\right) + a'(t), \quad \text{for } \lambda > 0 \text{ and } a \in W^{1,\infty}(\mathbb{R}).$$

The lack of regularizing effect in nonlocal diffusion equations

As discussed previously, there is no parabolic regularity theory for nonlocal diffusion equations. Although the regularity of solutions does not improve with time, we want to obtain some solutions with proper initial data in which the regularity does not worsen by the nonlocal dispersal operator. Note that in Chapter 3, we spend lots of pages to show that some solutions with suitable initial data are Lipschitz continuous. Our proof is based on some delicate constructions. Due to the special construction, we only prove some regularities of solutions for Logistic type equations. The spreading behaviours in such nonlocal diffusion KPP equations are only partially solved. The lack of regularity of solutions causes lots of difficulties in the analysis of large time behaviours of solutions.

Also, we are curious about how to prove the regularity of solutions to the following nonlocal diffusion KPP equations with time dependent kernel,

$$\partial_t u(t,x) = \int_{\mathbb{R}} K(t,y)\,[u(t,x-y) - u(t,x)]\,\mathrm{d}y + F(t,u).$$

If the regularity estimates of solutions to the above equation are obtained, we believe that our uniform persistence lemma developed in Chapter 3 can also be applied to study spreading behaviours of solutions to the above equation.

For prey-predator systems with nonlocal diffusion operators, how to show the solutions have good regularity is not answered completely as well. We refer the reader to [172] for investigation of spreading speeds in nonlocal diffusion prey-predator system with large diffusion rates. We also refer to [169] for studying competition system with nonlocal dispersal operators.

As long as one can show the regularity of solutions to prey-predator system with nonlocal diffusion, we believe that our methods developed in Chapter 4 and 5 can also be extended to study spreading speed for the following system:

$$\begin{cases} \partial_t u(t,x) = \int_{\mathbb{R}} K_1(t,y)\,[u(t,x-y)-u(t,x)]\,\mathrm{d}y + F(t,u,v), \\ \partial_t v(t,x) = \int_{\mathbb{R}} K_2(t,y)\,[v(t,x-y)-v(t,x)]\,\mathrm{d}y + G(t,u,v), \end{cases} \quad t>0, x\in\mathbb{R}.$$

Accelerating propagation

In this manuscript, we mainly focus on the nonlocal diffusion operator with thin-tailed kernel function. As we noticed in the literature review, the fat-tailed kernel might cause acceleration phenomena in KPP equations. Note that in our first work the kernel function $K = K(t,y)$ is dependent on time and K is assumed to be thin-tailed uniformly for $t\in\mathbb{R}$. If we couple the thin-tailed kernel and fat-tailed kernel function varying with time, for example we define for all $k\in\mathbb{Z}$, for some $T>0$,

$$J(t,y) = \begin{cases} e^{-|y|^2}, & \forall t\in[2kT,(2k+1)T], \\ \dfrac{1}{(1+|y|)^3}, & \forall t\in[(2k+1)T,(2k+2)T]. \end{cases}$$

Then it would be interesting to study the large time behaviours of solutions to nonlocal diffusion equation with such kernel function. Is it possible that the acceleration caused by the fat-tailed kernel will slow down due to the coupled thin-tailed kernel?

Spatial heterogeneous

Note that we only consider the time heterogeneities and one dimensional space in this manuscript. The propagation phenomena for reaction-diffusion equations in spatial heterogeneous environment have been attracted a lot of attentions in the last decades. To the best of our knowledge, there is no results in spreading speeds for prey-predator systems with spatial-time heterogeneities. Our local pointwise estimates between the prey and the predator derived in Chapter 4 might be able to extend to the heterogeneous time-space media. In the forthcoming works, we will try to understand spreading behaviours for the following prey-predator systems:

$$\begin{cases} \partial_t u - \nabla\cdot(A_1(t,x)\nabla u) + q_1(t,x)\cdot\nabla u = F(t,x,u,v), \\ \partial_t v - \nabla\cdot(A_2(t,x)\nabla v) + q_2(t,x)\cdot\nabla v = G(t,x,u,v), \end{cases} \quad t>0, x\in\mathbb{R}^N,$$

equipped with compactly supported initial data u_0 and v_0.

Chapter 2

Generalized travelling fronts for nonautonomous Fisher-KPP equations with nonlocal diffusion

This work in collaboration with Arnaud Ducrot is published in *Annali di Matematica Pura ed Applicata* [58].

Abstract

This work is concerned with the study of generalized travelling wave solutions for time heterogeneous Fisher-KPP equations with nonlocal diffusion. Here we consider general time heterogeneities both for the diffusion kernel and the reaction term. We investigate the existence and non existence of generalized travelling wave solutions for such a problem. Roughly speaking we prove that generalized travelling waves do exist for all sufficiently large wave speed function in some average sense, while such solutions do not exist for speed function with small average. In addition, under suitable assumptions on the time varying coefficients, we derive a sharp estimate for the average speed functions of the generalized travelling wave solutions.

2.1 Introduction

In this work we investigate the existence and non existence of the so-called generalized travelling wave solutions for the following non-autonomous nonlocal equation

$$\partial_t u(t,x) = \int_{\mathbb{R}} K(t,y)\,[u(t,x-y) - u(t,x)]\,\mathrm{d}y + F\big(t, u(t,x)\big), \tag{2.1.1}$$

where $(t,x) \in \mathbb{R} \times \mathbb{R}$.

Here $K = K(t,y)$ denotes a nonnegative time dependent dispersal kernel while the nonlinear term $F = F(t,u)$ is of Fisher-KPP type with

$$F(t,0) = F(t,1) = 0,\ \forall t \in \mathbb{R}.$$

See Assumption 2.2.3 below for our precise hypothesis.

This equation typically models the spatio-temporal evolution of an invading population into some empty environment. Here the motion of individuals is due to long range dispersal according to the time varying kernel K while the local population dynamics (birth and death processes) is described by the time varying Fisher-KPP nonlinearity F.

When the functions $K(t,y) = K(y)$ and $F(t,u) = F(u)$ are both independent of time t and the kernel K has a thin tail, namely there exists $\sigma > 0$ such that

$$\int_{\mathbb{R}} K(y) e^{\sigma y} \mathrm{d}y < \infty,$$

then Problem (2.1.1) is well known to admit travelling wave solutions, that is solution of the form

$$u(t,x) = U(x - ct),\ (t,x) \in \mathbb{R}^2$$

for some wave speed $c \in \mathbb{R}$.

Recall that travelling wave solutions have been widely studied in the last decades since the pioneer works of Fisher [70] and Kolmogorov, Petrovsky and Piskunov [97] for reaction-diffusion equations. Such solutions have also received a lot of interests for problems with nonlocal diffusion. We refer for instance the reader to [32, 44, 114] and the references therein for results on nonlocal diffusion equations with monostable nonlinearities and to [15, 43] for bistable nonlinearities.

As far as heterogeneous environments are concerned, the notion of travelling waves discussed above has to be generalized to take into account the lack of translation invariance of the medium. The case of periodic medium has also been widely studied, giving raise to the notion of the so-called pulsating waves (see [148]). We refer the reader for instance to [18, 123, 128] and the references therein for results on pulsating waves for monostable and ignition nonlinearities and local diffusion. See also [3, 50] for bistable nonlinearities, [54, 76] for multistable nonlinearities. We also refer to [52, 161, 170, 171] for systems of equations with local diffusion operators. For problems with nonlocal diffusion, we refer to [42, 93, 94] and the references cited therein for results on scalar problems and to [11, 13] for extensions to systems with nonlocal diffusion operators.

As far as general heterogeneous media are concerned, Berestycki and Hamel [19, 20] proposed a generalization of these notions introducing those of transition waves. We also refer to Matano [117] and Shen [138]. Here we follow the definition given in these works. To do so, we first introduce the following definition.

Definition 2.1.1. *A continuous function $u = u(t,x) : \mathbb{R}^2 \to [0,1]$ is said to be a transition wave of (2.1.1) if* *(i) For all $x \in \mathbb{R}$ the function $t \mapsto u(t,x)$ is absolutely continuous on $\mathbb{R}$ and satisfies (2.1.1) for almost every $t \in \mathbb{R}$;*

(ii) There exists some interface function $X : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x\to-\infty} u(t, x + X(t)) = 1 \text{ and } \lim_{x\to\infty} u(t, x + X(t)) = 0,$$

uniformly with respect to $t \in \mathbb{R}$.

From the above definition, we now state the definition of a generalized travelling wave for (2.1.1), which is a special case of transition waves defined above. We refer to [124] and [125] where this definition is used.

Definition 2.1.2. *A continuous function $u = u(t,x) : \mathbb{R}^2 \to [0,1]$ is said to be a generalized travelling wave of (2.1.1) with the wave speed function $c = c(t) \in L^\infty(\mathbb{R})$ if u is a transition wave of (2.1.1) with the interface function*

$$X(t) = \int_0^t c(s)\mathrm{d}s, \ \forall t \in \mathbb{R}.$$

In that case, we define its profile $\phi : \mathbb{R}^2 \to [0,1]$ by

$$\phi(t,z) = u\left(t, z + \int_0^t c(s)\mathrm{d}s\right), \ \forall (t,z) \in \mathbb{R}^2.$$

Note that the profile ϕ satisfies the following behaviours at $z = \pm\infty$:

$$\lim_{z\to-\infty} \phi(t,z) = 1 \text{ and } \lim_{z\to\infty} \phi(t,z) = 0 \text{ uniformly for } t \in \mathbb{R}.$$

Note that generalized travelling waves are nothing but transition waves associated to a globally Lipschitz continuous interface function. Let us also notice that when the profile ϕ of a generalized travelling wave $u = u(t,x)$ with a speed function $c = c(t)$ is rather smooth in space and time, say locally Lipschitz continuous, then it satisfies the following equation for almost every $(t,z) \in \mathbb{R}^2$:

$$\partial_t\phi(t,z) = c(t)\partial_z\phi(t,z) + \int_{\mathbb{R}} K(t,y)\left[\phi(t,z-y) - \phi(t,z)\right]\mathrm{d}y + F(t,\phi(t,z)), \tag{2.1.2}$$

together with the limit behaviours

$$\lim_{z\to-\infty} \phi(t,z) = 1 \text{ and } \lim_{z\to+\infty} \phi(t,z) = 0 \quad \text{uniformly in } t \in \mathbb{R}. \tag{2.1.3}$$

This generalized notion of waves has attracted of a lot interests and several recent works are devoted to the study of such front solutions. We may refer the reader to [87, 88, 120, 124, 125, 127, 141] for studies, including existence, non existence, uniqueness and stability, of scalar reaction-diffusion equations. See also [4, 9, 10] for extensions to systems with time heterogeneities. As far as nonlocal diffusion is concerned, we refer to [109] for results on spatially heterogeneous problems with monostable nonlinearities, to [31, 156] for heterogeneous lattice equations. For nonlocal equation with general time heterogeneous KPP nonlinearity we refer to [142, 143]. In these aforementioned works, the authors dealt with a dispersal kernel function independent of time and a general time heterogeneous KPP nonlinearity. They derived existence, uniqueness and stability properties.

In this work we extend some of these results by considering general time heterogeneities for both the nonlinear term and the dispersal kernel. Our aim is first to construct generalized travelling waves for (2.1.1) and secondly to derive lower estimates for the speed function of the generalized travelling waves. This latter estimate implies some non-existence results when the speed is too small (in some average sense). In addition we shall roughly show that under suitable assumptions on the time varying coefficients, there exists a minimal average speed of propagation. This somehow generalizes the well known results for the travelling waves of the Fisher-KPP equation both with local and nonlocal diffusion, for which we refer to [70, 97] and [44, 135].

2.2 Assumptions and main results

This section is devoted to the statement of our main assumptions for the functions, K and F, arising in (2.1.1) as well as to the statement of the main results presented in this note.

In order to state our assumptions for the kernel function $K = K(t, y)$, let us introduce the following definition, that will be referred along this work as the abscissa of convergence.

Definition 2.2.1. *Let $(X, \|\cdot\|_X)$ be a Banach space and $f \in L^1(\mathbb{R}; X)$. We define the quantity, denoted by $\sigma(f)$ and called the abscissa of convergence of f, as follows*

$$\sigma(f) = \sup\left\{\lambda \geq 0 : \text{ the improper integral } \int_{-\infty}^{\infty} e^{\lambda s} f(s)\mathrm{d}s \text{ converge in } X\right\}.$$

Since $f \in L^1(\mathbb{R}; X)$ the above quantity is also given by

$$\sigma(f) = \sup\left\{\lambda \geq 0 : \lim_{\tau\to\infty} \int_0^{\tau} e^{\lambda s} f(s)\mathrm{d}s \text{ exists in } X\right\}.$$

Using the above definition, we now state our main assumptions for the kernel K.

Assumption 2.2.2 (Kernel $K = K(t, y)$)**.** *The kernel $K : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ satisfies the following set of assumptions:*

(i) The function K is measurable, nonnegative and $K(\cdot, y) \in L^{\infty}_{+}(\mathbb{R})$ for almost every $y \in \mathbb{R}$;

(ii) The map $\tilde{K} : y \mapsto K(\cdot, y)$ from $\mathbb{R}$ into $L^{\infty}(\mathbb{R})$ is measurable and integrable, namely $\tilde{K} \in L^1(\mathbb{R}; L^{\infty}(\mathbb{R}))$;

(iii) Its abscissa of convergence, according to Definition 2.2.1 above, satisfies

$$\sigma\left(\tilde{K}\right) > 0.$$

In the following, for notational simplicity, we will simply use $\sigma(K)$ instead of $\sigma\left(\tilde{K}\right)$.

We now turn to our KPP assumptions for the nonlinear function $F = F(t, u)$.

Assumption 2.2.3 (KPP nonlinearity)**.** *We assume that the function F takes the form $F(t, u) = uf(t, u)$ where the function $f : \mathbb{R} \times [0, 1] \to \mathbb{R}$ satisfies the following set of hypotheses:*

(f1) $f(\cdot, u) \in L^{\infty}(\mathbb{R})$, for all $u \in [0, 1]$, and f is Lipschitz continuous with respect to $u \in [0, 1]$, uniformly with respect to $t \in \mathbb{R}$;

(f2) $f(t,0)=1, f(t,1)=0$ *for a.e.* $t\in\mathbb{R}$ *and*

$$h(u):=\inf_{t\in\mathbb{R}} f(t,u)>0 \text{ for all } u\in[0,1);$$

(f3) For almost every $t\in\mathbb{R}$*, the function* $u\mapsto f(t,u)$ *is nonincreasing on* $[0,1]$.

Remark 2.2.4. *Note that since* $f(t,0)\equiv 1$ *and* $(f1)$*, there exists some constant* $C>0$ *such that*

$$|f(t,0)-f(t,u)|\leq Cu,\ \forall (t,u)\in\mathbb{R}\times[0,1].$$

Hence due to $(f3)$ *we get*

$$1\geq f(t,u)\geq 1-Cu,\ \forall (t,u)\in\mathbb{R}\times[0,1]. \tag{2.2.4}$$

Remark 2.2.5. *In the above set of hypotheses (see Assumption 2.2.3), we have assumed, for simplicity, that* $f(t,0)\equiv 1$*. This assumption can be relaxed by using a change of variable in time to take into account more general KPP nonlinearity function* $F(t,u)=uf(t,u)$ *such that* $f(t,1)\equiv 0$ *and* $f(\cdot,0)=\mu\in L^\infty(\mathbb{R})$ *with*

$$\inf_{t\in\mathbb{R}}\mu(t)>0.$$

Indeed if $u=u(t,x)$ *is a solution of (2.1.1) then by setting*

$$t\mapsto\tau(t)=\int_0^t\mu(s)\mathrm{d}s \text{ and } \hat{u}(\tau(t),x)=u(t,x),$$

the function $\hat{u}$ *becomes a solution of the equation*

$$\partial_\tau\hat{u}(\tau,x)=\int_{\mathbb{R}}\hat{K}(\tau,y)\left[\hat{u}(\tau,x-y)-\hat{u}(\tau,x)\right]\mathrm{d}y+\hat{u}(\tau,x)\hat{f}(\tau,\hat{u}(\tau,x)),$$

wherein we have set

$$\hat{K}(\tau,y)=\frac{K(t,y)}{\mu(t)},\quad \hat{f}(\tau,\hat{u})=\frac{f\left(t,\hat{u}\right)}{\mu(t)}.$$

Hence $\hat{F}(\tau,u)=u\hat{f}(\tau,u)$ *becomes a KPP nonlinearity with* $\hat{f}(\tau,0)\equiv 1$*, while* $\hat{K}$ *satisfies Assumption 2.2.2 with* $\sigma(K)=\sigma\left(\hat{K}\right)$.

In order to state our main results, we need to recall the definitions of the so-called least mean and upper mean value for functions in $L^\infty(\mathbb{R})$. Such notions have been introduced and successfully used to study generalized travelling waves by Nadin and Rossi in [124, 125] (see also [4] for systems).

Definition 2.2.6. *The least mean (resp. the upper mean) value of a function* $g\in L^\infty(\mathbb{R})$ *is defined as follows*

$$\lfloor g\rfloor:=\sup_{T>0}\inf_{t\in\mathbb{R}}\frac{1}{T}\int_0^T g(t+s)\mathrm{d}s,\ \left(resp. \lceil g\rceil:=\inf_{T>0}\sup_{t\in\mathbb{R}}\frac{1}{T}\int_0^T g(t+s)\mathrm{d}s\right).$$

Let us also recall (see [124]) the following important reformulation of the least and upper mean value. For each function $g\in L^\infty(\mathbb{R})$ one has

$$\lfloor g\rfloor=\lim_{T\to+\infty}\inf_{t\in\mathbb{R}}\frac{1}{T}\int_0^T g(t+s)\mathrm{d}s=\sup_{A\in W^{1,\infty}(\mathbb{R})}\inf_{t\in\mathbb{R}}(A'+g)(t), \tag{2.2.5}$$

while

$$\lceil g\rceil=\lim_{T\to+\infty}\sup_{t\in\mathbb{R}}\frac{1}{T}\int_0^T g(t+s)\mathrm{d}s=\inf_{A\in W^{1,\infty}(\mathbb{R})}\sup_{t\in\mathbb{R}}(A'+g)(t). \tag{2.2.6}$$

These alternative reformulations will be used throughout this manuscript. The next definition will also be used at some point.

Definition 2.2.7. *A function $g \in L^\infty(\mathbb{R})$ is said to be uniquely ergodic if the least and the upper mean values coincide, namely*

$$\lfloor g \rfloor = \lceil g \rceil.$$

This means that some constant $[g] \in \mathbb{R}$ such that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g(t+s)\mathrm{d}s = [g] \text{ exists uniformorly for } t \in \mathbb{R}.$$

The main results of this work are strongly related to the following functions $L : [0,\sigma(K)) \to L^\infty(\mathbb{R})$ and $c : (0,\sigma(K)) \to L^\infty(\mathbb{R})$ given by

$$L(\lambda) := \int_{\mathbb{R}} K(\cdot, y)e^{\lambda y}\mathrm{d}y,\ \lambda \in [0,\sigma(K)), \tag{2.2.7}$$

and

$$c(\lambda)(t) := \lambda^{-1}\left[\int_{\mathbb{R}} K(t,y)\left[e^{\lambda y}-1\right]\mathrm{d}y + 1\right], \tag{2.2.8}$$

for $\lambda \in (0,\sigma(K))$ and $t \in \mathbb{R}$, as well as some of their properties, stated below.

Proposition 2.2.8. *Let Assumption 2.2.2 be satisfied. Then the following properties hold:*

(i) The maps L defined above in (2.2.7) is of class C^1 from $(0,\sigma(K))$ into $L^\infty(\mathbb{R})$.

(ii) Consider the sets

$$\Lambda = \{\lambda \in (0,\sigma(K)) :\ \exists \lambda' \in (\lambda,\sigma(K)),\ \forall k \in (\lambda,\lambda'],\ \lfloor c(\lambda) - c(k) \rfloor > 0\},$$

and

$$\tilde{\Lambda} = \{\lambda \in (0,\sigma(K)) :\ \exists \lambda' \in (\lambda,\sigma(K)),\ \lfloor c(\lambda) - c(\lambda') \rfloor > 0\}.$$

Then one has $\Lambda = \tilde{\Lambda}$ and there exists $\lambda^* \in (0,\sigma(K)]$ such that

$$\Lambda = (0,\lambda^*).$$

(iii) One also has:

$$\left\lfloor -\frac{\mathrm{d}c(\lambda)}{\mathrm{d}\lambda} \right\rfloor > 0,\ \forall \lambda \in (0,\lambda^*) \text{ and } \left\lfloor -\frac{\mathrm{d}c(\lambda^*)}{\mathrm{d}\lambda} \right\rfloor = 0 \text{ if } \lambda^* < \sigma(K).$$

(iv) The function $\lambda \mapsto \lfloor c(\lambda) \rfloor$ is decreasing on Λ.

Now for each $\lambda \in (0,\sigma(K))$ and $a \in W^{1,\infty}(\mathbb{R})$, we define $c_{\lambda,a} \in L^\infty(\mathbb{R})$ the function given by

$$c_{\lambda,a}(t) = c(\lambda)(t) + a'(t),\ t \in \mathbb{R}. \tag{2.2.9}$$

Using the above notation, our next result ensures the existence of generalized travelling waves for problem (2.1.1) with the speed function $c_{\lambda,a}$, for each $\lambda \in (0,\lambda^*)$ and $a \in W^{1,\infty}(\mathbb{R})$.

Theorem 2.2.9 (Existence)**.** *Let Assumption 2.2.2 and 2.2.3 be satisfied. Recalling that λ^* is defined in Proposition 2.2.8, for each $\lambda \in (0,\lambda^*)$ and each $a \in W^{1,\infty}(\mathbb{R})$, Problem (2.1.1) possesses a generalized travelling wave with the speed function $c_{\lambda,a} \in L^\infty(\mathbb{R})$, defined in (2.2.9). Furthermore, these travelling wave profiles are globally Lipschitz continuous on $\mathbb{R}^2$.*

Define $\mathscr{C} \subset L^\infty(\mathbb{R})$ the set of admissible speed function, that is the set of the functions $c \in L^\infty(\mathbb{R})$ such that there exists a generalized travelling wave, according to Definition 2.1.2, with the speed function c. Next the above theorem ensures that

$$\left\{t \mapsto c_{\lambda,a}(t),\quad \lambda \in (0,\lambda^*) \text{ and } a \in W^{1,\infty}(\mathbb{R})\right\} \subset \mathscr{C}.$$

As a consequence, recalling the definition of $c(\lambda)$ in (2.2.8) and Proposition 2.2.8 (iv) one also obtains that

$$\left(\lim_{\lambda\to\lambda^*} \lfloor c(\lambda)\rfloor, \infty\right) \subset \lfloor \mathscr{C} \rfloor := \{\lfloor c\rfloor\,,\ c \in \mathscr{C}\}\,. \tag{2.2.10}$$

Our next result provides further properties for the set of admissible wave speed function, $\mathscr{C}$. This result reads as follows.

Theorem 2.2.10 (Wave speed lower estimate)**.** *Let Assumption 2.2.2 and 2.2.3 be satisfied. Define for $\lambda \in (0,\sigma(K))$ the function $t \mapsto \underline{c}(\lambda)(t) \in L^\infty(\mathbb{R})$ by*

$$\underline{c}(\lambda)(t) := \int_{-\infty}^{\infty} zK(t,z)e^{\lambda z}\mathrm{d}z,$$

Then for any $c \in \mathscr{C}$ the following estimate holds

$$\lceil \underline{c}(\lambda)(\cdot) - c(\cdot)\rceil \le 0,\ \forall \lambda \in (0,\lambda^*)\,.$$

As a consequence one also has

$$\sup_{\lambda\in(0,\lambda^*)} \lfloor \underline{c}(\lambda)\rfloor \le \inf \lfloor \mathscr{C}\rfloor\,.$$

As a corollary of the above theorem, we now derive some conditions ensuring that the upper estimate of $\inf \lfloor \mathscr{C}\rfloor$ provided in (2.2.10) is sharp.

Corollary 2.2.11. *Under the same assumptions as in Theorem 2.2.10, assume that $\lambda^* < \sigma(K)$ and that*

$$\lceil c(\lambda^*)(\cdot) - \underline{c}(\lambda^*)(\cdot)\rceil \le 0. \tag{2.2.11}$$

Then $\lfloor \mathscr{C}\rfloor$ is an unbounded interval with

$$\lfloor c(\lambda^*)(\cdot)\rfloor = \inf \lfloor \mathscr{C}\rfloor\,.$$

Within the framework of the above corollary, note that due to (2.2.10) one obtains that the set $\lfloor \mathscr{C}\rfloor$ is given by

$$\text{either }\ (\lfloor c(\lambda^*)(\cdot)\rfloor\,, \infty)\ \text{ or }\ [\lfloor c(\lambda^*)(\cdot)\rfloor\,, \infty)\,.$$

By analogy with the usual Fisher-KPP equation, we suspect that $\lfloor \mathscr{C}\rfloor$ coincides with the closed interval. However we are not able to prove it for the moment. In other words, we cannot prove that $c_{\lambda^*,a}$ is an admissible wave speed function, for some $a \in W^{1,\infty}(\mathbb{R})$.

Let us comment on (2.2.11). To do so, observe that one has

$$-\lambda\frac{\mathrm{d}c(\lambda)}{\mathrm{d}\lambda} = c(\lambda) - \underline{c}(\lambda),\ \forall \lambda \in (0,\sigma(K)).$$

Hence, in view of Proposition 2.2.8 (iii), Condition (2.2.11) is equivalent to the unique ergodicity of the function $c(\lambda^*)(\cdot) - \underline{c}(\lambda^*)(\cdot)$, that is

$$\lfloor c(\lambda^*)(\cdot) - \underline{c}(\lambda^*)(\cdot)\rfloor = \lceil c(\lambda^*)(\cdot) - \underline{c}(\lambda^*)(\cdot)\rceil = 0.$$

As a special case, if for all λ closed λ^* the function $t \mapsto \int_{-\infty}^{\infty} K(t,y)[e^{\lambda y}-1]\mathrm{d}y$ is uniquely ergodic then (2.2.11) holds true and Corollary 2.2.11 applies.

Let us finally observe that the condition $\lambda^* < \sigma(K)$ holds if we assume that

$$\limsup_{\lambda \to \sigma(K)^-} \frac{1}{\lambda} \lfloor L(\lambda) \rfloor = \infty,$$

This property directly follows from the decreasing property of the map $\lambda \mapsto \lfloor c(\lambda) \rfloor$ on $(0, \lambda^*)$.

2.3 Comparison principle

In this section, we state a comparison principle for parabolic nonlocal diffusion equation, that will be used throughout this note.

Proposition 2.3.1 (Comparison principle)**.** *Let $t_0 \in \mathbb{R}$ and $T > 0$ be given. Let $K : (t_0, t_0+T) \times \mathbb{R} \to [0,\infty)$ be a measurable kernel such that the map $t \mapsto \int_{\mathbb{R}} K(t,y)\mathrm{d}y$ is bounded and let $F = F(t,u)$ be a function defined in $[t_0, t_0+T] \times [0,1]$ which is Lipschitz continuous with respect to $u \in [0,1]$, uniformly with respect to t. Let $\underline{u}$ and $\overline{u}$ be two uniformly continuous functions defined from $[t_0, t_0+T] \times \mathbb{R}$ into the interval $[0,1]$ such that for each $x \in \mathbb{R}$, the maps $\underline{u}(\cdot, x)$ and $\overline{u}(\cdot, x)$ both belong to $W^{1,1}(t_0, t_0+T)$, satisfying $\underline{u}(t_0, \cdot) \le \overline{u}(t_0, \cdot)$ and, for all $x \in \mathbb{R}$ and for almost every $t \in (t_0, t_0+T)$,*

$$\partial_t \overline{u}(t,x) \ge \int_{\mathbb{R}} K(t,y)\left[\overline{u}(t,x-y) - \overline{u}(t,x)\right]\mathrm{d}y + F(t, \overline{u}(t,x)),$$
$$\partial_t \underline{u}(t,x) \le \int_{\mathbb{R}} K(t,y)\left[\underline{u}(t,x-y) - \underline{u}(t,x)\right]\mathrm{d}y + F(t, \underline{u}(t,x)).$$

Then $\underline{u} \le \overline{u}$ on $[t_0, t_0+T] \times \mathbb{R}$.

Proof. Assume for notational simplicity that $t_0 = 0$. Next for $\delta \in \mathbb{R}$ to be chosen later, consider the function

$$v(t,x) = e^{\delta t}\left[\overline{u}(t,x) - \underline{u}(t,x)\right],$$

so that v is uniformly continuous on $[0,T] \times \mathbb{R}$, $v(0,\cdot) \ge 0$ and for all $x \in \mathbb{R}$ $t \mapsto v(t,x) \in W^{1,1}(0,T)$ so that for all $x \in \mathbb{R}$, $\partial_t v \in L^1(0,T)$.

Then the function v satisfies for all $x \in \mathbb{R}$ and *a.e.* $t \in (0,T)$

$$\partial_t v(t,x) \ge \int_{\mathbb{R}} K(t,y)\left[v(t,x-y) - v(t,x)\right]\mathrm{d}y + e^{\delta t}\left[F(t,\overline{u}(t,x)) - F(t,\underline{u}(t,x))\right] + \delta v.$$

Next there exists some function $a = a(t,x) \in L^\infty((0,T) \times \mathbb{R})$ such that

$$e^{\delta t}\left[F(t,\overline{u}(t,x)) - F(t,\underline{u}(t,x))\right] = a(t,x)v(t,x).$$

Hence setting $\bar{K}(t) = \int_{\mathbb{R}} K(t,y)\mathrm{d}y$, one obtains

$$\partial_t v(t,x) \ge \int_{\mathbb{R}} K(t,y)v(t,x-y)\mathrm{d}y + \left(a(t,x) - \bar{K}(t) + \delta\right)v(t,x).$$

Now choose $\delta > 0$ large enough such that $a(t,x) - \bar{K}(t) + \delta \ge 1$ for all $(t,x) \in (0,T) \times \mathbb{R}$. Next, consider the function $w(t) := \inf_{x \in \mathbb{R}} v(t,x)$ and observe that w is continuous since

v is bounded and uniformly continuous on $[0,T]\times\mathbb{R}$. Observe also that one has for all $(t,x)\in(0,T)\times\mathbb{R}$

$$\left(a(t,x)-\bar{K}(t)+\delta\right)v(t,x)\geq\left(a(t,x)-\bar{K}(t)+\delta\right)w(t)\geq G(t)w(t),$$

where the positive function $G\in L^\infty(0,T)$ is given by

$$G(t)=\begin{cases}\inf_{x\in\mathbb{R}}a(t,x)-K(t)+\delta \text{ if } w(t)\geq 0,\\ \sup_{x\in\mathbb{R}}a(t,x)-K(t)+\delta \text{ if } w(t)<0.\end{cases}$$

As a consequence, since v is continuous, v satisfies for all $x\in\mathbb{R}$ and all $t\in(0,T)$:

$$v(t,x)\geq v(0,x)+\int_0^t\int_{\mathbb{R}}K(s,y)v(s,x-y)\mathrm{d}y\mathrm{d}s+\int_0^t G(s)w(s)\mathrm{d}s.$$

so that, taking the infimum with respect to $x\in\mathbb{R}$ yields, for all $t\in[0,T]$,

$$w(t)\geq w(0)+\int_0^t\left[G(s)+\int_{\mathbb{R}}K(s,y)\mathrm{d}y\right]w(s)\mathrm{d}s.$$

Since $G(s)+\int_{\mathbb{R}}K(s,y)\mathrm{d}y\geq 0$ for all $s\in(0,T)$ and $w(0)\geq 0$, the above inequality ensures that $w(t)\geq 0$ for all $t\in[0,T]$ and completes the proof of the result. □

2.4 Proof of Proposition 2.2.8

In this section we are concerned with the proof of Proposition 2.2.8. We start by the proof of the first part, namely Proposition 2.2.8 (i).

Proof of Proposition 2.2.8 (i). Firstly, let us notice that due to Assumption 2.2.2 (ii) and Lebesgue dominated convergence theorem, the map $\lambda\mapsto\int_{-\infty}^0 K(\cdot,y)e^{\lambda y}\mathrm{d}y$ is continuous from the half complex plane $\{\lambda\in\mathbb{C}:\ \mathrm{Re}\,\lambda\geq 0\}$ into $L^\infty(\mathbb{R})$ and holomorphic on the half plane $\{\lambda\in\mathbb{C}:\ \mathrm{Re}\,\lambda>0\}$.

Next applying Theorem 1.5.1 in [6] the map $\lambda\mapsto\int_0^\infty K(\cdot,y)e^{\lambda y}\mathrm{d}y$ is holomorphic from the half space $\{\lambda\in\mathbb{C}:\ \mathrm{Re}\,\lambda<\sigma(K)\}$ into $L^\infty(\mathbb{R})$.

As a consequence the map L, the sum of the two above functions, is continuous from the strip $\{\lambda\in\mathbb{C}:\ 0\leq\mathrm{Re}\,\lambda<\sigma(K)\}$ into $L^\infty(\mathbb{R})$ while holomorphic on the open strip $\{\lambda\in\mathbb{C}:\ 0<\mathrm{Re}\,\lambda<\sigma(K)\}$. This completes the proof of Proposition 2.2.8 (i). □

Before proving Proposition 2.2.8(ii), we first give the following Lemma.

Lemma 2.4.1. *The least mean and upper mean operators are continuous from $L^\infty(\mathbb{R})$ into $\mathbb{R}$.*

Proof. We only prove it for the least mean operator, the continuity for the upper mean follows the same arguments. Let $g\in L^\infty(\mathbb{R})$ be given and let $(g_n)\subset L^\infty(\mathbb{R})$ be any sequence tending to g uniformly. Let $\varepsilon>0$ be given. Then there exists $N\geq 0$ large enough such that for all $n\geq N$ and for almost every $t\in\mathbb{R}$ one has

$$g_n(t)-\varepsilon\leq g(t)\leq g_n(t)+\varepsilon.$$

Hence we get for all $n\geq N$,

$$\lfloor g_n\rfloor-\varepsilon\leq\lfloor g\rfloor\leq\lfloor g_n\rfloor+\varepsilon,$$

so that $\lim\limits_{n\to\infty}\lfloor g_n\rfloor=\lfloor g\rfloor$, that proves the lemma. □

From the above lemma coupled with the continuity of L provided in Proposition 2.2.8 (i), one directly obtains that the maps $\lambda \mapsto \lfloor c(\lambda) \rfloor$ and $\lambda \mapsto \lceil c(\lambda) \rceil$ are both continuous on the interval $(0, \sigma(K))$. Moreover these maps enjoy the following behaviour when $\lambda \to 0^+$:

$$\lfloor c(\lambda) \rfloor \sim \lceil c(\lambda) \rceil \sim \frac{1}{\lambda} \text{ when } \lambda \to 0^+. \tag{2.4.12}$$

We now turn to the proof of Proposition 2.2.8 (ii). To that aim we adapt some arguments presented in [125] to our context.

Proof of Proposition 2.2.8 (ii). First observe that $\Lambda \subset \tilde{\Lambda}$. Next note that, for each fixed $t \in \mathbb{R}$, $\lambda \mapsto \lambda c(\lambda)(t)$ is convex. We now fix $\lambda_0, \lambda_1 \in (0, \sigma(K))$ and introduce, for $\tau \in [0,1]$, the point λ_τ given by

$$\lambda_\tau := (1-\tau)\lambda_0 + \tau\lambda_1.$$

Due to the convexity of the function $\lambda \mapsto \lambda c(\lambda)(t)$, we have for almost every $t \in \mathbb{R}$ and any $\tau \in [0,1]$

$$(1-\tau)\lambda_0 c(\lambda_0)(t) + \tau\lambda_1 c(\lambda_1)(t) \geq \lambda_\tau c(\lambda_\tau)(t),$$

that rewrites

$$\lambda_\tau c(\lambda_0)(t) + \tau\lambda_1 c(\lambda_1)(t) \geq \lambda_\tau c(\lambda_\tau)(t) + \tau\lambda_1 c(\lambda_0)(t).$$

As a consequence for all $T > 0$ and almost every $t \in \mathbb{R}$ one obtains

$$\lambda_\tau \int_0^T (c(\lambda_0)(t+s) - c(\lambda_\tau)(t+s))\, \mathrm{d}s \geq \tau\lambda_1 \int_0^T (c(\lambda_0)(t+s) - c(\lambda_1)(t+s))\, \mathrm{d}s,$$

that yields, dividing both sides by $T > 0$, taking the infimum of t and taking the limit as $T \to \infty$, the following inequality

$$\lfloor c(\lambda_0) - c(\lambda_\tau) \rfloor \geq \tau \frac{\lambda_1}{\lambda_\tau} \lfloor c(\lambda_0) - c(\lambda_1) \rfloor, \quad \forall \tau \in [0,1], \tag{2.4.13}$$

while, dividing by $-T$ ensures that

$$\lfloor c(\lambda_\tau) - c(\lambda_0) \rfloor \leq \tau \frac{\lambda_1}{\lambda_\tau} \lfloor c(\lambda_1) - c(\lambda_0) \rfloor, \quad \forall \tau \in [0,1]. \tag{2.4.14}$$

On the other hand, taking the supremum with respect to $t \in \mathbb{R}$ instead of the infimum, we get the following similar estimates for the upper mean,

$$\lceil c(\lambda_0) - c(\lambda_\tau) \rceil \geq \tau \frac{\lambda_1}{\lambda_\tau} \lceil c(\lambda_0) - c(\lambda_1) \rceil, \quad \forall \tau \in [0,1], \tag{2.4.15}$$

and

$$\lceil c(\lambda_\tau) - c(\lambda_0) \rceil \leq \tau \frac{\lambda_1}{\lambda_\tau} \lceil c(\lambda_1) - c(\lambda_0) \rceil, \quad \forall \tau \in [0,1]. \tag{2.4.16}$$

Now let us deduce from the above properties that $\tilde{\Lambda} \subset \Lambda$. To do so let $\lambda \in \tilde{\Lambda}$ be given, that is, there exists $\lambda' \in (\lambda, \sigma(K))$ such that $\lfloor c(\lambda) - c(\lambda') \rfloor > 0$. Applying (2.4.13) with $\lambda_0 = \lambda$, $\lambda_1 = \lambda'$, we obtain

$$\lfloor c(\lambda) - c(\lambda_\tau) \rfloor \geq \tau \frac{\lambda'}{\lambda_\tau} \lfloor c(\lambda) - c(\lambda') \rfloor > 0, \quad \forall \tau \in (0,1],$$

that is $\lfloor c(\lambda) - c(k) \rfloor > 0$ for any $k \in (\lambda, \lambda']$. Hence $\lambda \in \Lambda$ and $\tilde{\Lambda} \subset \Lambda$.

Next, we prove there exists $\lambda^* \in (0, \sigma(K)]$ such that $\Lambda = (0, \lambda^*)$. We split this proof into three steps.

Step1. Let us show that $\Lambda \neq \emptyset$.
Let $\lambda_0 \in (0, \sigma(K))$ be given. Then one has

$$\liminf_{\lambda \to 0^+} \lfloor c(\lambda) - c(\lambda_0) \rfloor \geq \lim_{\lambda \to 0^+} \lfloor c(\lambda) \rfloor - \lceil c(\lambda_0) \rceil = +\infty.$$

Hence there exists $0 < \lambda < \lambda_0 < \sigma(K)$ such that $\lfloor c(\lambda) - c(\lambda_0) \rfloor > 0$, that is $\lambda \in \tilde{\Lambda} = \Lambda$.

Step 2. In this step, let us show that if $\lambda \in \Lambda$, then $(0, \lambda] \subset \Lambda$.
To that aim, fix $0 < \lambda' < \lambda$ and $k \in (0, \sigma(K) - \lambda)$ such that $\lfloor c(\lambda) - c(\lambda + k) \rfloor > 0$. Then using successively (2.4.13) and (2.4.14), we get that there exists some positive constant $m > 0$ such that

$$\lfloor c(\lambda') - c(\lambda' + k) \rfloor \geq m \lfloor c(\lambda) - c(\lambda + k) \rfloor > 0,$$

and $\lambda' \in \Lambda$.

Step 3. We now define $\lambda^* \in (0, \sigma(K)]$ by $\lambda^* := \sup \Lambda$. Now to complete the proof of Proposition 2.2.8 (ii), let us show that if $\lambda^* < \sigma(K)$ then $\lambda^* \notin \Lambda$.
To check this property, let $k \in (0, \sigma(K) - \lambda^*)$ be given. Now from the definition of λ^*, for all $n \in \mathbb{N}^*$, $\lambda^* + \frac{1}{n} \notin \Lambda$ and there exists $0 < k_n < \frac{1}{n}$ such that

$$\lfloor c(\lambda^* + \frac{1}{n}) - c(\lambda^* + \frac{1}{n} + k_n) \rfloor \leq 0.$$

Hence for all n large enough such that $\frac{1}{n} + k_n < k$ one has

$$\begin{aligned} 0 &\geq \lfloor c(\lambda^* + \frac{1}{n}) - c(\lambda^* + \frac{1}{n} + k_n) \rfloor \\ &\geq \left(\frac{k_n}{k - \frac{1}{n}} \right) \left(\frac{\lambda^* + k}{\lambda^* + \frac{1}{n} + k_n} \right) \lfloor c(\lambda^* + \frac{1}{n}) - c(\lambda^* + k) \rfloor. \end{aligned}$$

On the other hand one also has for all n

$$\lfloor c(\lambda^*) - c(\lambda^* + k) \rfloor \leq \lfloor c(\lambda^* + \frac{1}{n}) - c(\lambda^* + k) \rfloor + \lceil c(\lambda^*) - c(\lambda^* + \frac{1}{n}) \rceil.$$

Coupling the two above inequalities yields that for all n large enough one has

$$\lfloor c(\lambda^*) - c(\lambda^* + k) \rfloor \leq \lceil c(\lambda^*) - c(\lambda^* + \frac{1}{n}) \rceil.$$

It follows from the continuity of the function $\lambda \mapsto c(\lambda)$ into $L^\infty(\mathbb{R})$ and the continuity of mean value (see Lemma 2.4.1) that the right hand side in the above inequality goes to 0 as $n \to \infty$, so that yields $\lfloor c(\lambda^*) - c(\lambda^* + k) \rfloor \leq 0$ for all $k \in (0, \sigma(K) - \lambda^*)$. We conclude that $\lambda^* \notin \Lambda$ and this completes the proof of Proposition 2.2.8 (ii). □

We now turn to the proof of Proposition 2.2.8 (iii).

Proof. Due to Proposition 2.2.8 (i) and recalling the definition of $c(\lambda)$, namely

$$c(\lambda)(t) = \frac{L(\lambda)(t) - \overline{K}(t) + 1}{\lambda}, \ a.e.\ t \in \mathbb{R},\ \lambda \in (0, \sigma(K)),$$

the map $\lambda \mapsto c(\lambda)(\cdot)$ is of the class C^1 from $(0, \sigma(K))$ into $L^\infty(\mathbb{R})$.

Now fix $\lambda \in \Lambda = (0, \lambda^*)$. Next from the definition of the set Λ, there exists $k \in (0, \lambda^* - \lambda)$ such that $\lfloor c(\lambda) - c(\lambda + k) \rfloor > 0$. Using (2.4.13), one has for all $\tau \in (0, 1)$

$$\lfloor c(\lambda) - c(\lambda + \tau k) \rfloor \geq \tau \frac{\lambda + k}{\lambda + \tau k} \lfloor c(\lambda) - c(\lambda + k) \rfloor > 0.$$

Hence taking the limit $\tau \to 0^+$ into the above inequality yields

$$\lim_{\tau\to 0^+} \frac{\lfloor c(\lambda) - c(\lambda+\tau k)\rfloor}{\tau}\frac{\lambda+\tau k}{\lambda+k} \geq \lfloor c(\lambda) - c(\lambda+k)\rfloor > 0.$$

Since $\lambda \mapsto c(\lambda)$ is of class C^1 with value in $L^\infty(\mathbb{R})$ and using the continuity stated in Lemma 2.4.1, this ensures that

$$k\left\lfloor -\frac{\mathrm{d}c(\lambda)}{\mathrm{d}\lambda}\right\rfloor = \left\lfloor \lim_{\tau\to 0^+}\frac{c(\lambda)-c(\lambda+\tau k)}{\tau}\right\rfloor = \lim_{\tau\to 0^+}\left\lfloor \frac{c(\lambda)-c(\lambda+\tau k)}{\tau}\right\rfloor.$$

Hence we obtain that

$$\left\lfloor -\frac{\mathrm{d}c(\lambda)}{\mathrm{d}\lambda}\right\rfloor > 0 \text{ for all } \lambda \in (0,\lambda^*).$$

Now, let us check that $\lfloor -\frac{\mathrm{d}c(\lambda^*)}{\mathrm{d}\lambda}\rfloor = 0$. To see this, letting $\lambda \to \lambda^*$ with $\lambda \in \Lambda$ into the above inequality and recalling that the map $\lambda \mapsto \frac{\mathrm{d}c(\lambda)}{\mathrm{d}\lambda}$ is continuous with values in $L^\infty(\mathbb{R})$, this yields $\lfloor -\frac{\mathrm{d}c(\lambda^*)}{\mathrm{d}\lambda}\rfloor \geq 0$.

On the other hand, since $\lambda^* \notin \Lambda$, then for all $h>0$, one has $\lfloor c(\lambda^*) - c(\lambda^*+h)\rfloor \leq 0$. Here again, since $\lambda \mapsto c(\lambda)$ is continuously differentiable into $L^\infty(\mathbb{R})$, we get

$$\lfloor \lim_{h\to 0^+}\frac{c(\lambda)-c(\lambda+h)}{h}\rfloor = \lim_{h\to 0^+}\lfloor\frac{c(\lambda)-c(\lambda+h)}{h}\rfloor,$$

so that $\lfloor -\frac{\mathrm{d}c(\lambda^*)}{\mathrm{d}\lambda}\rfloor \leq 0$ and the proof is completed.

□

Finally, we turn to the proof of Proposition 2.2.8 (iv).

Proof. By contradiction, assume that there exist $0<\lambda_1<\lambda_2<\lambda^*$ such that $\lfloor c(\lambda_1)\rfloor \leq \lfloor c(\lambda_2)\rfloor$. We have proved that $\lambda \mapsto \lfloor c(\lambda)\rfloor$ is continuous, hence it attains its minimum on $[\lambda_1,\lambda_2]$ at some λ. Since $\lfloor c(\lambda_1)\rfloor \leq \lfloor c(\lambda_2)\rfloor$, we can assume that $\lambda \in [\lambda_1,\lambda_2)$. From the definition of Λ, there exists $\lambda' \in (\lambda,\lambda_2)$ such that $\lfloor c(\lambda) - c(\lambda')\rfloor > 0$. Hence

$$\lfloor c(\lambda)\rfloor - \lfloor c(\lambda')\rfloor \geq \lfloor c(\lambda) - c(\lambda')\rfloor > 0.$$

This is impossible. □

2.5 Proof of Theorem 2.2.9

This section is devoted to the proof of Theorem 2.2.9, which is split into two main steps. We first construct suitable super-solutions and sub-solutions. They are used in the second step to construct a generalized travelling wave, by considering a suitable initial data Cauchy problem starting at time $t=-n$, for some integer n, and making use of a limiting argument letting $n\to\infty$.

2.5.1 Construction of sub and super solutions

Throughout this section, let $\lambda \in (0,\lambda^*)$ be given and fix $a \in W^{1,\infty}(\mathbb{R})$. Recall that $t \mapsto c_{\lambda,a}(t)$ is defined in (2.2.9).

First step: In this first step we construct a supersolution of (2.1.2) with the speed function $c(t) = c_{\lambda,a}(t)$. Set for $(t,z)\in\mathbb{R}^2$

$$\bar{\phi}(t,z) = \min\{1,\psi(t,z)\}, \text{ with } \psi(t,z) = e^{-\lambda(z+a(t))}.$$

Note that due to Assumption 2.2.3 (see Remark 2.2.4), we have $F(t,\phi) \leq \phi$ for all $t \in \mathbb{R}$ and $\phi \in [0,1]$. Hence to show that $\bar{\phi}$ is a supersolution, it is sufficient to check that

$$\partial_t \psi - c_{\lambda,a}(t)\partial_z \psi - \int_{\mathbb{R}} K(t,y)\left[\psi(t,z-y) - \psi(t,z)\right]\mathrm{d}y - \psi(t,z) \geq 0.$$

Plugging the expression of $\psi(t,z)$ into the above equation yields

$$\begin{aligned}\partial_t \psi - c_{\lambda,a}(t)\partial_z \psi - &\int_{\mathbb{R}} K(t,y)\left[\psi(t,z-y) - \psi(t,z)\right]\mathrm{d}y - \psi(t,z)\\ &= \psi(t,z)\left[-\lambda a'(t) + c_{\lambda,a}(t)\lambda - \int_{\mathbb{R}} K(t,y)\left[e^{\lambda y} - 1\right]\mathrm{d}y - 1\right] = 0,\end{aligned}$$

for all $(t,z) \in \mathbb{R}^2$.

Second step: The aim of this second step is to construct a suitable subsolution. To do that, let us start to define some quantities that will be used in our construction process. Recall that $\lambda \in (0,\lambda^*)$ is fixed. Hence due to Proposition 2.2.8, we can find $0 < k < \min\{\lambda, \sigma(K) - \lambda\}$ such that

$$\lfloor c(\lambda) - c(\lambda + k) \rfloor = \lfloor c_{\lambda,a} - c_{\lambda+k,a} \rfloor > 0. \tag{2.5.17}$$

Next due to (2.2.5), there exist a function $b_0 = b_0(t) \in W^{1,\infty}(\mathbb{R})$ and some $\varepsilon > 0$ such that for all $t \in \mathbb{R}$ one has

$$\left[(\lambda + k)c_{\lambda,a}(t) - \int_{\mathbb{R}} K(t,y)[e^{(\lambda+k)y} - 1]\mathrm{d}y - 1\right] + (b_0'(t) - \lambda a'(t)) \geq \varepsilon. \tag{2.5.18}$$

We now construct a subsolution for (2.1.2) with the speed function $c(t) = c_{\lambda,a}(t)$. To that aim, using the above notations, for $b_1 > 0$ (that will be chosen large enough below) we set

$$b(t) = b_0(t) + b_1 \in W^{1,\infty}(\mathbb{R})$$

as well as

$$\varphi(t,z) = e^{-\lambda(z+a(t))} - e^{-\lambda a(t) + b(t)}e^{-(\lambda+k)z}, \quad (t,z) \in \mathbb{R}^2. \tag{2.5.19}$$

Define also the set

$$\mathcal{O} := \{(t,z) \in \mathbb{R}^2 : \ \varphi(t,z) \geq 0\} = \left\{(t,z) \in \mathbb{R}^2 : \ z \geq \frac{b(t)}{k}\right\},$$

and observe that

$$0 \leq \varphi(t,z) \leq e^{-\lambda(z+a(t))} \leq e^{-\frac{\lambda}{k}(b(t)+ka(t))}, \ \forall (t,z) \in \mathcal{O}. \tag{2.5.20}$$

Now let us determined b_1 large enough such that φ is a subsolution for (2.1.2) with $c(t) = c_{\lambda,a}(t)$ on $\mathcal{O}$. Recalling the definition of $C > 0$ in (2.2.4), choose b_1 large enough such that

$$e^{-\frac{\lambda}{k}(b_0(t)+b_1+ka(t))} \leq 1, \ \forall t \in \mathbb{R}.$$

Next to prove that φ is a sub-solution on the set $\mathcal{O}$, we only need to check that for all $(t,z) \in \mathcal{O}$, one has

$$\left(\partial_t - c_{\lambda,a}(t)\partial_z\right)\varphi(t,z) - \int_{\mathbb{R}} K(t,y)[\varphi(t,z-y) - \varphi(t,z)]\mathrm{d}y \leq \varphi(t,z) - C\varphi^2(t,z). \tag{2.5.21}$$

Using straightforward algebra and setting $A(t) = e^{-\lambda a(t)+b(t)}$, this rewrites as for all $(t,z) \in \mathcal{O}$,

$$A'(t)e^{-(\lambda+k)z} + A(t)e^{-(\lambda+k)z}\left[(\lambda+k)c_{\lambda,a}(t) - \int_{\mathbb{R}} K(t,y)[e^{(\lambda+k)y}-1]\mathrm{d}y - 1\right] \geq C\varphi^2(t,z).$$

Due to (2.5.20), it is sufficient to have for $(t,z) \in \mathcal{O}$,

$$\left[(\lambda+k)c_{\lambda,a}(t) - \int_{\mathbb{R}} K(t,y)[e^{(\lambda+k)y}-1]\mathrm{d}y - 1\right] + (b_0'(t) - \lambda a'(t)) \geq CA^{-1}(t)e^{(-\lambda+k)z-2\lambda a(t)}.$$

Since $k < \lambda$, it is thus sufficient to have for all $t \in \mathbb{R}$

$$\left[(\lambda+k)c_{\lambda,a}(t) - \int_{\mathbb{R}} K(t,y)[e^{(\lambda+k)y}-1]\mathrm{d}y - 1\right] + (b_0'(t) - \lambda a'(t)) \geq Ce^{-\frac{\lambda}{k}(b_0(t)+b_1)-\lambda a(t)}.$$

Finally in view of the definition of b_0 and ε in (2.5.18), fix b_1 larger if necessary so that

$$Ce^{-\frac{\lambda}{k}(b_0(t)+b_1)-\lambda a(t)} \leq \frac{\varepsilon}{2}, \ \forall t \in \mathbb{R},$$

and with such a choice, one may observe that (2.5.21) is satisfied for $(t,z) \in \mathcal{O}$.

As a conclusion of the above analysis, the function $\underline{\phi}$ given by

$$\underline{\phi}(t,z) := \max\{0, \varphi(t,z)\}, \ \forall (t,z) \in \mathbb{R}^2, \tag{2.5.22}$$

is a sub-solution of (2.1.2) in $\mathbb{R}^2$.

2.5.2 Construction of a solution by a limiting procedure

For any integer $n \geq 1$, we consider the following initial value problem, posed for $t \geq -n$ and $z \in \mathbb{R}$,

$$\begin{cases} \partial_t \phi = c_{\lambda,a}(t)\partial_z\phi(t,z) + \int_{\mathbb{R}} K(t,y)\left[\phi(t,z-y) - \phi(t,z)\right]\mathrm{d}y + F(t,\phi), \\ \phi(-n,z) = \overline{\phi}(-n,z). \end{cases} \tag{2.5.23}$$

We denote by $\phi^n = \phi^n(t,z)$ the solution of the above equation and define the function $u^n = u^n(t,z)$ by

$$u^n(t,z) = \phi^n(t, z - \int_0^t c_{\lambda,a}(s)\mathrm{d}s).$$

One may observe that the function $u^n(t,z)$ satisfies the following equation without the drift term $c_{\lambda,a}(t)\partial_z$

$$\begin{cases} \partial_t u(t,z) = \int_{\mathbb{R}} K(t,y)\left[u(t,z-y) - u(t,z)\right]\mathrm{d}y + F(t,u), \ t \geq -n, \ z \in \mathbb{R}, \\ u(-n,z) = \overline{\phi}\left(-n, z - \int_0^{-n} c_{\lambda,a}(s)\mathrm{d}s\right), \ z \in \mathbb{R}. \end{cases} \tag{2.5.24}$$

Now note that the comparison principle stated in Proposition 2.3.1 applies and ensures that the solution $u^n(t,z)$ of (2.5.24) for all $t \geq -n$ and $z \in \mathbb{R}$ satisfies the estimates

$$\underline{\phi}\left(t, z - \int_0^t c_{\lambda,a}(s)\mathrm{d}s\right) \leq u^n(t,z) \leq \overline{\phi}\left(t, z - \int_0^t c_{\lambda,a}(s)\mathrm{d}s\right).$$

Moreover since the function $z \mapsto \overline{\phi}(-n, z - \int_0^{-n} c_{\lambda,a}(s)\mathrm{d}s)$ is nonincreasing on $\mathbb{R}$, then the function $z \mapsto u^n(t,z)$ is also nonincreasing with respect to $z \in \mathbb{R}$ for each given $t \geq -n$.

Our aim now is to pass to the limit $n \to \infty$ in the sequence of function $\{u^n = u^n(t,z)\}$ to construct a generalized travelling wave of (2.1.1). To do so, we first discuss in the following lemma some important Lipschitz regularity estimates, inspired by [143].

Lemma 2.5.1. *There exists some constant $m > \lambda$ large enough such that for all $n \geq 1$ one has*

$$|u^n(t,z+h) - u^n(t,z)| \leq \min\left\{1, e^{m|h|} - 1\right\},\ \forall t \geq -n,\ \forall z \in \mathbb{R}.$$

For all $n \geq 1$ one has $\partial_t u^n \in L^\infty((-n,\infty) \times \mathbb{R})$ and the following estimate holds

$$\|\partial_t u^n\|_\infty \leq 2\int_{\mathbb{R}} \|K(\cdot,y)\|_\infty \mathrm{d}y + 1,\ \forall n \geq 1.$$

In other words, the sequence $\{u^n = u^n(t,z)\}$ is uniformly bounded (with respect to n) in the Lipschitz norm on the set $[-n,\infty) \times \mathbb{R}$.

Proof. Since for all $n \geq 1$, the function u^n is between 0 and 1 and since it is nonincreasing with respect to z, in order to prove the first estimate, we only need to prove that for all $n \geq 1$ one has

$$|u^n(t,z+h) - u^n(t,z)| \leq e^{m|h|} - 1,\quad \forall t \geq -n, \forall z \in \mathbb{R}.$$

In the following we prove it for $h > 0$. The case where $h < 0$ can be proved similarly.

For any n set $c_n = \int_0^{-n} c_{\lambda,a}(s)\mathrm{d}s$. Note that one has

$$u^n(-n,z) = \begin{cases} e^{-\lambda(z-c_n+a(-n))}, & if\ \ z \geq c_n - a(-n), \\ 1, & if\ \ z < c_n - a(-n), \end{cases} \tag{2.5.25}$$

while for $h > 0$, one has

$$u^n(-n,z+h) = \begin{cases} e^{-\lambda(z+h-c_n+a(-n))}, & if\ \ z+h \geq c_n - a(-n), \\ 1, & if\ \ z+h < c_n - a(-n). \end{cases} \tag{2.5.26}$$

Then we infer from these formulas

$$\frac{u^n(-n,z+h)}{u^n(-n,z)} = \begin{cases} e^{-\lambda h}, & if\ \ z \geq c_n - a(-n), \\ e^{-\lambda(z+h-c_n+a(-n))}, & if\ \ c_n - a(-n) \geq z \geq c_n - a(-n) - h, \\ 1, & if\ \ z+h < c_n - a(-n). \end{cases} \tag{2.5.27}$$

Hence one can choose $m > \lambda$ large enough such that for all n one has

$$e^{-mh} \leq \frac{u^n(-n,z+h)}{u^n(-n,z)} \leq 1.$$

Now fix n and $h > 0$ and consider the function $v^h = v^h(t,z)$ given by

$$v^h(t,z) := \frac{u^n(t,z+h)}{e^{-mh}}.$$

It satisfies the problem

$$\begin{cases} \partial_t v^h(t,z) = \int_{\mathbb{R}} K(t,y)\left[v^h(t,z-y) - v^h(t,z)\right]\mathrm{d}y + e^{mh}F(t, e^{-mh}v^h), \\ v^h(-n,z) = e^{mh}\overline{\phi}(-n, z+h-c_n). \end{cases} \tag{2.5.28}$$

Next Assumption 2.2.3 (see $(f3)$) ensures that

$$e^{mh}F(t,e^{-mh}v^h)=v^hf(t,e^{-mh}v^h)\geq v^hf(t,v^h)=F(t,v^h). \tag{2.5.29}$$

Hence v^h becomes the super-solution of (2.5.24) and the following ordering holds at $t=-n$

$$v^h(-n,z)=e^{mh}u^n(-n,z+h)\geq u^n(-n,z),\ \forall z\in\mathbb{R}.$$

As a consequence, the comparison principle applies and provides

$$u^n(t,z)\leq v^h(t,z),\ \ \forall (t,z)\in[-n,\infty)\times\mathbb{R}. \tag{2.5.30}$$

Now since for all $t\geq -n$ the function $z\mapsto u^n(t,z)$ is nonincreasing, for all $t\geq -n$ and $z\in\mathbb{R}$ one gets

$$\begin{aligned}|u^n(t,z+h)-u^n(t,z)|&=u^n(t,z)-u^n(t,z+h)\\ &\leq(1-e^{-mh})v^h(t,z)\leq e^{mh}-1.\end{aligned} \tag{2.5.31}$$

Hence, since $u^n\leq 1$, we have obtained that, for all $n\geq 1$ and for all $h>0$,

$$|u^n(t,z+h)-u^n(t,z)|\leq\min\left\{1,e^{m|h|}-1\right\},\forall t\geq -n,\forall z\in\mathbb{R}.$$

As mentioned above, the case of $h<0$ can be handled similarly and this completes the proof of the first estimate in Lemma 2.5.1.

Finally due to (2.5.24) and Remark 2.2.4, we have

$$\|\partial_t u^n\|_{L^\infty}\leq 2\int_{\mathbb{R}}\|K(\cdot,y)\|_\infty \mathrm{d}y+1,\ \ \forall n\geq 1.$$

This proves the second estimate of the lemma and this completes the proof of the result. □

The Lipschitz continuous estimate provided in Lemma 2.5.1 allows us to make use of Arzelà-Ascoli theorem, which ensures that there exists a subsequence of $\{u^n\}$, still denoted with the same indexes, and a globally Lipschitz continuous function $u=u(t,z):\mathbb{R}^2\to\mathbb{R}$ such that

$$u^n(t,z)\to u(t,z)\text{ as } n\to\infty$$

locally uniformly for $(t,z)\in\mathbb{R}^2$.

This allows us to define the Lipschitz continuous function $\phi=\phi(t,z)$ by

$$\phi(t,z)=u\left(t,z+\int_0^t c_{\lambda,a}(s)\mathrm{d}s\right),\ \forall(t,z)\in\mathbb{R}^2. \tag{2.5.32}$$

We summarize in the next proposition some important properties satisfied by the function ϕ, directed inherited from those of sequence of function $\{u^n\}$.

Proposition 2.5.2. *The function $\phi=\phi(t,z)$ enjoys the following properties.*

(i) It is nonincreasing with respect to $z\in\mathbb{R}$, for all $t\in\mathbb{R}$, and is globally Lipschitz continuous on $\mathbb{R}^2$;

(ii) It satisfies the following estimate for all $(t,z)\in\mathbb{R}^2$

$$\underline{\phi}\,(t,z)\leq\phi\,(t,z)\leq\overline{\phi}\,(t,z)\,. \tag{2.5.33}$$

(iii) It satisfies (2.1.2) with $c(t) = c_{\lambda,a}(t)$ for any $z \in \mathbb{R}$ and for a.e. $t \in \mathbb{R}$.

Note that (2.5.33) ensures the following behaviour at $z = \infty$

$$\phi(t,z) \sim e^{-\lambda(z+a(t))} \text{ as } z \to \infty, \text{ uniformly for } t \in \mathbb{R},$$

so that as a special case, one has

$$\lim_{z\to\infty} \|\phi(\cdot,z)\|_{L^\infty(\mathbb{R})} = 0.$$

Hence to complete the proof of Theorem 2.2.9, it remains to study the behaviour of ϕ as $z \to -\infty$. We will more precisely prove the following lemma.

Lemma 2.5.3. *The function $\phi = \phi(t,z)$ defined in (2.5.32) satisfies the following behaviour for $z = -\infty$*

$$\lim_{z\to-\infty} \sup_{t\in\mathbb{R}} |1 - \phi(t,z)| = 0.$$

Proof. To prove the above lemma, let us show that

$$\lim_{z\to-\infty} u\left(t, z + \int_0^t c_{\lambda,a}(s)\mathrm{d}s\right) = 1 \text{ uniformly for } t \in \mathbb{R}.$$

Recalling the definition of $\underline{\phi}$ in (2.5.22), there exists z_0 large enough such that

$$\inf_{t\in\mathbb{R}} \underline{\phi}(t,z_0) > 0.$$

Hence due to (2.5.33) and since ϕ is nonincreasing with respect to $z \in \mathbb{R}$, it follows that

$$\Theta := \lim_{z\to-\infty} \inf_{t\in\mathbb{R}} \phi(t,z) > 0.$$

Due to (2.5.32) this also rewrites as

$$\Theta = \lim_{z\to-\infty} \inf_{t\in\mathbb{R}} u\left(t, z + \int_0^t c_{\lambda,a}(s)\mathrm{d}s\right).$$

Now, since $u \leq 1$, to prove the lemma, it is sufficient to check that $\Theta = 1$. To do so, let us consider two sequences $\{t_n\} \subset \mathbb{R}$ and $\{z_n\} \subset \mathbb{R}$ such that $z_n \to -\infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} u\left(t_n, z_n + \int_0^{t_n} c_{\lambda,a}(s)\mathrm{d}s\right) = \Theta.$$

Consider now the sequence of functions $\{u_n = u_n(t,z)\}$ given for $n \geq 1$ by

$$u_n(t,z) = u(t + t_n, z + z_n + c_n) \text{ with } c_n = \int_0^{t_n} c_{\lambda,a}(s)\mathrm{d}s.$$

Note that the sequence $\{u_n\}$ is uniformly bounded in the Lipschitz norm on $\mathbb{R}^2$ so that one may assume, possibly along a subsequence, that $u_n(t,z) \to u_\infty(t,z)$ locally uniformly for $(t,z) \in \mathbb{R}^2$. Moreover the limit function satisfies $u_\infty(0,0) = \Theta$. We now claim that

$$u_\infty(t,z) \geq \Theta, \ \forall (t,z) \in \mathbb{R}^2. \tag{2.5.34}$$

To see this, note that

$$u_n(t,z) = u\left(t + t_n, z + z_n - \int_0^t c_{\lambda,a}(t_n + s)\mathrm{d}s + \int_0^{t+t_n} c_{\lambda,a}(s)\mathrm{d}s\right).$$

Now, since one has locally uniformly for $(t, z) \in \mathbb{R}^2$

$$z + z_n - \int_0^t c_{\lambda,a}(t_n + s)\mathrm{d}s \to -\infty,$$

it follows from the definition of Θ that (2.5.34) holds true.

We now derive an equation satisfied by u_∞. To that aim, observe that since the function u satisfies the following equation for all $(t, z) \in \mathbb{R}^2$,

$$\partial_t u(t, z) = \int_{\mathbb{R}} K(t, y) \left[u(t, z - y) - u(t, z)\right] \mathrm{d}y + u(t, z) f(t, u(t, z)).$$

We obtain that for any $n \geq 1$ the function u_n satisfies the shifted equation

$$\partial_t u_n(t, z) = \int_{\mathbb{R}} K(t + t_n, y) \left[u_n(t, z - y) - u_n(t, z)\right] \mathrm{d}y + u_n(t, z) f(t + t_n, u_n(t, z)).$$

In order to pass to the limit $n \to \infty$ and obtain a suitable equation for u_∞, we first investigate the shifted kernel $(t, y) \mapsto K(t + t_n, y)$. For that purpose recall that $y \mapsto K(\cdot, y) \in L^1(\mathbb{R}; L^\infty(\mathbb{R}))$ so that Dunford-Pettis theorem applies and ensures that the sequence $\{(t, y) \mapsto K(t + t_n, y)\}$ is relatively weakly compact in $L^1((-T, T) \times \mathbb{R})$ for any $T > 0$. Hence, there exists a subsequence, still denoted with the same notation, and $\overline{K} = \overline{K}(t, y) \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ with

$$0 \leq \overline{K}(t, y) \leq \|K(\cdot, y)\|_{L^\infty(\mathbb{R})} \; a.e.(t, y) \in \mathbb{R}^2,$$

and such that for all $T > 0$ and any $\varphi \in L^\infty((-T, T) \times \mathbb{R})$ the following convergence holds

$$\lim_{n \to \infty} \int_{-T}^{T} \int_{\mathbb{R}} K(t + t_n, y) \varphi(t, y) \mathrm{d}t\mathrm{d}y = \int_{-T}^{T} \int_{\mathbb{R}} \overline{K}(t, y) \varphi(t, y) \mathrm{d}t\mathrm{d}y.$$

As a special case, taking $\varphi(t, y) \equiv 1$ yields

$$\int_{\mathbb{R}} K(t + t_n, y)\mathrm{d}y \to \int_{\mathbb{R}} \overline{K}(t, y)\mathrm{d}y \text{ weakly in } L^1_{\mathrm{loc}}(\mathbb{R}),$$

so that

$$u_n(t, x) \int_{\mathbb{R}} K(t + t_n, y)\mathrm{d}y \to u_\infty(t, x) \int_{\mathbb{R}} \overline{K}(t, y)\mathrm{d}y,$$

weakly in $L^1_{\mathrm{loc}}(\mathbb{R})$ with respect to t and locally uniformly with respect to $x \in \mathbb{R}$.

Using the above convergence for the sequence of the shifted kernels , we now claim that

Claim 2.5.4. *The following holds*

$$\lim_{n \to \infty} \int_{\mathbb{R}} K(t + t_n, y) u_n(t, x - y)\mathrm{d}y = \int_{\mathbb{R}} \overline{K}(t, y) u_\infty(t, x - y)\mathrm{d}y,$$

weakly $L^1_{\mathrm{loc}}(\mathbb{R})$ with respect to t and locally uniformly with respect to $x \in \mathbb{R}$. In other words, for any $T > 0$ and any $\psi \in L^\infty(-T, T)$ one has

$$\lim_{n \to \infty} \int_{-T}^{T} \int_{\mathbb{R}} \psi(t) K(t + t_n, y) u_n(t, x - y)\mathrm{d}t\mathrm{d}y = \int_{-T}^{T} \int_{\mathbb{R}} \psi(t) \overline{K}(t, y) u_\infty(t, x - y)\mathrm{d}t\mathrm{d}y,$$

locally uniformly with respect to $x \in \mathbb{R}$.

Proof. Let us first observe that

$$\int_{\mathbb{R}} K(t+t_n,y)u_\infty(t,x-y)\mathrm{d}y \to \int_{\mathbb{R}} \overline{K}(t,y)u_\infty(t,x-y)\mathrm{d}y \text{ as } n\to\infty, \tag{2.5.35}$$

locally uniformly for $x\in\mathbb{R}$ and weakly in $L^1_{\mathrm{loc}}(\mathbb{R})$ with respect to the $t-$variable.

Next note that for any n one has

$$\begin{aligned}\int_{\mathbb{R}} K(t+t_n,y)u_n(t,x-y)\mathrm{d}y &- \int_{\mathbb{R}} \overline{K}(t,y)u_\infty(t,x-y)\mathrm{d}y\\ &= \int_{\mathbb{R}} K(t+t_n,y)\left[u_n(t,x-y)-u_\infty(t,x-y)\right]\mathrm{d}y\\ &+ \int_{\mathbb{R}} \left[K(t+t_n,y)-\overline{K}(t,y)\right]u_\infty(t,x-y)\mathrm{d}y.\end{aligned}$$

Due to (2.5.35) to prove Claim 2.5.4 it is sufficient to check that

$$\int_{\mathbb{R}} K(t+t_n,y)\left[u_n(t,x-y)-u_\infty(t,x-y)\right]\mathrm{d}y \to 0 \text{ as } n\to\infty, \tag{2.5.36}$$

locally uniformly for $(t,x)\in\mathbb{R}^2$. To do so, note that for any $B>0$ one has

$$\begin{aligned}&\left|\int_{\mathbb{R}} K(t+t_n,y)\left[u_n(t,x-y)-u_\infty(t,x-y)\right]\mathrm{d}y\right|\\ &\quad\le \int_{|y|\le B} K(t+t_n,y)\left|u_n(t,x-y)-u_\infty(t,x-y)\right|\mathrm{d}y\\ &\quad+ \int_{|y|\ge B} K(t+t_n,y)\left|u_n(t,x-y)-u_\infty(t,x-y)\right|\mathrm{d}y.\end{aligned}$$

Since $0\le u_n\le 1$ the above inequality implies that for all $B>0$, any n and any $(t,x)\in\mathbb{R}^2$ one has

$$\begin{aligned}&\left|\int_{\mathbb{R}} K(t+t_n,y)\left[u_n(t,x-y)-u_\infty(t,x-y)\right]\mathrm{d}y\right|\\ &\quad\le \int_{\mathbb{R}} \|K(\cdot,y)\|_{L^\infty(\mathbb{R})}\mathrm{d}y \sup_{|y|\le B}\left|u_n(t,x-y)-u_\infty(t,x-y)\right|\\ &\quad+2\int_{|y|\ge B} \|K(\cdot,y)\|_{L^\infty(\mathbb{R})}\mathrm{d}y.\end{aligned}$$

Next since $u_n(t,z)\to u_\infty(t,z)$ locally uniformly for $(t,z)\in\mathbb{R}^2$, one obtains for each $A>0$ and any $B>0$

$$\begin{aligned}\limsup_{n\to\infty}\sup_{(t,x)\in[-A,A]^2}&\left|\int_{\mathbb{R}} K(t+t_n,y)\left[u_n(t,x-y)-u_\infty(t,x-y)\right]\mathrm{d}y\right|\\ &\le 2\int_{|y|\ge B} \|K(\cdot,y)\|_{L^\infty(\mathbb{R})}\mathrm{d}y.\end{aligned}$$

Finally since $y\mapsto \|K(\cdot,y)\|_{L^\infty(\mathbb{R})}\in L^1(\mathbb{R})$, letting $B\to\infty$ ensures that (2.5.36) holds and this completes the proof of Claim 2.5.4. □

Now consider the sequence of function $g_n(t,z)=f(t+t_n,u_n(t,z))$. It is a bounded sequence in $L^\infty(\mathbb{R}^2)$ so that up to a subsequence, one may assume that it converges for the weak-$\star$ topology of $L^\infty(\mathbb{R}^2)$ to some function $g_\infty = g_\infty(t,z) \in L^\infty(\mathbb{R}^2)$. Note that due to Assumption $(f1)$ and $(f2)$ the function g_∞ satisfies

$$h(u_\infty(t,z)) \le g_\infty(t,z) \le K(1-u_\infty(t,z)),\ \forall (t,z) \in \mathbb{R}^2, \tag{2.5.37}$$

where $K>0$ denotes the Lipschitz constant of f with respect to $u \in [0,1]$.

As a consequence the Lipschitz continuous function u_∞ satisfies the equation for *a.e.* $(t,z) \in \mathbb{R}^2$

$$\partial_t u_\infty(t,z) = \int_{\mathbb{R}} \overline{K}(t,y)\,[u_\infty(t,z-y) - u_\infty(t,z)]\,\mathrm{d}y + u_\infty(t,z) g_\infty(t,z),$$

together with $0 < \Theta \le u_\infty(t,z) \le 1$ for all $(t,z) \in \mathbb{R}^2$ and $u_\infty(0,0) = \Theta$.

Now let us complete the proof of the lemma by showing that $\Theta = 1$. To do so let us consider the function $U = U(t)$ defined for $t \ge 0$ by

$$U'(t) = h\left(U(t)\right) U(t),\ \forall t \ge 0 \text{ and } \ U(0) = \Theta.$$

Then due to (2.5.37), the comparison principle applies and ensures that

$$U(t) \le u_\infty(s+t,z) \le 1,\ \forall t \ge 0,\ \forall s \in \mathbb{R},\ \forall z \in \mathbb{R}.$$

As a consequence, we obtain that

$$U(t) \le u_\infty(0,0) = \Theta \le 1,\ \forall t \ge 0.$$

Since $\Theta > 0$, $U(t) \to 1$ as $t \to \infty$. This implies that $\Theta = 1$ and this completes the proof of the lemma.

□

2.6 Lower speed estimates

In this section we derive some lower speed estimates for generalized travelling wave solutions of (2.1.1), proving as a by-product some non-existence result for such solutions. Throughout this section we assume that Assumption 2.2.2 and 2.2.3 are satisfied.

Consider a generalized travelling wave $u = u(t,x)$ of (2.1.1) with speed function $c = c(t) \in \mathscr{C}$, according to Definition 2.1.2. We also denote by $\phi = \phi(t,z)$ its profile. As mentioned above, in this section we focus on deriving of a lower estimate for the speed function c, proving Theorem 2.2.10. Our analysis is based on the construction of a suitable sub-solution on some large bounded interval coupled with a comparison argument on a moving spatial domain, inspired by [169].

We split this section into two subsections. We first construct a suitable sub-solution on some large interval for the problem with a compactly supported convolution kernel. Then we make use of such a sub-solution to complete the proof of Theorem 2.2.10 and those of its corollary as well.

2.6.1 A sub-solution

This subsection is devoted to the construction on a suitable sub-solution on some large interval for the integro-differential operator

$$\partial_t - v(t)\partial_x - K(t,\cdot) * \cdot + (\overline{K}(t) - 1 + \theta),$$

wherein $*$ devoted the convolution product in $\mathbb{R}$, $\overline{K}(t) := \int_{\mathbb{R}} K(t,y)\mathrm{d}y$ while $v = v(t)$ is a suitable speed function and $\theta > 0$ is some given parameter. The construction presented in this section extends some preliminary ideas used in [1, 4, 48, 114] for nonlocal diffusion.

To that aim we define, for $B > 0$, $R > 0$ and $\gamma \in \mathbb{R}$, the following function $t \mapsto c_{R,B}(\gamma)(t) \in L^\infty(\mathbb{R})$ given by

$$c_{R,B}(\gamma)(t) := \frac{2R}{\pi} \int_{-B}^{B} K(t,z) e^{\gamma z} \sin\left(\frac{\pi z}{2R}\right) \mathrm{d}z. \tag{2.6.38}$$

Using the above notation our lemma reads as follows.

Lemma 2.6.1. *Let $\gamma \in \Lambda$ be given. Then there exists $B_0 > 0$ large enough and $\theta_0 > 0$ such that for all $B > B_0$ there exists $R_0 = R_0(B) > 0$ large enough enjoying the following properties:*
for all $B > B_0$ and $R > \max(R_0(B), B)$, there exists some function $a \in W^{1,\infty}(\mathbb{R})$ such that the function

$$u_{R,B}(t,x) = \begin{cases} e^{a(t)} e^{-\gamma x} \cos(\frac{\pi x}{2R}) \text{ if } t \in \mathbb{R} \text{ and } x \in [-R,R], \\ 0 \text{ else,} \end{cases}$$

satisfies, for all $\theta \le \theta_0$, for all $x \in [-R,R]$ and for any $t \in \mathbb{R}$,

$$\left(\partial_t - c_{R,B}(\gamma)(t)\partial_x\right) u(t,x) \le \int_{\mathbb{R}} K(t,x-y) u(t,y) \mathrm{d}y + \left(1 - \theta - \overline{K}(t)\right) u(t,x).$$

Herein $c_{R,B}(\gamma)(t)$ is defined above in (2.6.38).

To prove this result we set, for all $B > 0$, the compactly supported kernel $K_B = K_B(t,y)$ given by

$$K_B(t,y) = \begin{cases} K(t,y) \text{ for } (t,y) \in \mathbb{R} \times (-B,B), \\ 0 \text{ else.} \end{cases}$$

Proof. Let $\gamma \in \Lambda$ be given. Recall that from Proposition 2.2.8 one has

$$\left\lfloor -\frac{\mathrm{d}c(\gamma)}{\mathrm{d}\lambda} \right\rfloor > 0.$$

This rewrites as

$$\theta_\infty := \left\lfloor \int_{-\infty}^{\infty} K(t,z) e^{\gamma z} (1 - \gamma z) \, \mathrm{d}z + 1 - \overline{K}(t) \right\rfloor > 0,$$

Define $\theta_0 > 0$ by

$$\theta_0 := \frac{\theta_\infty}{8}.$$

Next observe that since $\gamma < \sigma(K)$, one has

$$\int_{-\infty}^{\infty} K(\cdot,z) e^{\gamma z} (1 - \gamma z) \, \mathrm{d}z = \lim_{B \to \infty} \int_{-B}^{B} K(\cdot,z) e^{\gamma z} (1 - \gamma z) \, \mathrm{d}z \text{ in } L^\infty(\mathbb{R}).$$

so that there exists $B_0 = B_0(\gamma) > 0$ large enough such that for all $B > B_0$ one has

$$\left\lfloor \int_{-\infty}^{\infty} K_B(t,z) e^{\gamma z} (1 - \gamma z) \, \mathrm{d}z + 1 - \overline{K}(t) \right\rfloor > \frac{\theta_\infty}{2}.$$

Now for any given $B > B_0$ let us observe that, due to Lebesgue convergence theorem, it holds

$$\lim_{R\to\infty}\int_{-\infty}^{\infty} K_B(t,z)e^{\gamma z}\left(\cos(\frac{\pi z}{2R}) - \gamma\frac{2R}{\pi}\sin(\frac{\pi z}{2R})\right)\mathrm{d}z = \int_{-\infty}^{\infty} K_B(t,z)e^{\gamma z}\,(1-\gamma z)\,\mathrm{d}z,$$

uniformly for $t \in \mathbb{R}$.

Hence for any $B > B_0$ there exists $R_0 = R_0(B)$ large enough such that for all $R > R_0(B)$ one has

$$\lfloor\int_{-\infty}^{\infty} K_B(t,z)e^{\gamma z}\left(\cos(\frac{\pi z}{2R}) - \gamma\frac{2R}{\pi}\sin(\frac{\pi z}{2R})\right)\mathrm{d}z + 1 - \overline{K}(t)\rfloor > 2\theta_0 > 0.$$

In the rest of this proof we fix $B > B_0$ and $R > \max(R_0(B), B)$. Then using the formulation of the least mean value recalled in (2.2.5), there exists some function $a = a(t) \in W^{1,\infty}(\mathbb{R})$ (depending upon B and R) such that for *a.e.* $t \in \mathbb{R}$

$$a'(t) + \int_{-\infty}^{\infty} K_B(t,z)e^{\gamma z}\left(\gamma\frac{2R}{\pi}\sin(\frac{\pi z}{2R}) - \cos(\frac{\pi z}{2R})\right)\mathrm{d}z - 1 + \overline{K}(t) \le -\theta_0. \tag{2.6.39}$$

Next with such a function $a \in W^{1,\infty}(\mathbb{R})$. Fix $\theta \in (0,\theta_0)$ and let us compute

$$\left[\partial_t - c_{R,B}(\gamma)(t)\partial_x - K(t,\cdot) * \cdot + (\overline{K}(t) - 1 + \theta)\right] u_{R,B}(t,x),$$

for $t \in \mathbb{R}$ and $x \in [-R,R]$. For notational simplicity we let L be the above integro-differential operator, namely

$$L = \partial_t - c_{R,B}(\gamma)(t)\partial_x - K(t,\cdot) * \cdot + (\overline{K}(t) - 1 + \theta).$$

Then setting $\gamma_R = -\gamma + \mathrm{i}\frac{\pi}{2R} \in \mathbb{C}$ and denoting by $\operatorname{Re} z$ the real part of a complex number $z \in \mathbb{C}$, observe that for all $t \in \mathbb{R}$ and $x \in [-R,R]$ we have

$$\begin{aligned} e^{-a(t)}Lu_{R,B}(t,x) =& \operatorname{Re}\left\{e^{\gamma_R x}\left[a'(t) - c_{R,B}(\gamma)(t)\gamma_R - (1-\theta) + \overline{K}(t)\right]\right\} \\ &- \int_{-R}^{R} K(t,x-y)e^{-\gamma y}\cos(\frac{\pi y}{2R})\mathrm{d}y. \end{aligned}$$

Now to estimate the last term of the above expression, we make use of some arguments developed by Diekmann in [48]. First let us observe that for all $t \in \mathbb{R}$ and any $|x| \le R$ one has

$$\begin{aligned} \int_{-R}^{R} K(t,x-y)e^{-\gamma y}\cos(\frac{\pi y}{2R})\mathrm{d}y &\ge \int_{-R}^{R} K_B(t,x-y)e^{-\gamma y}\cos(\frac{\pi y}{2R})\mathrm{d}y \\ &\ge \int_{-\infty}^{\infty} K_B(t,x-y)e^{-\gamma y}\cos(\frac{\pi y}{2R})\mathrm{d}y. \end{aligned} \tag{2.6.40}$$

Indeed to see this, first note that $K \ge K_B$. Next observe that since $\cos(\frac{\pi y}{2R}) \le 0$ for $R \le |y| \le 2R$, one already obtains

$$\int_{-R}^{R} K_B(t,x-y)e^{-\gamma y}\cos(\frac{\pi y}{2R})\mathrm{d}y \ge \int_{-2R}^{2R} K_B(t,x-y)e^{-\gamma y}\cos(\frac{\pi y}{2R})\mathrm{d}y,$$

and when $x \in [-R,R]$ and $|y| \ge 2R$ then $|x-y| \ge R \ge B$ and $K_B(t,x-y) = 0$ that completes the above estimate.

Next note that for all $t \in \mathbb{R}$ and $x \in [-R, R]$ one has

$$\int_{-R}^{R} K_B(t, x-y)e^{-\gamma y}\cos(\frac{\pi y}{2R})\mathrm{d}y \geq \int_{-\infty}^{\infty} K_B(t, x-y)e^{-\gamma y}\cos(\frac{\pi y}{2R})\mathrm{d}y$$
$$= \int_{-\infty}^{\infty} K_B(t,z)e^{-\gamma(x-z)}\cos(\frac{\pi(x-z)}{2R})\mathrm{d}z$$
$$= \int_{-\infty}^{\infty} K_B(t,z)e^{-\gamma(x-z)}\cos(\frac{\pi x}{2R})\cos(\frac{\pi z}{2R})\mathrm{d}z + \int_{-\infty}^{\infty} K_B(t,z)e^{-\gamma(x-z)}\sin(\frac{\pi x}{2R})\sin(\frac{\pi z}{2R})\mathrm{d}z.$$

We thus infer from the above estimate that for all $t \in \mathbb{R}$ and $x \in [-R, R]$

$$\begin{aligned} e^{-a(t)}Lu_{R,B}(t,x) \leq &\mathrm{Re}\left\{e^{\gamma_R x}\left(a'(t) - c_{R,B}(\gamma)(t)\gamma_R - 1 + \theta + \overline{K}(t)\right)\right\} \\ &- \int_{-\infty}^{\infty} K_B(t,z)e^{-\gamma(x-z)}\cos(\frac{\pi x}{2R})\cos(\frac{\pi z}{2R})\mathrm{d}z \\ &- \int_{-\infty}^{\infty} K_B(t,z)e^{-\gamma(x-z)}\sin(\frac{\pi x}{2R})\sin(\frac{\pi z}{2R})\mathrm{d}z. \end{aligned}$$

Hence this yields for all $(t, x) \in \mathbb{R} \times [-R, R]$

$$\begin{aligned} e^{-a(t)}Lu_{R,B}(t,x) \leq e^{-\gamma x}\cos(\frac{\pi x}{2R})&\Big[a'(t) + \gamma c_{R,B}(\gamma)(t) - (1-\theta) + \overline{K}(t) \\ &- \int_{-\infty}^{\infty} K_B(t,z)e^{\gamma z}\cos(\frac{\pi z}{2R})\mathrm{d}z\Big] \\ - e^{-\gamma x}\sin(\frac{\pi x}{2R})&\left[\int_{-\infty}^{\infty} K_B(t,z)e^{\gamma z}\sin(\frac{\pi z}{2R})\mathrm{d}z - \frac{\pi}{2R}c_{R,B}(\gamma)(t)\right]. \end{aligned}$$

Note that due to the choice of the speed (see (2.6.38)), the last term vanishes and we end-up with

$$\begin{aligned} e^{-a(t)}Lu_{R,B}(t,x) \leq e^{-\gamma x}\cos(\frac{\pi x}{2R})&\Big[a'(t) + \gamma c_{R,B}(\gamma)(t) - (1-\theta) + \overline{K}(t) \\ &- \int_{-\infty}^{\infty} K_B(t,z)e^{\gamma z}\cos(\frac{\pi z}{2R})\mathrm{d}z\Big], \end{aligned}$$

for all $(t, x) \in \mathbb{R} \times [-R, R]$. Finally coupling the above computation with (2.6.39) yields for any $(t, x) \in \mathbb{R} \times [-R, R]$

$$e^{-a(t)}Lu_{R,B}(t,x) \leq e^{-\gamma x}\cos(\frac{\pi x}{2R})\left(\theta - \theta_0\right) \leq 0.$$

This completes the proof of the lemma.

□

2.6.2 Proof of Theorem 2.2.10

As introduced at the beginning of this section, recall that $u = u(t, x)$ denotes a generalized travelling wave of (2.1.1) with speed function $c = c(t) \in \mathscr{C}$ while $\phi = \phi(t, z)$ denotes its profile. We focus in this section on the proof of Theorem 2.2.10 and its corollary, namely Corollary 2.2.11. Our lower estimate analysis for the speed function is related to the following lemma.

Lemma 2.6.2. *Let $v = v(t) \in L^\infty(\mathbb{R})$ be a function such that*

$$\limsup_{t\to\infty} \inf_{\tau\in\mathbb{R}} \phi\left(t-\tau, \int_0^t [v(l-\tau) - c(l-\tau)]\,\mathrm{d}l\right) > 0, \tag{2.6.41}$$

then one has

$$\lceil v(\cdot) - c(\cdot) \rceil \leq 0.$$

Proof. Define $\alpha > 0$ by

$$\limsup_{t\to\infty} \inf_{\tau\in\mathbb{R}} \phi\left(t-\tau, \int_0^t [v(l-\tau) - c(l-\tau)]\,\mathrm{d}l\right) = \alpha.$$

Letting $\tau = t - s$ the above limit rewrites as

$$\limsup_{t\to\infty} \inf_{s\in\mathbb{R}} \phi\left(s, \int_0^t [v(l-t+s) - c(l-t+s)]\,\mathrm{d}l\right) = \alpha.$$

Next let us argue by contradiction by assuming that

$$\lceil v(\cdot) - c(\cdot) \rceil > 0. \tag{2.6.42}$$

Now set $\Gamma(t, s)$ the function given by

$$\Gamma(t,s) = \frac{1}{t}\int_0^t [v(l-t+s) - c(l-t+s)]\mathrm{d}l.$$

Consider a sequence $\{t_n\}$ with $t_n \to \infty$ such that

$$\inf_{s\in\mathbb{R}} \phi\left(s, \int_0^{t_n} [v(l-t_n+s) - c(l-t_n+s)]\,\mathrm{d}l\right) \to \alpha. \tag{2.6.43}$$

Next due to (2.6.42), there exists a sequence $\{s_n\}$ such that one has

$$\liminf_{n\to\infty} \Gamma(t_n, s_n) > 0,$$

so that $t_n\Gamma(t_n, s_n) \to \infty$ as $n \to \infty$. On the other hand, note that

$$t_n\Gamma(t_n,s_n) = \int_0^{t_n} [v(l-t_n+s_n) - c(l-t_n+s_n)]\mathrm{d}l \to \infty.$$

As a consequence, since $\phi(t, z) \to 0$ as $z \to \infty$, uniformly with respect to $t \in \mathbb{R}$, one obtains

$$\phi\left(s_n, \int_0^{t_n} [v(l-t_n+s_n) - c(l-t_n+s_n)]\mathrm{d}l\right) \to 0 \text{ as } n \to \infty,$$

that contradicts (2.6.43) and completes the proof of the lemma. □

Using the above lemma we first complete the proof of Theorem 2.2.10.

Proof of Theorem 2.2.10. To prove this result let $\gamma \in \Lambda$ be given and fixed. Let B_0 and θ_0 be the constants provided by Lemma 2.6.1. Recalling the definition of the function $t \mapsto c_{R,B}(\gamma)(t) \in L^\infty(\mathbb{R})$ in (2.6.38), let us first show that for all $B > B_0$ and any $R > \max(B, R_0(B))$ (where $R_0(B)$ is defined in Lemma 2.6.1 as well) one has

$$\lceil c_{R,B}(\gamma)(\cdot) - c(\cdot) \rceil \leq 0. \tag{2.6.44}$$

To do so, fix $B > B_0$ and $R > \max(B, R_0(B))$. Now consider for $t \in \mathbb{R}$, $x \in \mathbb{R}$ and $\tau \in \mathbb{R}$ the function u given by

$$u(t,x;\tau) = \phi\left(t-\tau, x - \int_0^t c(l-\tau)\mathrm{d}l\right).$$

It satisfies the equation

$$\partial_t u(t,x;\tau) = \int_{\mathbb{R}} K(t-\tau,y)[u(t,x-y;\tau) - u(t,x;\tau)]\mathrm{d}y + F(t-\tau,u).$$

Consider also the function $\underline{u} = \underline{u}(t,x;\tau)$ given by

$$\underline{u}(t,x;\tau) = \eta u_{R,B}\left(t-\tau, x - \int_0^t c_{R,B}(\gamma)(l-\tau)\mathrm{d}l\right), \ \forall (t,\tau,x) \in \mathbb{R}^3,$$

wherein the function $u_{R,B}$ is defined in Lemma 2.6.1, using some function $a = a(t) \in W^{1,\infty}(\mathbb{R})$.

Next define the constant $M_R > 0$ by

$$M_R := e^{\|a\|_\infty} \max_{z\in[-R,R]} e^{\gamma z}.$$

Now since $\phi(\tau, z) \to 1$ as $z \to -\infty$ uniformly for $\tau \in \mathbb{R}$, there exists $z_0 \in \mathbb{R}$ such that

$$\inf_{\tau\in\mathbb{R}, z\leq z_0} \phi(\tau,z) > 0.$$

Up to work with a shift in z of ϕ, we assume that

$$\inf_{\tau\in\mathbb{R}, z\leq R} \phi(\tau,z) > 0.$$

Now choose $\eta_0 > 0$ such that one has

$$\phi(\tau,x) \geq \eta_0 M_R \chi_{[-R,R]}(x), \ \forall \tau \in \mathbb{R}, \ x \in \mathbb{R},$$

wherein $\chi_{[-R,R]}(x)$ denotes the characteristic function of the interval $[-R,R]$.

Next note (see (2.2.4)) that for each $\eta \in (0,\eta_0)$ one has

$$F(t, \eta u_{R,B}(t,x)) \geq \eta u_{R,B}(t,x)\left[1 - C\eta u_{R,B}\right] \geq \eta u_{R,B}(t,x)\left[1 - C\eta M_R\right].$$

Now choose $\eta < \min(\eta_0, \frac{\theta_0}{CM_R})$ so that the function

$$\underline{u}(t,x;\tau) = \eta u_{R,B}\left(t-\tau, x - \int_0^t c_{R,B}(\gamma)(l-\tau)\mathrm{d}l\right)$$

satisfies, for all $t \in \mathbb{R}$, $\tau \in \mathbb{R}$ and $x \in [-R + X(t;\tau), R + X(t;\tau)]$ with

$$X(t;\tau) = \int_0^t c_{R,B}(\gamma)(l-\tau)\mathrm{d}l,$$

the following integro-differential inequality

$$\begin{aligned}
&\partial_t \underline{u}(t,x;\tau) = \eta\partial_t u_{R,B} - \eta c_{R,B}(\gamma)(t-\tau)\partial_x u_{R,B} \\
&\leq \int_{\mathbb{R}} K(t-\tau,y)[\underline{u}(t,x-y;\tau) - \underline{u}(t,x;\tau)]\mathrm{d}y + (1 - C\eta M_R)\underline{u}(t,x;\tau) \\
&\leq \int_{\mathbb{R}} K(t-\tau,y)[\underline{u}(t,x-y;\tau) - \underline{u}(t,x;\tau)]\mathrm{d}y + F(t-\tau, \underline{u}(t,x;\tau)).
\end{aligned}$$

Now let us prove that

$$\underline{u}(t,x;\tau) \le u(t,x;\tau),\ \forall t \ge 0, \tau \in \mathbb{R},\ \forall x \in \mathbb{R}, \tag{2.6.45}$$

that rewrites as

$$\eta u_{R,B}(t-\tau,0) \le \phi\left(t-\tau, X(t;\tau) - \int_0^t c(l-\tau)\mathrm{d}l\right),\ \forall t \ge 0, \tau \in \mathbb{R}.$$

Note that in view of Lemma 2.6.2, the above estimate ensures that (2.6.44) holds true.

To complete the proof of (2.6.44), it remains to prove (2.6.45). To prove this inequality, fix $\tau \in \mathbb{R}$ and consider the open set Ω given by

$$\Omega = \left\{(t,x) \in \mathbb{R}^2 : 0 < t \text{ and } x \in [-R + X(t;\tau), R + X(t;\tau)]\right\},$$

as well as the function $v = v(t,x)$ given by

$$v(t,x) = u(t,x;\tau) - \underline{u}(t,x;\tau).$$

Note that $v(0,x) > 0$ on $[-R + X(0;\tau), R + X(0;\tau)] = [-R, R]$ and that one has

$$v(t,x) \ge 0,\ \forall t \ge 0,\ \forall x \notin [-R + X(t;\tau), R + X(t;\tau)].$$

Furthermore, the function v is continuous on $[0,\infty) \times \mathbb{R}$, for all $x \in \mathbb{R}$ the map $t \mapsto v(t,x) \in W^{1,1}_{\text{loc}}([0,\infty))$ and for almost every $(t,x) \in \Omega$ one has

$$\partial_t v(t,x) \ge \int_{\mathbb{R}} K(t-\tau,y)[v(t,x-y) - v(t,x)]\mathrm{d}y + F(t-\tau, u(t,x;\tau)) - F(t-\tau, \underline{u}(t,x;\tau)).$$

The above differential inequality rewrites as

$$\partial_t v(t,x) \ge \int_{\mathbb{R}} K(t-\tau,y) v(t,x-y)\mathrm{d}y + g(t,x)v(t,x),\ a.e.\ (t,x) \in \Omega,$$

for some bounded function $g = g(t,x)$.

Choose $\delta > 0$ large enough so that $g(t,x) + \delta \ge 1$. Hence, the function $w(t,x) = e^{\delta t}v(t,x)$ satisfies

$$\partial_t w(t,x) \ge \int_{\mathbb{R}} K(t-\tau,y) w(t,x-y)\mathrm{d}y + [g(t,x) + \delta]\, w(t,x),\ a.e.\ (t,x) \in \Omega. \tag{2.6.46}$$

Assume now by contradiction that there exists $(t,x) \in \Omega$ such that $v(t,x) < 0$. Consider the time $t_* > 0$ defined by

$$t_* = \sup\{t \ge 0 : \min_{x \in [-R+X(t;\tau), R+X(t;\tau)]} v(t,x) > 0\}.$$

Since $v(t, \pm R + X(t;\tau)) > 0$, there exists $x_* \in (-R + X(t;\tau), R + X(t;\tau))$ such that $v(t_*, x_*) = 0$. Moreover, since $X(\cdot;\tau)$ is continuous, there exists $\varepsilon > 0$ small enough such that

$$[t_* - \varepsilon, t_*] \times \{x_*\} \subset \Omega.$$

Hence integrating (2.6.46) with $x = x_*$ and from $t_* - \varepsilon$ and t_* yields

$$0 = w(t_*, x_*) \ge w(t_* - \varepsilon, x_*) + \int_{t_*-\varepsilon}^{t_*} \int_{\mathbb{R}} K(t-\tau,y) w(t, x_* - y)\mathrm{d}y\mathrm{d}t + \int_{t_*-\varepsilon}^{t_*} w(t,x_*)\mathrm{d}t > 0,$$

a contradiction, that completes the proof of (2.6.45) and thus the proof of (2.6.44).

Finally observe that due to Lebesgue convergence theorem, for any $\gamma \in \mathbb{R}$ and for all $B > 0$ one has

$$\lim_{R\to\infty} c_{R,B}(\gamma)(\cdot) = c_B(\gamma)(\cdot) := \int_{-B}^{B} zK(\cdot,z)e^{\gamma z}\mathrm{d}z \text{ in } L^\infty(\mathbb{R}).$$

As a consequence (2.6.44) yields, for all $\gamma \in \Lambda$ and all $B > B_0(\gamma)$:

$$\lceil c_B(\gamma)(\cdot) - c(\cdot)\rceil \leq 0.$$

Finally, observe that for each $\gamma \in (0, \sigma(K))$ one also has

$$\lim_{B\to\infty} c_B(\gamma)(\cdot) = \underline{c}(\gamma)(\cdot) := \int_{-\infty}^{\infty} zK(\cdot,z)e^{\gamma z}\mathrm{d}z \text{ in } L^\infty(\mathbb{R}),$$

and the above estimate ensures that

$$\lceil \underline{c}(\gamma)(\cdot) - c(\cdot)\rceil \leq 0, \ \forall \gamma \in \Lambda. \tag{2.6.47}$$

From the above estimate we also have for all $\gamma \in (0, \lambda^*)$,

$$\lfloor \underline{c}(\gamma)(\cdot)\rfloor \leq \lceil \underline{c}(\gamma)(\cdot) - c(\cdot)\rceil + \lfloor c(\cdot)\rfloor \leq \lfloor c(\cdot)\rfloor.$$

This completes the proof of Theorem 2.2.10. □

We now turn to the proof of Corollary 2.2.11.

Proof of Corollary 2.2.11. Note that the map $\gamma \mapsto \underline{c}(\gamma)$ is continuous from $(0, \sigma(K))$ into $L^\infty(\mathbb{R})$. Hence when $\lambda^* < \sigma(K)$, this map is in particular continuous at $\gamma = \lambda^*$. Hence letting $\gamma \to \lambda^*$ with $\gamma \in \Lambda$ into (2.6.47) yields for all $c \in \mathscr{C}$

$$\lceil \underline{c}(\lambda^*)(\cdot) - c(\cdot)\rceil \leq 0 \text{ and } \lfloor \underline{c}(\lambda^*)(\cdot)\rfloor \leq \inf\lfloor \mathscr{C}\rfloor.$$

Now due to the assumption

$$\lceil c(\lambda^*)(\cdot) - \underline{c}(\lambda^*)(\cdot)\rceil \leq 0,$$

note that one has

$$\lfloor c(\lambda^*)(\cdot)\rfloor \leq \lceil c(\lambda^*)(\cdot) - \underline{c}(\lambda^*)(\cdot)\rceil + \lfloor \underline{c}(\lambda^*)(\cdot)\rfloor .$$

Hence this implies that

$$\lfloor c(\lambda^*)(\cdot)\rfloor \leq \lfloor \underline{c}(\lambda^*)(\cdot)\rfloor ,$$

so that

$$\lfloor c(\lambda^*)(\cdot)\rfloor \leq \inf \lfloor \mathscr{C}\rfloor .$$

The upper estimate follows from (2.2.10) and this completes the proof of the corollary. □

Chapter 3

Spreading properties for nonautonomous Fisher-KPP equations with nonlocal diffusion

This is a joint work with Arnaud Ducrot, submitted [59].

Abstract

We investigate spreading properties of solutions to a non-autonomous Fisher-KPP equation with nonlocal diffusion, driven by a thin-tailed kernel. In this paper, we are concerned with both compactly supported and exponentially decaying initial data. For general time heterogeneity, we provide lower and upper estimates for the location of the propagating front, which is expressed in term of the least mean of the time varying coefficients of the problem. Under some stronger time averaging assumptions for these coefficients, we prove that these solutions propagate with some determined speed. In this analysis, an important difficulty comes from the lake of regularization for the solutions arising with nonlocal diffusion. Through delicate analysis we derive some regularity estimates (of uniform continuity type for the large time) for some solutions of the logistic equation equipped with suitable initial data. Such regularity estimates are coupled with the construction of appropriated propagating paths to derive spreading speed estimates. These results are then used to handle more general KPP nonlinearities.

3.1 Introduction and main results

In this paper we study the spreading speeds for the solutions of the following non-autonomous and nonlocal one-dimensional equation

$$\partial_t u(t,x) = \int_{\mathbb{R}} K(y)\left[u(t,x-y) - u(t,x)\right]\mathrm{d}y + F\left(t, u(t,x)\right), \tag{3.1.1}$$

posed for time $t \geq 0$ and $x \in \mathbb{R}$. This evolution problem is supplemented with an appropriated initial data, that will be discussed below. Here $K = K(y)$ is a nonnegative dispersal kernel with thin-tailed (see Assumption 3.1.3 below), while $F = F(t,u)$ stands for the nonlinear term, which depends on time t and that will be assumed in this note to be Fisher-KPP type (see Assumption 3.1.5). The above problem typically describes the spatial invasion of a population (see for instance [17, 114] and the references therein) with the following features:
1) the individuals exhibit long distance dispersal according to the kernel K, in other words the quantity $K(x-y)$ corresponds to the probability to jump from y to x;
2) time varying birth and death processes modeled by the nonlinear Fisher-KPP type function $F(t,u)$. The time variations may stand for seasonality and/or external events (see [93]).

The similar equation with local diffusion operator and posed in a time homogeneous medium reads as

$$\partial_t u(t,x) = \partial_{xx} u(t,x) + F(u(t,x)). \tag{3.1.2}$$

As mentioned above, this problem arises as a basic model in many different fields, especially in biology and ecology. It can be used for instance to describe the spatio-temporal evolution of an invading species into an empty environment.
The above equation (3.1.2) was first introduced separately by Fisher [70] and Kolmogorov, Petrovsky and Piskunov [97], when the nonlinear function F satisfies the Fisher-KPP conditions. Recall that a typical example of such Fisher-KPP nonlinearity is given by the logistic function $F(u) = u(1-u)$.

There is a large amount of literature related to this equation (3.1.2) and to generalizations. To study propagation phenomena generated by reaction diffusion equations in quantitatively, in addition to the existence of travelling wave solution, the asymptotic speed of spread (or spreading speed) was introduced and studied by Aronson and Weinberger in [8]. Roughly speaking if u_0 is a nontrival and nonnegative initial data with compact support, then the solution of (3.1.2) associated to this initial data u_0 spreads with the speed $c^* > 0$ (the minimal wave speed of the traveling waves) in the sense that

$$\lim_{t\to\infty} \sup_{|x|\leq ct} |u(t,x) - 1| = 0, \forall\, 0 \leq c < c^* \text{ and } \lim_{t\to\infty} \sup_{|x|\geq ct} u(t,x) = 0,\ \forall\, c > c^*.$$

This concept of spreading speed has been further developed by several researchers in the last decades from different view points including PDE's argument, dynamical systems theory, probability theory and mathematical biology etc. Spreading speeds of KPP-type reaction diffusion equations with homogeneous and periodic media have been extensively studied (see [22, 66, 103, 104, 162, 163] and the references therein). There are also some results about spreading phenomena for reaction diffusion systems (see [4, 55, 77] and the references therein).

Recently the spreading speeds for KPP-type reaction diffusion equations in more complicated structures of media obtained more and more attention, see [21, 23, 140] and the references cited therein. Particularly, Nadin and Rossi [124] studied lower and upper spreading speeds of KPP equation with general time heterogeneity. Furthermore, they showed that if the coefficients are uniquely ergodic, then these two speeds equal.

The spreading properties of nonlocal diffusion equation as (3.1.1) has attracted a lot of interest in the last decades. Since the semi-flow generated by nonlocal diffusion equations are non-compact and the solution without priori estimates, these bring more difficulties. Fisher-KPP equation or monostable problem in homogeneous environment has been studied from various point of views: wave front propagation (see [43, 135] and the reference cited therein), hair trigger effect and spreading speed (see [2, 29, 48, 68, 114, 167] and the reference cited therein). For the thin-tailed kernel, we refer for instance to [114] and the recent work [167] where a new sub-solution has been constructed to provide a lower bound of the spreading speed. Note also that the aforementioned work deals with possibly non-symmetric kernel so that the propagation speed on the left and the right hand side of the domain can be different. For the fat-tailed dispersion kernels the propagation behavior of the solution can be very different from the one observed with thin-tailed kernel. Acceleration may occur. We refer to [69, 73] and to [29] for fractional Laplace type dispersal.
Recently, wave propagation and spreading speeds for nonlocal diffusion problem incorporated time and/or space heterogeneity have been considered, the existence and nonexistence of travelling wave solutions see [58, 94, 109, 143] and the references cited therein. For the spreading speeds results, we refer to [93, 94, 107, 146] and the references cited therein. For the analysis of the spreading speed for systems with nonlocal diffusion, we refer the reader to [12, 166, 169] and references cited therein.

In this work, we extend some of these spreading properties for (3.1.1) with both fast and slow decaying initial data by considering general time heterogeneity for the nonlinear term. For the general time heterogeneity, we provide a new approach, based on what we call a persistence lemma (see Lemma 3.2.6 below) for uniformly continuous solutions, to obtain lower estimate of the propagation speed. It is different from the well developed monotone semi-flow method for which we refer the reader to [162, 104, 93, 94]. Moreover, we expect our key persistence lemma may also be applied to study the acceleration phenomena for fat-tailed dispersal kernel. However the uniform continuity property for the solutions remains complicated to check. Here we are able to prove such a property for some specific initial data and logistic type nonlinearities. Note that in [101] the authors consider this regularity problem. They show that when the nonlinear term satisfies $F_u(u) < \overline{K}$ for any $u \geq 0$, where $\overline{K} = \int_{\mathbb{R}} K(y)\mathrm{d}y$, then solutions of the homogeneous problem inherit the Lipschitz continuity property from those of their initial data. In this note, we prove the uniform continuity of some solutions when the above condition fails (see Assumption 3.1.5 $(f4)$). This point is studied in Section 3.3.1, where we provide a class of initial data for which the solutions (of the nonlocal logistic equation) are uniformly continuous on $[0, \infty) \times \mathbb{R}$.

Now to state our results, we first introduce some notations and present our main assumptions. Now, we define the important notion of the least mean value for a bounded function.

Definition 3.1.1. *Along this work, for any given function $h \in L^\infty(0, \infty; \mathbb{R})$, we define*

$$\lfloor h \rfloor := \lim_{T \to +\infty} \inf_{s>0} \frac{1}{T} \int_0^T h(t+s)\mathrm{d}t. \tag{3.1.3}$$

In that case the quantity $\lfloor h \rfloor$ is called the least mean of the function h (over $(0, \infty)$).

If h admits a *mean value* $\langle h \rangle$, that is, there exists

$$\langle h \rangle := \lim_{T \to +\infty} \frac{1}{T} \int_0^T h(t+s)\mathrm{d}t, \text{ uniformly with respect to } s \geq 0. \tag{3.1.4}$$

Then $\lfloor h \rfloor = \langle h \rangle$.

An equivalent and useful characterization for the least mean of the function, as above, is given in the next lemma.

Lemma 3.1.2. *[124, 125] Let $h \in L^\infty(0,\infty;\mathbb{R})$ be given. Then one has*

$$\lfloor h \rfloor = \sup_{a \in W^{1,\infty}(0,\infty)} \inf_{t>0} \left(a' + h\right)(t).$$

We are now able to present the main assumptions we shall need in this note. First we assume that the kernel $K = K(y)$ enjoys the following set of properties:

Assumption 3.1.3 (Kernel $K = K(y)$)**.** *We assume that the kernel $K : \mathbb{R} \to [0,\infty)$ satisfies the following set of assumptions:*

(i) The function $y \mapsto K(y)$ is non-negative, continuous and integrable;

(ii) There exists $\alpha > 0$ such that

$$\int_{\mathbb{R}} K(y) e^{\alpha y} dy < \infty.$$

(iii) We also assume that $K(0) > 0$.

Remark 3.1.4. *Note that here we do not impose that the kernel function is symmetric. There exist $\delta > 0$ and $k : \mathbb{R} \to [0,\infty)$, continuous, even and compactly supported such that*

$$\begin{aligned} &\text{supp } k = [-\delta, \delta],\ k(y) > 0,\ \forall y \in (-\delta,\delta), \\ &k(y) \le K(y) \text{ and } k(y) = k(-y),\ \forall y \in \mathbb{R}. \end{aligned} \tag{3.1.5}$$

This is due to $K(y)$ is continuous and $K(0) > 0$.

Now we discuss our Fisher-KPP assumptions for the nonlinear term $F = F(t,u)$.

Assumption 3.1.5 (KPP nonlinearity)**.** *We assume that the function F from $[0,\infty) \times [0,1]$ to $\mathbb{R}$ takes the form $F(t,u) = uf(t,u)$ where the function $f : [0,\infty) \times [0,1] \to \mathbb{R}$ satisfies the following set of hypotheses:*

(f1) $f(\cdot,u) \in L^\infty(0,\infty;\mathbb{R})$, for all $u \in [0,1]$, and f is Lipschitz continuous with respect to $u \in [0,1]$, uniformly with respect to $t \ge 0$;

(f2) We assume that $f(t,1) = 0$ for a.e. $t \ge 0$. Setting $\mu(t) := f(t,0)$ we assume that $\mu(\cdot)$ is bounded and uniformly continuous. Also, we assume that

$$h(u) := \inf_{t \ge 0} f(t,u) > 0 \text{ for all } u \in [0,1);$$

(f3) For almost every $t \ge 0$, the function $u \mapsto f(t,u)$ is nonincreasing on $[0,1]$;

(f4) Set $\overline{K} := \int_{\mathbb{R}} K(y)\mathrm{d}y$. Assume that the least mean of the function μ satisfies

$$\lfloor \mu \rfloor > \overline{K}.$$

Remark 3.1.6. *From the above assumption, one can note that*

$$\inf_{t\geq 0}\mu(t)=h(0)>0.$$

Next this assumption also implies that there exists some constant $C>0$ such that for all $u\in[0,1]$ and $t\geq 0$ one has

$$\mu(t)\geq f(t,u)\geq \mu(t)-Cu\geq \mu(t)(1-Hu), \tag{3.1.6}$$

where we have set $H:=\sup_{t\geq 0}\frac{C}{\mu(t)}=\frac{C}{h(0)}$.

Assumption $(f4)$ is imposed for some technical reasons. It will be used when we prove the hair trigger effect property in (3.1.1).

Let us now define some notations related to the speed function that will be used in the following. We define $\sigma(K)$, the abscissa of convergence of K, by

$$\sigma(K):=\sup\left\{\gamma>0:\int_{\mathbb{R}}K(y)e^{\gamma y}dy<\infty\right\}.$$

Assumption 3.1.3 (ii) yields that $\sigma(K)\in(0,\infty]$. We set

$$L(\lambda):=\int_{\mathbb{R}}K(y)[e^{\lambda y}-1]\mathrm{d}y,\ \lambda\in[0,\sigma(K)), \tag{3.1.7}$$

as well as for $\lambda\in(0,\sigma(K))$ and $t\geq 0$,

$$c(\lambda)(t):=\lambda^{-1}L(\lambda)+\lambda^{-1}\mu(t). \tag{3.1.8}$$

For a given function $a\in W^{1,\infty}(0,\infty)$, denote $c_{\lambda,a}$ that the function given by

$$c_{\lambda,a}(t):=c(\lambda)(t)+a'(t),\ \lambda\in(0,\sigma(K)),\ t\geq 0. \tag{3.1.9}$$

Obviously, it follows from Definition 3.1.1 that $\lfloor c_{\lambda,a}(\cdot)\rfloor=\lfloor c(\lambda)(\cdot)\rfloor$ for each $\lambda\in(0,\sigma(K))$. Next note that

$$\lfloor c(\lambda)(\cdot)\rfloor=\lambda^{-1}L(\lambda)+\lambda^{-1}\lfloor\mu\rfloor.$$

Now we state some properties of $\lfloor c(\lambda)(\cdot)\rfloor$ in the following proposition.

Proposition 3.1.7. *Let Assumption 3.1.3 and 3.1.5 be satisfied. Then the following properties hold:*

(i) The map $\lambda\mapsto\lfloor c(\lambda)(\cdot)\rfloor$ from $(0,\sigma(K))$ to $\mathbb{R}$ is of class C^1 from $(0,\sigma(K))$ into $\mathbb{R}$.

(ii) Set $c_r^*:=\inf_{\lambda\in(0,\sigma(K))}\lfloor c(\lambda)(\cdot)\rfloor$. There exists $\lambda_r^*\in(0,\sigma(K)]$ such that

$$\lim_{\lambda\to(\lambda_r^*)^-}\lfloor c(\lambda)(\cdot)\rfloor=c_r^*.$$

Moreover, one has $c_r^*>0$ and the map $\lambda\mapsto\lfloor c(\lambda)(\cdot)\rfloor$ is decreasing on $(0,\lambda_r^*)$.

(iii) Assume that $\lambda_r^*<\sigma(K)$. One has

$$c_r^*=\int_{\mathbb{R}}K(y)e^{\lambda_r^* y}y\mathrm{d}y. \tag{3.1.10}$$

The above Proposition 3.1.7 has been mostly proved in [58] (see Proposition 2.8 in [58]) with a more general kernel which depends on t.

Here we only explain that $c_r^* > 0$. To see this, note that for $\lambda \in (0, \sigma(K))$ one has

$$\lambda c(\lambda)(t) = \int_{\mathbb{R}} K(y) e^{\lambda y} \mathrm{d}y + \mu(t) - \overline{K}, \ \forall t \geq 0.$$

Next due to Assumption 3.1.5 ($f4$) and Lemma 3.1.2, there exists some function $a \in W^{1,\infty}(0,\infty)$ such that $\mu(t) - \overline{K} + a'(t) \geq 0$ for all $t \geq 0$. This yields for all $\lambda \in (0, \sigma(K))$ and $t \geq 0$

$$\lambda c(\lambda)(t) + a'(t) = \int_{\mathbb{R}} K(y) e^{\lambda y} \mathrm{d}y + \mu(t) - \overline{K} + a'(t) \geq \int_{\mathbb{R}} K(y) e^{\lambda y} \mathrm{d}y > 0,$$

that rewrites $c_r^* > 0$ since $\lfloor a' \rfloor = 0$. The result follows.

Remark 3.1.8. *Let us point out that the assumption $\lambda_r^* < \sigma(K)$ needed for (iii) to hold is satisfied for instance if we have*

$$\limsup_{\lambda \to \sigma(K)^-} \frac{L(\lambda)}{\lambda} = +\infty. \tag{3.1.11}$$

Indeed, one can observe that

$$\lfloor c(\lambda)(\cdot) \rfloor \sim \frac{\lfloor \mu \rfloor}{\lambda} \to \infty \ \text{as } \lambda \to 0^+.$$

In addition, if (3.1.11) holds then the decreasing property of the map $\lambda \mapsto \lfloor c(\lambda)(\cdot) \rfloor$ on $(0, \lambda_r^*)$ as stated in Proposition 3.1.7 (ii) ensures that $\lambda_r^* < \sigma(K)$.

To state our spreading result, we impose in the following that the condition discussed in the previous remark is satisfied, that means λ_r^* is different from the convergence abscissa.

Assumption 3.1.9. *In addition to Assumption 3.1.3, we assume that $\lambda_r^* < \sigma(K)$.*

Using the above properties for the speed function $c(\lambda)(\cdot)$ and its least mean value, we are now able to state the spreading properties.

Theorem 3.1.10 (Upper bounds)**.** *Let Assumption 3.1.3, 3.1.5 and 3.1.9 be satisfied. Let $u = u(t,x)$ denote the solution of (3.1.1) equipped with a continuous initial data u_0, with $0 \leq u_0(\cdot) \leq 1$ and $u_0(\cdot) \not\equiv 0$.*
Then the following upper estimate for the propagation set holds: if $u_0(x) = O(e^{-\lambda x})$ as $x \to \infty$ for some $\lambda > 0$, then one has

$$\lim_{t \to \infty} \sup_{x \geq \int_0^t c^+(\lambda)(s)\mathrm{d}s + \eta t} u(t,x) = 0, \ \forall \eta > 0,$$

where the function $c^+(\lambda)(\cdot)$ is defined by

$$c^+(\lambda)(\cdot) := \begin{cases} c(\lambda_r^*)(\cdot) & \text{if } \lambda \geq \lambda_r^*, \\ c(\lambda)(\cdot) & \text{if } \lambda \in (0, \lambda_r^*). \end{cases}$$

For our lower estimate of the propagation set, we first state our result for a specific function $f = f(t,u)$ of the form $f(t,u) = \mu(t)(1-u)$. In other words, we are considering the following non-autonomous logistic equation

$$\partial_t u(t,x) = \int_{\mathbb{R}} K(y) \left[u(t,x-y) - u(t,x) \right] \mathrm{d}y + \mu(t) u(t,x) \left(1 - u(t,x) \right). \tag{3.1.12}$$

To enter the framework of Assumption 3.1.5, we assume that the function μ satisfies following conditions:

$$\begin{aligned} & t \mapsto \mu(t) \text{ is uniformly continuous and bounded with } \inf_{t\geq 0}\mu(t) > 0, \\ & \text{and the least mean of } \mu(\cdot) \text{ satisfies } \lfloor \mu \rfloor > \overline{K}. \end{aligned} \tag{3.1.13}$$

For this problem, our propagation result reads as follows.

Theorem 3.1.11 (Lower bounds). *Let Assumption 3.1.3, 3.1.9 be satisfied and assume furthermore that μ satisfies (3.1.13). Let $u = u(t,x)$ denote the solution of (3.1.12) equipped with a continuous initial data u_0, with $0 \leq u_0(\cdot) \leq 1$ and $u_0(\cdot) \not\equiv 0$. Then the following propagation occurs:*

(i) **(Fast exponential decay case)** *If $u_0(x) = O(e^{-\lambda x})$ as $x \to \infty$ for some $\lambda \geq \lambda_r^*$ then one has*

$$\lim_{t\to\infty} \sup_{x\in[0,ct]} |1 - u(t,x)| = 0, \ \forall c \in (0, c_r^*);$$

(ii) **(Slow exponential decay case)** *If $\liminf_{x\to\infty} e^{\lambda x}u_0(x) > 0$ for some $\lambda \in (0, \lambda_r^*)$ then it holds that*

$$\lim_{t\to\infty} \sup_{x\in[0,ct]} |1 - u(t,x)| = 0, \ \forall c \in (0, \lfloor c(\lambda) \rfloor).$$

Next as a consequence of the comparison principle, one obtains the following lower estimate of the propagation set to the right for more general nonlinearity satisfying Assumption 3.1.5.

Corollary 3.1.12 (Inner propagation). *Let Assumption 3.1.3, 3.1.5 and 3.1.9 be satisfied. Let $u = u(t,x)$ denote the solution of (3.1.1) supplemented with a continuous initial data u_0, with $0 \leq u_0(\cdot) \leq 1$ and $u_0(\cdot) \not\equiv 0$. Then the following propagation result holds true:*

(i) **(Fast exponential decay case)** *If $u_0(x) = O(e^{-\lambda x})$ as $x \to \infty$ for some $\lambda \geq \lambda_r^*$ then one has*

$$\lim_{t\to\infty} \inf_{x\in[0,ct]} u(t,x) > 0, \ \forall c \in (0, c_r^*);$$

(ii) **(Slow exponential decay case)** *If $\liminf_{x\to\infty} e^{\lambda x}u_0(x) > 0$ for some $\lambda \in (0, \lambda_r^*)$ then one has*

$$\lim_{t\to\infty} \inf_{x\in[0,ct]} u(t,x) > 0, \ \forall c \in (0, \lfloor c(\lambda) \rfloor).$$

Remark 3.1.13. *If $\lim_{t\to\infty} \frac{1}{t}\int_0^t c^+(\lambda)(s)\mathrm{d}s = \lfloor c^+(\lambda) \rfloor$, then Theorem 3.1.10 and Corollary 3.1.12 provide the exact spreading speed $\lfloor c^+(\lambda) \rfloor$. This condition holds for instance if $\mu(\cdot)$ has a mean value.*
If one has $\lfloor c^+(\lambda) \rfloor < \liminf_{t\to\infty} \frac{1}{t}\int_0^t c^+(\lambda)(s)\mathrm{d}s$, then behavior of $u(t,\beta t)$ for $t \gg 1$ with

$$\lfloor c^+(\lambda) \rfloor < \beta < \liminf_{t\to\infty} \frac{1}{t}\int_0^t c^+(\lambda)(s)\mathrm{d}s,$$

is unknown. This open problem is similar to the Fisher-KPP equation with local diffusion [124].

In the above result we only consider the propagation to the right hand-side of the real line and obtain a propagation result on some interval of the form $[0, ct]$ for suitable speed c and for $t \gg 1$. To study the propagation of the left hand side, it is sufficient to change x to $-x$ and impose K is thin-tailed in the left-hand side. Note also that the kernel is not assumed to be even, so that the minimal spreading speeds on the right and on the left can be different.

The results stated in this section and more precisely the lower bounds for the propagation follows from the derivation of suitable regularity estimate for the solution. Here we show that the solutions of (3.1.12) with suitable initial data are uniformly continuous. Next Theorem 3.1.11 follows from the application of a general persistence lemma (see Lemma 3.2.6) for uniformly continuous solutions. This key lemma roughly ensures that if there a uniformly continuous solution $u = u(t, x)$ admits a propagating path $t \mapsto X(t)$, then $[0, kX(t)]$ with any $k \in (0, 1)$ is a propagating interval, that is u stays uniformly far from 0 on this interval, in the large time.

This paper is organized as follows. In Section 2, we recall comparison principles and derive our general key persistence Lemma. Section 3 is devoted to the derivation of some regularity estimates for the solutions of (3.1.12) with suitable initial data. With all these materials, we conclude the proofs of theorems and the corollary.

3.2 Preliminary and Key Lemma

This section is devoted to the statement of the comparison principle and a key lemma that will be used to prove the inner propagation theorem, namely Theorem 3.1.11.

3.2.1 Comparison principle and strong maximum principle

We start this section by recalling the following more general comparison principle.

Proposition 3.2.1. *(See [58, Proposition 3.1])[Comparison principle] Let $t_0 \in \mathbb{R}$ and $T > 0$ be given. Let $K : \mathbb{R} \to [0, \infty)$ be an integrable kernel and let $F = F(t, u)$ be a function defined in $[t_0, t_0+T] \times [0, 1]$ which is Lipschitz continuous with respect to $u \in [0, 1]$, uniformly with respect to t. Let $\underline{u}$ and $\overline{u}$ be two uniformly continuous functions defined from $[t_0, t_0 + T] \times \mathbb{R}$ into the interval $[0, 1]$ such that for each $x \in \mathbb{R}$, the maps $\underline{u}(\cdot, x)$ and $\overline{u}(\cdot, x)$ both belong to $W^{1,1}(t_0, t_0 + T)$, satisfying $\underline{u}(t_0, \cdot) \leq \overline{u}(t_0, \cdot)$, and for all $x \in \mathbb{R}$ and for almost every $t \in (t_0, t_0 + T)$,*

$$\partial_t \overline{u}(t, x) \geq \int_{\mathbb{R}} K(y) \left[\overline{u}(t, x - y) - \overline{u}(t, x)\right] \mathrm{d}y + F(t, \overline{u}(t, x)),$$
$$\partial_t \underline{u}(t, x) \leq \int_{\mathbb{R}} K(y) \left[\underline{u}(t, x - y) - \underline{u}(t, x)\right] \mathrm{d}y + F(t, \underline{u}(t, x)).$$

Then $\underline{u} \leq \overline{u}$ on $[t_0, t_0 + T] \times \mathbb{R}$.

We also need some comparison principle on moving domain as follows (this can be proved similarly as Lemma 5.4 in [1] and Lemma 4.7 in [169]).

Proposition 3.2.2. *Assume that $K : R \to [0, \infty)$ is integrable. Let $t_0 > 0$ and $T > 0$ be given, let $b(t, x)$ be a uniformly bounded function from $[t_0, t_0 + T] \times \mathbb{R} \to \mathbb{R}$. Assume that $u(t, x)$ is uniformly continuous defined from $[t_0, t_0 + T] \times \mathbb{R}$ into the interval $[0, 1]$ such that for each $x \in \mathbb{R}$, $u(\cdot, x) \in W^{1,1}(t_0, t_0+T)$. Assume that X and Y are continuous* *functions on* $[t_0, t_0 + T]$ *with* $X < Y$. *If* u *satisfies*

$$\begin{cases} \partial_t u \geq \int_{\mathbb{R}} K(y)\,[u(t,x-y) - u(t,x)]\,\mathrm{d}y + b(t,x)u, & \forall t \in [t_0, t_0+T], x \in (X(t), Y(t)), \\ u(t,x) \geq 0, & \forall t \in (t_0, t_0+T], x \in \mathbb{R} \setminus (X(t), Y(t)), \\ u(t_0, x) \geq 0, & \forall x \in (X(t_0), Y(t_0)). \end{cases}$$

Then

$$u(t,x) \geq 0 \text{ for all } t \in [t_0, t_0+T], x \in [X(t), Y(t)].$$

We continue this section by the following strong maximum principle. We refer the reader to [95] for the proof of following proposition.

Proposition 3.2.3 (Strong maximum principle). *Let Assumption 3.1.3, 3.1.5 be satisfied. Let* $u = u(t,x)$ *be the solution of (3.1.1) supplemented with some continuous initial data* u_0, *such that* $0 \leq u_0 \leq 1$ *and* $u_0 \not\equiv 0$. *Then* $u(t,x) > 0$ *for all* $t > 0, x \in \mathbb{R}$.

3.2.2 Key lemma

In this section, we derive an important lemma that will be used in the next section to prove our main inner propagation result, namely Theorem 3.1.11. In this section we only let Assumption 3.1.3 (i), (iii) and Assumption 3.1.5 be satisfied.

Definition 3.2.4 (Limit orbits set). *Let* $u = u(t,x)$ *be a uniformly continuous function on* $[0,\infty) \times \mathbb{R}$ *into* $[0,1]$, *solution of (3.1.1). We define* $\omega(u)$, **the set of the limit orbits**, *as the set of bounded and uniformly continuous functions* $\tilde{u} : \mathbb{R}^2 \to \mathbb{R}$ *where exist sequences* $(x_n) \subset \mathbb{R}$ *and* (t_n) *such that* $t_n \to \infty$ *as* $n \to \infty$ *and*

$$\tilde{u}(t,x) = \lim_{n\to\infty} u(t + t_n, x + x_n),$$

uniformly for (t,x) *in bounded sets of* $\mathbb{R}^2$.

Let us observe that since u is assumed to be bounded and uniformly continuous on $[0,\infty) \times \mathbb{R}$, Arzelà-Ascoli theorem ensures that $\omega(u)$ is not empty. Indeed, for each sequence (t_n) with $t_n \to \infty$ and $(x_n) \subset \mathbb{R}$ the sequence of functions $(t,x) \mapsto u(t+t_n, x+x_n)$ is equi-continuous and thus has a converging subsequence with respect to the local uniform topology. In addition, it is a compact set with respect to the compact open topology, that is with respect to the local uniform topology.

Before going to our key lemma, we claim that the set $\omega(u)$ enjoys the following property:

Claim 3.2.5. *Let* $\tilde{u} \in \omega(u)$ *be given. Then one has:*

$$\text{Either } \tilde{u}(t,x) > 0 \text{ for all } (t,x) \in \mathbb{R}^2 \text{ or } \tilde{u}(t,x) \equiv 0 \text{ on } \mathbb{R}^2.$$

Proof. Let $u = u(t,x)$ be a uniformly continuous solution of (3.1.1). Note that due to Assumption 3.1.5 (see Remark 3.1.6), the function u satisfies the following differential inequality for all $t \geq 0$ and $x \in \mathbb{R}$

$$\partial_t u(t,x) \geq K * u(t,\cdot)(x) - \overline{K} u(t,x) + u(t,x)(\mu(t) - C u(t,x)).$$

Since the function $\mu(\cdot)$ is bounded, for each $\tilde{u} \in \omega(u)$, there exists $\tilde{\mu} = \tilde{\mu}(t) \in L^\infty(\mathbb{R})$, a weak star limit of some shifted function $\mu(t_n + \cdot)$, for some suitable time sequence (t_n), such that $\tilde{u}$ satisfies

$$\begin{aligned} \partial_t \tilde{u}(t,x) &\geq K * \tilde{u}(t,\cdot)(x) - \overline{K}\tilde{u}(t,x) + \tilde{u}(t,x)(\tilde{\mu}(t) - C\tilde{u}(t,x)) \\ &\geq K * \tilde{u}(t,\cdot)(x) + \left(-\overline{K} + \inf_{t\in\mathbb{R}} \tilde{\mu}(t) - C\right)\tilde{u}(t,x), \ \forall (t,x) \in \mathbb{R}^2. \end{aligned}$$

Herein $\partial_t \tilde{u}$ is a weak star limit of $\partial_t u(\cdot + t_n, \cdot + x_n)$ for some suitable sub-sequence of $(x_n)_n$ and $(t_n)_n$. This is due to $\partial_t u \in L^\infty([0,\infty) \times \mathbb{R})$.

Next the claim follows from the same arguments as for the proof of the strong maximum principle, see [95].

□

Using the above definition and its properties we are now able to state and prove the following key lemma.

Lemma 3.2.6. *Let $u = u(t,x) : [0,\infty) \times \mathbb{R} \to [0,1]$ be a uniformly continuous solution of (3.1.1). Let $t \mapsto X(t)$ from $[0,\infty)$ to $[0,\infty)$ be a given continuous function. Let the following set of hypothesis be satisfied:*

(H1) Assume that $\liminf_{t\to\infty} u(t,0) > 0$;

(H2) There exists some constant $\tilde{\varepsilon}_0 > 0$ such that for all $\tilde{u} \in \omega(u) \setminus \{0\}$, one has

$$\liminf_{t\to\infty} \tilde{u}(t,0) > \tilde{\varepsilon}_0;$$

(H3) The map $t \mapsto X(t)$ is a propagating path for u, in the sense that

$$\liminf_{t\to\infty} u(t, X(t)) > 0.$$

Then for any $k \in (0,1)$, one has

$$\liminf_{t\to\infty} \inf_{0 \le x \le kX(t)} u(t,x) > 0.$$

Remark 3.2.7. *The above result holds without assuming that the convolution kernel is exponential bounded. We expect this key lemma may also be useful to study the spatial propagation for Fisher-KPP equation with fat-tailed dispersion kernel, which may occur acceleration, see [29, 69, 73].*

To prove the above lemma, we make use of ideas coming from uniform persistence theory, somehow close to those developed in [53, 55].

Proof. To prove the lemma we argue by contradiction by assuming that there exists $k \in (0,1)$, a sequence (t_n) with $t_n \to \infty$ and a sequence (k_n) with $0 \le k_n \le k$ such that

$$u(t_n, k_n X(t_n)) \to 0 \ \text{ as } \ n \to \infty. \tag{3.2.14}$$

First we claim that one has

$$\lim_{n\to\infty} k_n X(t_n) = \infty. \tag{3.2.15}$$

To prove this claim we argue by contradiction by assuming that $\{k_n X(t_n)\}$ has a bounded subsequence. Hence there exists $x_\infty \in \mathbb{R}$ such that possibly along a subsequence still denoted with the index n such that $k_n X(t_n) \to x_\infty$ as $n \to \infty$.

Now let us consider the sequence of functions $u_n(t,x) := u(t+t_n, x)$. Since $u = u(t,x)$ is uniformly continuous, possibly up to a sub-sequence still denoted with the same index n, there exists $u_\infty \in \omega(u)$ such that

$$u_n(t,x) \to u_\infty(t,x) \text{ locally uniformly for } (t,x) \in \mathbb{R}^2.$$

Next since $k_nX(t_n) \to x_\infty$, (3.2.14) ensures that

$$u_\infty(0, x_\infty) = \lim_{n\to\infty} u(t_n, k_nX(t_n)) = 0.$$

Since $u_\infty \in \omega(u)$, Claim 3.2.5 ensures that $u_\infty(t, x) \equiv 0$. On the other hand, $(H1)$ ensures that for all $t \in \mathbb{R}$, one has

$$u_\infty(t, 0) \geq \liminf_{t\to\infty} u(t, 0) > 0,$$

a contradiction, so that (3.2.15) holds.

Now due to (3.2.15), there exists N such that

$$X(0) < k_nX(t_n),\ \forall n \geq N.$$

Hence due to $k_n < 1$ we have

$$X(0) < k_nX(t_n) < X(t_n),\ \forall n \geq N.$$

And since $t \mapsto X(t)$ is continuous, then for each $n \geq N$ there exists $t'_n \in (0, t_n)$ such that $t'_n \to \infty$ and

$$X(t'_n) = k_nX(t_n),\ \forall n \geq N.$$

From the above definition of t'_n, one has

$$u(t'_n, k_nX(t_n)) = u(t'_n, X(t'_n)),\ \forall n \geq N.$$

So that $(H3)$ ensures that for all n large enough, there exists $\varepsilon_3 > 0$ such that

$$u(t'_n, k_nX(t_n)) = u(t'_n, X(t'_n)) \geq \varepsilon_3.$$

Recall that Assumption $(H2)$. Now for all n large enough, we define

$$t''_n := \inf\left\{t \leq t_n;\ \forall s \in (t, t_n),\ u(s, k_nX(t_n)) \leq \frac{\min\{\tilde{\varepsilon}_0, \varepsilon_3\}}{2}\right\} \subset (t'_n, t_n).$$

Since $u(t_n, k_nX(t_n)) \to 0$ as $n \to \infty$, then one may assume that, for all n large enough one has

$$\begin{cases} u(t''_n, k_nX(t_n)) = \frac{\min\{\tilde{\varepsilon}_0, \varepsilon_3\}}{2}, \\ u(t, k_nX(t_n)) \leq \frac{\min\{\tilde{\varepsilon}_0, \varepsilon_3\}}{2},\ \forall t \in [t''_n, t_n], \\ u(t_n, k_nX(t_n)) \leq \frac{1}{n}. \end{cases}$$

Next we claim that $t_n - t''_n \to \infty$ as $n \to \infty$. Indeed, if (a subsequence of) $t_n - t''_n$ converge to $\sigma \in \mathbb{R}$, define the sequence of functions $\tilde{u}_n(t, x) := u(t + t''_n, x + k_nX(t_n))$, that converge, possibly along a subsequence, locally uniformly to some function $\tilde{u}_\infty = \tilde{u}_\infty(t, x) \in \omega(u)$ that satisfies

$$\tilde{u}_\infty(0, 0) = \frac{\min\{\tilde{\varepsilon}_0, \varepsilon_3\}}{2} > 0,$$

and

$$\tilde{u}_\infty(\sigma, 0) = \lim_{n\to\infty} \tilde{u}_n(t_n - t''_n, 0) = \lim_{n\to\infty} u(t_n, k_nX(t_n)) = 0.$$

Hence since $\tilde{u}_\infty \in \omega(u)$ the two above values of $\tilde{u}_\infty$ contradict the dichotomy stated in Claim 3.2.5 and this proves that $t_n - t''_n \to \infty$ as $n \to \infty$.

As a consequence one obtains that the function $\tilde{u}_\infty \in \omega(u)$ satisfies

$$\tilde{u}_\infty(0, 0) = \frac{\min\{\tilde{\varepsilon}_0, \varepsilon_3\}}{2} > 0,$$

together with

$$\tilde{u}_\infty(t, 0) \leq \frac{\min\{\tilde{\varepsilon}_0, \varepsilon_3\}}{2},\ \forall t \geq 0. \tag{3.2.16}$$

Due to Claim 3.2.5, the above equality yields $\tilde{u}_\infty \in \omega(u) \setminus \{0\}$ and (3.2.16) contradicts $(H2)$. The proof is completed. □

3.3 Proof of spreading properties

In this section, we shall make use of the key lemma (see Lemma 3.2.6) to prove Theorem 3.1.11. To do this, we first derive some important regularity properties of the solutions of the Logistic equation (3.1.12) associated with suitable initial data. Next we prove Theorem 3.1.10 by constructing suitable exponentially decaying super-solutions for (3.1.1). Finally we turn to the proof of Theorem 3.1.11. As already mentioned we crucially make use of Lemma 3.2.6 and construct a suitable propagating path $t \mapsto X(t)$, that depends on the decay rate of the initial data $u_0 = u_0(x)$ for $x \gg 1$. As a corollary, we conclude the propagation results for (3.1.1).

3.3.1 Uniform continuity estimate

This subsection is devoted to give some regularity estimates for the solutions of the following Logistic equation (recalling (3.1.12)) when endowed with suitable initial data,

$$\partial_t u(t,x) = \int_{\mathbb{R}} K(y)u(t,x-y)\mathrm{d}y - \overline{K}u(t,x) + \mu(t)u(t,x)\left(1-u(t,x)\right).$$

Here we focus on two types of initial data, that will be used to prove Theorem 3.1.11: initial data with a compact support and initial data with support on a right semi-infinite interval and with some prescribed exponential decay on this right-hand side (that is for $x \gg 1$).

Our first lemma is concerned with the compactly supported case.

Lemma 3.3.1. *Let Assumption 3.1.3 and (3.1.13) be satisfied. Let $u = u(t,x)$ be the solution of (3.1.12) equipped with the initial data $v_0 = v_0(x)$, where v_0 is Lipschitz continuous in $\mathbb{R}$, and $0 < v_0(x) < 1$ for all $x \in (0,A)$, for some constant $A > 0$ while $v_0 = 0$ outside of $(0,A)$. Then, the function $(t,x) \mapsto u(t,x)$ is uniformly continuous on $[0,\infty) \times \mathbb{R}$.*

Proof. Firstly, since $0 \leq u \leq 1$, then one has

$$\|\partial_t u\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})} \leq M := 2\overline{K} + \|\mu\|_\infty. \tag{3.3.17}$$

As a consequence, the map $(t,x) \mapsto u(t,x)$ is Lipchitz continuous for the variable $t \in [0,\infty)$, uniformly with respect to $x \in \mathbb{R}$, that is

$$|u(t,x) - u(s,x)| \leq M|t-s|,\ \forall (t,s) \in [0,\infty)^2,\ \forall x \in \mathbb{R}. \tag{3.3.18}$$

Next we investigate the regularity with respect to the spatial variable $x \in \mathbb{R}$. To do so we claim that the following holds true:

Claim 3.3.2. *For all $h > 0$ sufficiently small, there exists $0 < \sigma(h) < 1$ such that $\sigma(h) \to 1$ as $h \to 0$ and*

$$u(\sqrt{h},x) \geq \sigma(h)v_0(x-h),\ \forall x \in \mathbb{R}.$$

Proof of Claim 3.3.2. Let us first observe that since $u(t,.) > 0$ for all $t > 0$, it is sufficient to look at $x - h \in [0,A]$, that is $h \leq x \leq A + h$.

Next to prove this claim, note that one has for all $h > 0$ and $x \in \mathbb{R}$:

$$\begin{aligned} u(\sqrt{h},x) &= v_0(x) + \int_0^{\sqrt{h}} \partial_t u(l,x)\mathrm{d}l \\ &= v_0(x) + \int_0^{\sqrt{h}} \left\{\int_{\mathbb{R}} K(y)\left[u(l,x-y) - u(l,x)\right]\mathrm{d}y + \mu(l)u(l,x)\left(1-u(l,x)\right)\right\}\mathrm{d}l \end{aligned}$$

Now coupling (3.3.18) and $0 \leq u \leq 1$, one gets, for all $h > 0$ small enough and uniformly for $x \in \mathbb{R}$

$$u(\sqrt{h}, x) \geq v_0(x) + \int_0^{\sqrt{h}} \left\{ \int_{\mathbb{R}} K(y) v_0(x-y) \mathrm{d}y - \overline{K} v_0(x) \right\} \mathrm{d}l + o(\sqrt{h}),$$

that is

$$u(\sqrt{h}, x) \geq v_0(x)\left(1 - \overline{K}\sqrt{h}\right) + \sqrt{h}\left(\int_{\mathbb{R}} K(y) v_0(x-y) \mathrm{d}y + o(1)\right).$$

Now observe that Assumption 3.1.3 (see (i) and (iii)) ensures that there exists $\varepsilon > 0$ such that

$$\min_{x \in [0,A]} \int_{\mathbb{R}} K(y) v_0(x-y) \mathrm{d}y \geq 2\varepsilon,$$

so that for $h > 0$ small enough one has

$$\min_{x \in [h,A+h]} \int_{\mathbb{R}} K(y) v_0(x-y) \mathrm{d}y \geq \varepsilon,$$

Now to prove the claim, it is sufficiently to reach, for all $h > 0$ small enough and $x \in [h, A+h]$,

$$v_0(x)\left(1 - \overline{K}\sqrt{h}\right) + \sqrt{h}\,(o(1) + \varepsilon) \geq \sigma(h) v_0(x-h). \tag{3.3.19}$$

Now set $\sigma(h) = 1 - 2\overline{K}\sqrt{h}$ and let us show that Claim 3.3.2 follows.

Since v_0 is Lipschitz continuous, then there exists some constant $L > 0$ such that

$$|v_0(x) - v_0(x-h)| \leq Lh, \ \forall x \in \mathbb{R}.$$

Hence to reach (3.3.19) it is sufficient to reach for all $x \in [h, A+h]$ and all $h > 0$ small enough

$$\overline{K}\sqrt{h} v_0(x-h) + \sqrt{h}\,(o(1) + \varepsilon) \geq Lh\left(1 - \overline{K}\sqrt{h}\right). \tag{3.3.20}$$

Dividing by $\sqrt{h}$ the above inequality holds whenever

$$\overline{K} v_0(x-h) + (o(1) + \varepsilon) \geq L\sqrt{h}\left(1 - \overline{K}\sqrt{h}\right), \tag{3.3.21}$$

which holds true for all $h > 0$ small enough. So the claim is proved.

□

Now we come back to the proof of Lemma 3.3.1. For each $h > 0$ small enough, let us introduce the following function

$$b_h(t) = b_h(0) \exp\left\{ \int_0^t \left[\mu(s + \sqrt{h}) - \mu(s) \right] \mathrm{d}s \right\}, \quad \text{for all } t \geq 0, \tag{3.3.22}$$

where $b_h(0)$ is some constant depending on h and that satisfies the following three conditions:

$$0 < b_h(0) \leq \sigma(h) < 1,$$

$b_h(0) \to 1$ as $h \to 0$ and for all $h > 0$ small enough

$$b_h(0) \leq \inf_{t \geq 0} \frac{\mu(t)}{\mu(t + \sqrt{h})} \exp\left\{ \int_0^t \left[\mu(s) - \mu(s + \sqrt{h}) \right] \mathrm{d}s \right\}.$$

For the later condition, one can observe that it is feasible since one has

$$\begin{aligned}\left|\int_0^t \left[\mu(s+\sqrt{h})-\mu(s)\right]\mathrm{d}s\right| &= \left|\int_{\sqrt{h}}^{t+\sqrt{h}} \mu(s)\mathrm{d}s - \int_0^t \mu(s)\mathrm{d}s\right| \\ &= \left|\int_{t}^{t+\sqrt{h}} \mu(s)\mathrm{d}s - \int_0^{\sqrt{h}} \mu(s)\mathrm{d}s\right| \\ &\le 2\|\mu\|_\infty\sqrt{h}.\end{aligned}$$

As a consequence, recalling (3.1.13), $\mu(\cdot)$ is uniformly continuous and we end-up with

$$\frac{\mu(t)}{\mu(t+\sqrt{h})}\exp\left\{\int_0^t \left[\mu(s)-\mu(s+\sqrt{h})\right]\mathrm{d}s\right\} \to 1, \text{ as } h\to 0, \text{ uniformly for } t\ge 0.$$

Hence $b_h(0)$ is well defined and $b_h(t)\to 1$ as $h\to 0$ uniformly for $t\ge 0$.

Now, setting $w_h = w_h(t,x)$ the function given by

$$w_h(t,x) := u(t+\sqrt{h},x) - b_h(t)u(t,x-h),$$

one obtains that it becomes a solution of the following equation

$$\begin{aligned}\partial_t w_h(t,x) &= K * w_h(t,x) - \overline{K}w_h(t,x) \\ &\quad + \mu(t+\sqrt{h})\left[w_h(t,x)+b_h(t)u(t,x-h)\right]\left[1-(w_h(t,x)+b_h(t)u(t,x-h))\right] \\ &\quad - \mu(t)b_h(t)u(t,x-h)\left[1-u(t,x-h)\right] - b_h'(t)u(t,x-h) \\ &= K * w_h(t,x) - \overline{K}w_h(t,x) \\ &\quad + \mu(t+\sqrt{h})w_h(t,x)\Big(1-w_h(t,x)-2b_h(t)u(t,x-h)\Big) \\ &\quad + b_h(t)u(t,x-h)\left(\mu(t+\sqrt{h})-\mu(t)-\frac{b_h'(t)}{b_h(t)}\right) \\ &\quad + b_h(t)u^2(t,x-h)\left(\mu(t)-b_h(t)\mu(t+\sqrt{h})\right).\end{aligned}$$

It follows from the definition of $b_h(t)$ (see (3.3.22) above) that $w_h(t,x)$ satisfies

$$\partial_t w_h(t,x) \ge K*w_h(t,x)-\overline{K}w_h(t,x)+w_h(t,x)\mu(t+\sqrt{h})\Big(1-w_h(t,x)-2b_h(t)u(t,x-h)\Big).$$

The Claim 3.3.2 together with $b_h(0)<\sigma(h)$ ensure that $w_h(0,\cdot)\ge 0$. Then the comparison principle applies and implies that $w_h(t,x)\ge 0$ for all $t\ge 0, x\in\mathbb{R}$, that rewrites as $u(t+\sqrt{h},x)\ge b_h(t)u(t,x-h)$ for all $t\ge 0, x\in\mathbb{R}$, for $h>0$ small enough. Recalling (3.3.18), for $h>0$ sufficiently small, one has for all $t\ge 0$ and $x\in\mathbb{R}$,

$$u(t,x-h)-u(t,x) \le \left(\frac{1}{b_h(t)}-1\right)u(t+\sqrt{h},x)+M\sqrt{h} \le \left(\frac{1}{b_h(t)}-1\right)+M\sqrt{h}. \quad (3.3.23)$$

Since for $h>0$ small enough one has

$$\min_{x\in[-h,A-h]}\int_{\mathbb{R}} K(y)v_0(x-y)\mathrm{d}y \ge \varepsilon,$$

then one can similarly prove that for sufficiently small $h>0$, there exists $\sigma(h)=1-2\overline{K}\sqrt{h}$ such that

$$u(\sqrt{h},x)\ge\sigma(h)v_0(x+h),\ \forall x\in\mathbb{R}.$$

This rewrites as

$$u(\sqrt{h}, x-h) \geq \sigma(h)v_0(x), \ \forall x \in \mathbb{R}.$$

Then as above one can choose a suitable function $b_h(t)$ and obtain that

$$u(t+\sqrt{h}, x-h) \geq b_h(t)u(t,x), \ \forall t \geq 0, x \in \mathbb{R}.$$

Recalling (3.3.18), for $h>0$ sufficiently small, one obtains for all $t \geq 0$ and $x \in \mathbb{R}$,

$$\begin{aligned} u(t,x) - u(t,x-h) &\leq \left(\frac{1}{b_h(t)} - 1\right) u(t+\sqrt{h}, x-h) + M\sqrt{h} \\ &\leq \left(\frac{1}{b_h(t)} - 1\right) + M\sqrt{h}. \end{aligned} \tag{3.3.24}$$

Since estimates (3.3.23) and (3.3.24) are uniform with respect to the spatial variable $x \in \mathbb{R}$, one also obtains a similar estimates for $u(t,x) - u(t,x+h)$ and $u(t,x+h) - u(t,x)$. From these estimates one has reached that $u = u(t,x)$ is uniformly continuous for all $t \geq 0, x \in \mathbb{R}$, which completes the proof of the lemma. □

In the following we derive regularity estimates for the solutions to (3.1.12) coming from an initial data with a prescribed exponential decay rate of the right, that for $x \gg 1$. To do this, we show that such solutions to (3.1.12) decay with the same rate as the initial data, at least in short time.

Let us introduce some function spaces. Recalling that λ_r^* is defined in Proposition 3.1.7, for $\lambda \in (0, \lambda_r^*)$ let us define the space $BC_\lambda(\mathbb{R})$ by

$$BC_\lambda(\mathbb{R}) := \left\{ \phi \in C(\mathbb{R}) : \sup_{x\in\mathbb{R}} e^{\lambda x}|\phi(x)| < \infty \right\},$$

equipped with the weighted norm

$$\|\phi\|_{BC_\lambda} := \sup_{x\in\mathbb{R}} e^{\lambda x}|\phi(x)|.$$

Recall that $BC_\lambda(\mathbb{R})$ is a Banach space when endowed with the above norm.

Define also the subset E by

$$E := \{\phi \in BC_\lambda(\mathbb{R}) : 0 \leq \phi \leq 1\}, \tag{3.3.25}$$

and let us observe that it is a closed subset of $BC_\lambda(\mathbb{R})$.

Using these notations, we turn to the proof of the following lemma.

Lemma 3.3.3. *Let Assumption 3.1.3 and 3.1.9 and (3.1.13) be satisfied. Let $\lambda \in (0, \lambda_r^*)$ and $u_0 \in E$ be given. Then the solution of (3.1.12) with initial data u_0, denoted by $u = u(t,x)$, satisfies*

$$\lim_{t\to 0^+} \sup_{x\in\mathbb{R}} e^{\lambda x}|u(t,x) - u_0(x)| = 0.$$

Proof. Fix $\alpha > \overline{K} + 2\|\mu\|_\infty$. Let us introduce for each $\phi \in E$ and $t \geq 0$, the operator given by

$$Q_t[\phi](\cdot) := \alpha\phi(\cdot) + \int_{\mathbb{R}} K(y)\phi(\cdot - y)\mathrm{d}y - \overline{K}\phi(\cdot) + \mu(t)\phi(\cdot)\left(1 - \phi(\cdot)\right).$$

Note that one has

$$\left\| \int_{\mathbb{R}} K(y)\phi(\cdot - y)\mathrm{d}y \right\|_{BC_\lambda} = \sup_{x\in\mathbb{R}} \left| \int_{\mathbb{R}} K(y)e^{\lambda y}e^{\lambda(x-y)}\phi(x-y)\mathrm{d}y \right| \leq \left| \int_{\mathbb{R}} K(y)e^{\lambda y}\mathrm{d}y \right| \|\phi\|_{BC_\lambda}.$$

Let us observe that $\left|\int_{\mathbb{R}} K(y)e^{\lambda y}\mathrm{d}y\right| < \infty$ due to $0 < \lambda < \lambda_r^* < \sigma(K)$. Since $0 \le \phi \le 1$ then one has

$$\|Q_t[\phi](\cdot)\|_{BC_\lambda} \le \left(\alpha + \left|\int_{\mathbb{R}} K(y)e^{\lambda y}\mathrm{d}y\right| + \overline{K} + \|\mu\|_\infty\right)\|\phi\|_{BC_\lambda} < \infty.$$

Thus for each $\phi(\cdot) \in E$, for all $t \ge 0$, $Q_t[\phi](\cdot) \in BC_\lambda(\mathbb{R})$.

Next let us observe that $Q_t[\phi]$ is nondecreasing with respect to $\phi \in E$. Indeed, if for any $\phi, \psi \in E$ and $\phi(x) \ge \psi(x)$ for all $x \in \mathbb{R}$, then for each given $t \ge 0, x \in \mathbb{R}$

$$\begin{aligned} Q_t[\phi](x) - Q_t[\psi](x) &= \alpha(\phi(x) - \psi(x)) + \int_{\mathbb{R}} K(y)[\phi(x-y) - \psi(x-y)]\mathrm{d}y - \overline{K}(\phi - \psi)(x) \\ &\quad + \mu(t)\phi(x)(1-\phi(x)) - \mu(t)\psi(x)(1-\psi(x)) \\ &\ge \left(\alpha - \overline{K} - 2\|\mu\|_\infty\right)(\phi(x) - \psi(x)) \\ &\ge 0. \end{aligned}$$

The last inequality comes from $\alpha > \overline{K} + 2\|\mu\|_\infty$. So that for any $t \ge 0$, the map $\phi \mapsto Q_t[\phi]$ is nondecreasing on E.

For each given $u_0 \in E$ and any fixed $h > 0$, we define the following space

$$W := \{t \mapsto u(t,\cdot) \in C([0,h], BC_\lambda(\mathbb{R})) :\ 0 \le u \le 1, u(0,x) = u_0(x)\}.$$

Let us rewrite (3.1.12) to

$$\partial_t u(t,x) + \alpha u(t,x) = Q_t[u(t,\cdot)](x),$$

then one has

$$u(t,\cdot) = e^{-\alpha t}u_0(\cdot) + \int_0^t e^{\alpha(s-t)}Q_s[u(s,\cdot)](\cdot)\mathrm{d}s =: T[u](t,\cdot).$$

Next we show that for each $u \in W$, one has $T[u] \in W$. Let $u \in W$ be given, firstly we show that $Q_t[u](\cdot) \in BC_\lambda(\mathbb{R})$ uniformly for $t \in [0,h]$. Since $t \mapsto u(t,\cdot) \in C([0,h], BC_\lambda(\mathbb{R}))$, then one has

$$\sup_{t\in[0,h]} \|u(t,\cdot)\|_{BC_\lambda} < \infty.$$

Thus

$$\sup_{t\in[0,h]} \|Q_t[u(t,\cdot)](\cdot)\|_{BC_\lambda} \le \left(\alpha + \left|\int_{\mathbb{R}} K(y)e^{\lambda y}\mathrm{d}y\right| + \overline{K} + \|\mu\|_\infty\right)\sup_{t\in[0,h]} \|u(t,\cdot)\|_{BC_\lambda} < \infty.$$

Moreover, one can observe that for each $t \in [0,h]$,

$$\|T[u](t,\cdot)\|_{BC_\lambda} \le \|u_0\|_{BC_\lambda} + \frac{1}{\alpha}\sup_{t\in[0,h]} \|Q_t[u(t,\cdot)]\|_{BC_\lambda} < \infty.$$

That is $T[u](t,\cdot) \in BC_\lambda(\mathbb{R})$, for each $t \in [0,h]$.

Then we show that $t \mapsto T[u](t,\cdot)$ is continuous. To see this, fix $t_0 \in [0,h]$ and observe that one has

$$\begin{aligned}\|T[u](t,\cdot) - T[u](t_0,\cdot)\|_{BC_\lambda} &\le \left|e^{-\alpha t} - e^{-\alpha t_0}\right| \|u_0\|_{BC_\lambda} \\ &\quad + \sup_{x\in\mathbb{R}} e^{\lambda x} \left| \int_0^{t_0} \left[e^{\alpha(s-t)} - e^{\alpha(s-t_0)}\right] Q_s[u(s,\cdot)](x)\mathrm{d}s \right| \\ &\quad + \sup_{x\in\mathbb{R}} e^{\lambda x} \left| \int_{t_0}^{t} e^{\alpha(s-t)} Q_s[u(s,\cdot)](x)\mathrm{d}s \right| \\ &\le \left|e^{-\alpha t} - e^{-\alpha t_0}\right| \|u_0\|_{BC_\lambda} \\ &\quad + \left|e^{-\alpha t} - e^{-\alpha t_0}\right| \sup_{s\in[0,h]} \|Q_s[u(s,\cdot)]\|_{BC_\lambda} \int_0^{t_0} e^{\alpha s}\mathrm{d}s \\ &\quad + \sup_{s\in[0,h]} \|Q_s[u(s,\cdot)]\|_{BC_\lambda} \left| \frac{1-e^{\alpha(t_0-t)}}{\alpha} \right|.\end{aligned}$$

So that $t \mapsto T[u](t,\cdot) \in C([0,h], BC_\lambda(\mathbb{R}))$ and $T[u](0,\cdot) = u_0(\cdot)$.

Also, note that due to for each $t \in [0,h]$, $Q_t[u(t,\cdot)]$ is nondecreasing with $u(t,\cdot) \in E$, then we get

$$0 \le T[u](t,\cdot) \le e^{-\alpha t} + \frac{1}{\alpha}(1-e^{-\alpha t})\alpha \le 1, \quad \forall t \in [0,h].$$

Hence, for each $u \in W$, then $T[u] \in W$.

For each $u, v \in W$ and a given $\gamma > 0$ large enough, we introduce a metric on W defined by

$$d(u,v) := \sup_{t\in[0,h]} \sup_{x\in\mathbb{R}} e^{\lambda x}|u(t,x) - v(t,x)|e^{-\gamma t}.$$

Note that

$$\begin{aligned}&d(T[u],T[v]) = \sup_{t\in[0,h]} \sup_{x\in\mathbb{R}} e^{\lambda x} \left| \int_0^t e^{\alpha(s-t)} \left(Q[u](s,x) - Q[v](s,x)\right)\mathrm{d}s \right| e^{-\gamma t} \\ &\le \sup_{t\in[0,h]} \sup_{x\in\mathbb{R}} \left| \int_0^t e^{(\alpha+\gamma)(s-t)} \left[\alpha + \int_{\mathbb{R}} K(y)e^{\lambda y}\mathrm{d}y + \overline{K} + 3\|\mu\|_\infty\right] e^{-\gamma s} e^{\lambda x}|u(s,x) - v(s,x)|\mathrm{d}s \right| \\ &\le \left[\alpha + \int_{\mathbb{R}} K(y)e^{\lambda y}\mathrm{d}y + \overline{K} + 3\|\mu\|_\infty\right] \sup_{t\in[0,h]} \int_0^t e^{(\alpha+\gamma)(s-t)}\mathrm{d}s \cdot d(u,v) \\ &\le \frac{\alpha + \int_{\mathbb{R}} K(y)e^{\lambda y}\mathrm{d}y + \overline{K} + 3\|\mu\|_\infty}{\alpha+\gamma} \cdot d(u,v).\end{aligned}$$

So that $T[u]$ is a contraction map on W endowed with the metric $d = d(u,v)$, as long as $\gamma > 0$ sufficiently large such that

$$\frac{\alpha + \int_{\mathbb{R}} K(y)e^{\lambda y}\mathrm{d}y + \overline{K} + 3\|\mu\|_\infty}{\alpha+\gamma} < 1.$$

Finally since (W,d) is a complete metric space, by Banach fixed point theorem ensures that $T[u]$ has a unique fixed point in W which is the solution of (3.1.12) with $u(0,\cdot) = u_0(\cdot)$. Since $t \mapsto u(t,\cdot) \in C([0,h], BC_\lambda(\mathbb{R}))$, then one has obtained

$$\lim_{t\to 0^+} \sup_{x\in\mathbb{R}} e^{\lambda x}|u(t,x) - u_0(x)| = 0,$$

that completes the proof of the lemma. □

Lemma 3.3.4. *Let Assumption 3.1.3 and 3.1.9 and (3.1.13) be satisfied. Let $u = u(t,x)$ be the solution of (3.1.12) supplemented with the initial data v_0 satisfying the following properties:*
assume v_0 is Lipschitz continuous in $\mathbb{R}$, there is $A > 0$ large enough, $\alpha > 0$, $p \in (0,1)$ and $\lambda \in (0, \lambda_r^*)$ such that

$$v_0(x) = \begin{cases} \text{increasing function}, & x \in [0,\alpha], \\ \beta := pe^{-\lambda A}, & x \in [\alpha, A], \\ pe^{-\lambda x}, & x \in [A, \infty), \\ 0, & x \in (-\infty, 0]. \end{cases} \tag{3.3.26}$$

Then the function $u = u(t,x)$ is uniformly continuous on $[0,\infty) \times \mathbb{R}$.

Proof. As in the proof of Lemma 3.3.1, $u = u(t,x)$ also satisfies (3.3.18).
Now from the definition of v_0, for $h > 0$ small enough, one can observe that there exists $m > \lambda$ such that

$$v_0(x) \geq e^{-mh} v_0(x-h), \ \forall x \in \mathbb{R}.$$

Indeed, for $x \leq A$, due to v_0 is nondecreasing on this interval, then $v_0(x) \geq v_0(x-h)$ for $x \leq A$. Note that $e^{-mh} < 1$, then

$$v_0(x) \geq e^{-mh} v_0(x-h), \ \forall x \leq A.$$

For $A \leq x \leq A + h$, since $m > \lambda$, then

$$v_0(x) \geq v_0(A+h) = pe^{-\lambda(A+h)} \geq pe^{-mh} e^{-\lambda A} = e^{-mh} v_0(x-h), \ \forall x \in [A, A+h].$$

For $x \geq A + h$, one has

$$v_0(x) = pe^{-\lambda x} \geq pe^{-mh} e^{-\lambda(x-h)} = e^{-mh} v_0(x-h), \ \forall x \geq A + h.$$

Thus we have obtained

$$v_0(x) \geq e^{-mh} v_0(x-h), \ \ \forall x \in \mathbb{R}.$$

Now, let us show that the function $v^h(t,x) := e^{-mh} u(t, x-h)$ (with $v^h(0,x) = e^{-mh} v_0(x-h)$) is a sub-solution of (3.1.12). To see this, note that $v^h(t,x)$ satisfies

$$\begin{aligned} \partial_t v^h(t,x) &= \int_{\mathbb{R}} K(y) v^h(t, x-y) \mathrm{d}y - \overline{K} v^h(t,x) + \mu(t) v^h(t,x) \left(1 - e^{mh} v^h(t,x)\right) \\ &\leq \int_{\mathbb{R}} K(y) v^h(t, x-y) \mathrm{d}y - \overline{K} v^h(t,x) + \mu(t) v^h(t,x) \left(1 - v^h(t,x)\right). \end{aligned}$$

Hence $v^h(t,x)$ becomes a sub-solution of (3.1.12).
Since $v^h(0,\cdot) \leq v_0(\cdot)$, the comparison principle implies that

$$u(t,x) \geq e^{-mh} u(t, x-h), \ \forall t \geq 0, x \in \mathbb{R}.$$

Similarly as in (3.3.23), one also has, for all $h > 0$ sufficiently small,

$$u(t, x-h) - u(t,x) \leq \left(1 - e^{-mh}\right) u(t, x-h) \leq 1 - e^{-mh}, \forall t \geq 0, x \in \mathbb{R}, \tag{3.3.27}$$

and changing x to $x + h$ yields for all $h > 0$ sufficiently small,

$$u(t,x) - u(t, x+h) \leq \left(1 - e^{-mh}\right) u(t,x) \leq 1 - e^{-mh}, \ \forall t \geq 0, x \in \mathbb{R}. \tag{3.3.28}$$

Next we show that there exists $0 < \alpha(h) < 1$, $\alpha(h) \to 1$ as $h \to 0$ such that for all $h > 0$ small enough

$$u(\sqrt{h}, x) \geq \alpha(h) v_0(x+h), \ \forall x \in \mathbb{R}.$$

Since $v_0(x+h) = 0$ for $x \leq -h$, it is sufficiently to consider the above inequality for $x \geq -h$. As in the proof of Lemma 3.3.1, note that for all $h > 0$ sufficiently small and uniformly for $x \in \mathbb{R}$, one has

$$u(\sqrt{h}, x) \geq v_0(x)\left(1 - \overline{K}\sqrt{h}\right) + \sqrt{h}\left(\int_{\mathbb{R}} K(y) v_0(x-y) \mathrm{d}y + o(1)\right).$$

One may now observe that for all $2A \geq x \geq -h$, there exists $\varepsilon > 0$ such that

$$\int_{\mathbb{R}} K(y) v_0(x-y) \mathrm{d}y \geq \varepsilon > 0.$$

As in the proof of Claim 3.3.2, set $\alpha_1(h) = 1 - 2\overline{K}\sqrt{h}$. Then one has

$$u(\sqrt{h}, x) \geq \alpha_1(h) v_0(x+h), \ \forall x \leq 2A.$$

Let us now prove that there exists $0 < \alpha_2(h) < 1$ and $\alpha_2(h) \to 1$, as $h \to 0$ such that $u(\sqrt{h}, x) \geq \alpha_2(h) v_0(x+h)$ for $x \geq 2A$. From Lemma 3.3.3, one has

$$\lim_{h \to 0^+} \sup_{x \geq 2A} e^{\lambda x} |u(\sqrt{h}, x) - p e^{-\lambda x}| = 0.$$

Set

$$\gamma(h) := \sup_{x \geq 2A} e^{\lambda x} |u(\sqrt{h}, x) - p e^{-\lambda x}|,$$

and observe that, for h sufficiently small, for all $x \geq 2A$, one has

$$\begin{aligned} \left(1 - \frac{\gamma(h)}{p}\right) v_0(x) = -\gamma(h) e^{-\lambda x} + p e^{-\lambda x} &\leq u(\sqrt{h}, x) \\ &\leq \gamma(h) e^{-\lambda x} + p e^{-\lambda x} \\ &= \left(\frac{\gamma(h)}{p} + 1\right) v_0(x). \end{aligned}$$

So that one can set $\alpha_2(h) := 1 - \frac{\gamma(h)}{p}$ to obtain $0 < \alpha_2(h) < 1$, $\alpha_2(h) \to 1$ as $h \to 0$ and

$$u(\sqrt{h}, x) \geq \alpha_2(h) v_0(x), \ \forall x \geq 2A.$$

Then since v_0 is non-increasing for $x \geq A$, one has

$$u(\sqrt{h}, x) \geq \alpha_2(h) v_0(x) \geq \alpha_2(h) v_0(x+h), \ \forall x \geq 2A.$$

Now, set $\alpha(h) := \min\{\alpha_1(h), \alpha_2(h)\}$, so that we get

$$u(\sqrt{h}, x) \geq \alpha(h) v_0(x+h), \ \forall x \in \mathbb{R}.$$

As in the proof of Lemma 3.3.1, one can also construct a function $\tilde{b}_h(t) \to 1$ as $h \to 0$ uniformly for $t \geq 0$ with $0 < \tilde{b}_h(0) < \alpha(h)$ and such that for all $h > 0$ small enough one has

$$u(t + \sqrt{h}, x) \geq \tilde{b}_h(t) u(t, x+h), \ \forall t \geq 0, x \in \mathbb{R}.$$

With such a choice, for all $h > 0$ small enough, for all $t \geq 0$ and $x \in \mathbb{R}$, one obtains that

$$u(t, x+h) - u(t, x) \leq \left(\frac{1}{\tilde{b}_h(t)} - 1\right) u(t+\sqrt{h}, x) + M\sqrt{h} \leq \left(\frac{1}{\tilde{b}_h(t)} - 1\right) + M\sqrt{h}. \quad (3.3.29)$$

As well as, for all $t \geq 0$ and $x \in \mathbb{R}$, one has

$$u(t, x) - u(t, x-h) \leq \left(\frac{1}{\tilde{b}_h(t)} - 1\right) u(t+\sqrt{h}, x-h) + M\sqrt{h} \leq \left(\frac{1}{\tilde{b}_h(t)} - 1\right) + M\sqrt{h}. \quad (3.3.30)$$

Combined with (3.3.27) and (3.3.28), this ensures that u is uniformly continuous on $[0, \infty) \times \mathbb{R}$ and completes the proof of the lemma.

□

Remark 3.3.5. *Here we point out Problem (3.1.1) is invariant with respect to spatial translation, so that spatial shift on the initial data $v_0(\cdot)$, induces the same spatial shift on the solution and does not change the uniform continuity on $[0, \infty) \times \mathbb{R}$.*

3.3.2 Proof of Theorem 3.1.10

In this subsection, we construct a suitable exponentially decaying super-solution and prove Theorem 3.1.10.

Proof of Theorem 3.1.10. For each given $\lambda > 0$ and sufficiently large $A > 0$, let us firstly construct the following function

$$\overline{u}(t, x) := \begin{cases} Ae^{-\lambda_r^*\left(x - \int_0^t c(\lambda_r^*)(s)\mathrm{d}s\right)}, & \text{if } \lambda \geq \lambda_r^*, \\ Ae^{-\lambda\left(x - \int_0^t c(\lambda)(s)\mathrm{d}s\right)}, & \text{if } 0 < \lambda < \lambda_r^*. \end{cases}$$

Here we let $A > 0$ large enough such that $\overline{u}(0, \cdot) \geq u_0(\cdot)$ and recall that the speed function $t \mapsto c(\lambda)(t)$ is defined in (3.1.8).

Since $f(t, u) \leq \mu(t)$ for all $t \geq 0$ and $u \in [0, 1]$, then one readily obtains that $\overline{u}$ is super-solution of (3.1.1). So that the comparison principle implies that

$$\lim_{t\to\infty} \sup_{x \geq \int_0^t c^+(\lambda)(s)\mathrm{d}s + \eta t} u(t, x) \leq \lim_{t\to\infty} \sup_{x \geq \int_0^t c^+(\lambda)(s)\mathrm{d}s + \eta t} \overline{u}(t, x) = 0, \quad \forall \eta > 0.$$

This completes the proof of the upper estimate as stated in Theorem 3.1.10.

□

3.3.3 Proof of Theorem 3.1.11

In this section we first discuss some properties of the solution of the following autonomous Fisher-KPP equation:

$$\partial_t u(t, x) = \int_{\mathbb{R}} k(y) u(t, x-y)\mathrm{d}y - \bar{k} u(t, x) + u(t, x)(m - bu(t, x)), \; t \geq 0, x \in \mathbb{R}. \quad (3.3.31)$$

Here $k(\cdot)$ is a given symmetric kernel as defined in Remark 3.1.4, $\bar{k} = \int_{\mathbb{R}} k(y)\mathrm{d}y > 0$ while m and b are given positive constants.

Define

$$c_0 := \inf_{\lambda > 0} \frac{\int_{\mathbb{R}} k(y) e^{\lambda y}\mathrm{d}y - \bar{k} + m}{\lambda}.$$

Note that $c_0 > 0$ since $k(\cdot)$ is a symmetric function (see also [167] where the sign of the (right and left) wave speed is investigated). Next our first important result reads as follows.

Lemma 3.3.6. *Let $u = u(t,x)$ be the solution of (3.3.31) supplemented with a continuous initial data $0 \le u_0(\cdot) \le \frac{m}{b}$ and $u_0 \not\equiv 0$ with compact support. Let us furthermore assume that u is uniformly continuous for all $t \ge 0$, $x \in \mathbb{R}$. Then one has*

$$\lim_{t\to\infty} \sup_{|x|\le ct} \left| \frac{m}{b} - u(t,x) \right| = 0, \ \forall 0 < c < c_0.$$

Remark 3.3.7. *For the kernel function with* $\operatorname{supp}(k) = \mathbb{R}$ *and without the uniform continuity assumption, the above propagating behavior is already known. We refer to [114, Theorem 3.2]. For the reader convenience, we give a short proof of Lemma 3.3.6, with the help of Theorem 3.3 in [167] and the additional regularity assumption of solution.*

Proof. Let $0 < c < c_0$ be given and fixed. To prove the lemma let us argue by contradiction by assuming that there exists a sequence (t_n, x_n) and $|x_n| \le ct_n$ such that

$$\limsup_{n\to\infty} u(t_n, x_n) < \frac{m}{b}.$$

Denote for $n \ge 0$ the sequence of functions u_n by $u_n(t,x) := u(t+t_n, x+x_n)$. Since $u = u(t,x)$ is uniformly continuous on $[0,\infty) \times \mathbb{R}$ and $0 \le u \le \frac{m}{b}$, then Arzelà-Ascoli theorem applies and ensures that as $n \to \infty$ one has $u_n(t,x) \to u_\infty(t,x)$ locally uniformly for $(t,x) \in \mathbb{R}^2$, for some function $u_\infty = u_\infty(t,x)$ defined in $\mathbb{R}^2$ and such that $u_\infty(0,0) < \frac{m}{b}$.

Now fix $c' \in (c, c_0)$. Recall that Theorem 3.3 in [167] ensures that there exists some constant $q_{c'} \in \left(0, \frac{m}{b}\right]$ such that

$$\liminf_{t\to\infty} \inf_{|x|\le c't} u(t,x) \ge q_{c'}.$$

Hence there exists $T > 0$ such that

$$\inf_{|x|\le c't} u(t,x) \ge \frac{q_{c'}}{2}, \ \forall t \ge T.$$

This implies that for all $n \ge 0$ and $t \in \mathbb{R}$ such that $t + t_n \ge T$ one has

$$\inf_{|x+x_n|\le c'(t+t_n)} u(t+t_n, x+x_n) \ge q_{c'}/2.$$

Since one has $|x_n| \le ct_n$ for all $n \ge 0$, this implies that for all $n \ge 0$ and $t \in \mathbb{R}$ with $t + t_n \ge T$:

$$\inf_{|x|\le (c'-c)t_n + c't} u(t+t_n, x+x_n) \ge q_{c'}/2.$$

Finally since $c' > c$ and $t_n \to \infty$ as $n \to \infty$, so that for all $(t,x) \in \mathbb{R}^2$, $u_\infty(t,x) \ge \frac{q_{c'}}{2} > 0$.

Next, we consider $U = U(t)$ with $U(0) = q_{c'}/2 > 0$ the solution of the ODE

$$U'(t) = U(t)\left(m - bU(t)\right), \forall t \ge 0.$$

Since $u_\infty(s,x) \ge q_{c'}/2$ for all $(s,x) \in \mathbb{R}^2$, then comparison principle implies that

$$u_\infty(t+s, x) \ge U(t), \forall t \ge 0, s \in \mathbb{R}, x \in \mathbb{R}.$$

So that

$$u_\infty(0,0) \ge U(t), \forall t \ge 0.$$

On the other hand, since $U(0) > 0$, one gets $U(t) \to \frac{m}{b}$ as $t \to \infty$. Hence this yields $u_\infty(0,0) \ge \frac{m}{b}$, a contradiction with $u_\infty(0,0) < \frac{m}{b}$, which completes the proof. □

Now we apply the key lemma to prove our inner propagation result Theorem 3.1.11.

Proof of Theorem 3.1.11 (i). Here we assume that the initial data u_0 has a fast decay and we aim at proving that

$$\lim_{t\to\infty}\sup_{x\in[0,ct]}|1-u(t,x)|=0,\ \forall c\in(0,c_r^*).$$

One can construct a initial data v_0 alike in Lemma 3.3.1, through choosing proper parameter and spatial shifting (see Remark 3.3.5) such that $v_0(x)\le u_0(x)$ for all $x\in\mathbb{R}$. Let $v(t,x)$ be the solution of (3.1.12) with initial data v_0, Lemma 3.3.1 ensures that $v(t,x)$ is uniformly continuous for all $t\ge 0, x\in\mathbb{R}$. Since $v_0(\cdot)\le u_0(\cdot)$, then the comparison principle implies that $v(t,x)\le u(t,x)$ for all $t\ge 0, x\in\mathbb{R}$. Note that $u(t,x)\le 1$, it is sufficiently to prove that

$$\lim_{t\to\infty}\inf_{x\in[0,ct]}v(t,x)=1,\ \forall c\in(0,c_r^*).$$

Firstly, let us prove that

$$\liminf_{t\to\infty}\inf_{x\in[0,ct]}v(t,x)>0,\ \forall c\in(0,c_r^*).$$

To do this, for all $B,R>0$, $\gamma\in\mathbb{R}$, we define $c_{R,B}(\gamma)$ by

$$c_{R,B}(\gamma):=\frac{2R}{\pi}\int_{-B}^{B}K(z)\mathrm{e}^{\gamma z}\sin(\frac{\pi z}{2R})\mathrm{d}z. \tag{3.3.32}$$

Note that $\gamma\mapsto c_{R,B}(\gamma)$ is continuous and recalling (3.1.10) one has

$$\lim_{\gamma\to\lambda_r^*}\lim_{\substack{R\to\infty\\ B\to\infty}}c_{R,B}(\gamma)=c_r^*.$$

So for each $c'\in(c,c_r^*)$, one can choose proper γ close to λ_r^* such that for $R,B>0$ large enough,

$$c'\le c_{R,B}(\gamma).$$

Then for all $\frac{c}{c'}<k<1$,

$$\frac{ct}{k}\le X(t):=c_{R,B}(\gamma)t.$$

Now, we apply Lemma 3.2.6 to show that

$$\liminf_{t\to+\infty}\inf_{0\le x\le kX(t)}v(t,x)>0.$$

Note that $t\mapsto X(t)$ is continuous for $t\ge 0$, and Lemma 3.3.1 ensures that $v=v(t,x)$ is uniformly continuous for all $t\ge0, x\in\mathbb{R}$. We only need to check that $v=v(t,x)$ satisfies the conditions $(H1)-(H3)$ in Lemma 3.2.6.

To show $(H1)$, recalling (3.1.5) and (3.1.6), one may observe that $v=v(t,x)$ satisfies

$$\partial_t v(t,x)\ge\int_{\mathbb{R}}k(y)v(t,x-y)\mathrm{d}y-\overline{K}v(t,x)+v(t,x)\left(\mu(t)-Cv(t,x)\right).$$

Recalling Assumption 3.1.5 $(f4)$ and Lemma 3.1.2, there exists $a\in W^{1,\infty}(0,\infty)$ such that $\mu(t)-\overline{K}+a'(t)\ge 0$ for all $t\ge 0$. Set $w(t,x):=e^{a(t)}v(t,x)$ so that w satisfies

$$\begin{aligned}\partial_t w(t,x)\ge&\int_{\mathbb{R}}k(y)w(t,x-y)\mathrm{d}y-\bar{k}w(t,x)\\&+w(t,x)\left(\bar{k}+\mu(t)-\overline{K}+a'(t)-Ce^{-a(t)}w(t,x)\right)\\ \ge&\int_{\mathbb{R}}k(y)w(t,x-y)\mathrm{d}y-\bar{k}w(t,x)+w(t,x)\left(m-Ce^{\|a\|_\infty}w(t,x)\right),\end{aligned}$$

where $m := \inf_{t\geq 0}\left(\bar{k}+\mu(t)-\overline{K}+a'(t)\right)\geq \bar{k}>0$. Now we consider $\underline{w}=\underline{w}(t,x)$ the solution of following equation

$$\partial_t \underline{w}(t,x) = k*\underline{w}(t,x) - \bar{k}\underline{w}(t,x) + \underline{w}(t,x)\left(m - Ce^{\|a\|_\infty}\underline{w}(t,x)\right). \tag{3.3.33}$$

supplemented with the initial data $\underline{w}(0,x) = e^{-\|a\|_\infty}v_0(x)$. Thus note that one has $\underline{w}(0,x)\leq w(0,x)$ for all $x\in\mathbb{R}$ and the comparison principle implies that

$$w(t,x) = e^{a(t)}v(t,x) \geq \underline{w}(t,x), \quad \forall t\geq 0, x\in\mathbb{R}.$$

Lemma 3.3.6 implies that there exists $\tilde{c}>0$ such that

$$\lim_{t\to\infty}\sup_{|x|\leq ct}\left|\underline{w}(t,x) - \frac{m}{Ce^{\|a\|_\infty}}\right| = 0, \ \forall c\in(0,\tilde{c}). \tag{3.3.34}$$

Since $a\in W^{1,\infty}(0,\infty)$, we end-up with

$$\liminf_{t\to\infty} v(t,0) \geq \lim_{t\to\infty} e^{-\|a\|_\infty}\underline{w}(t,0) = \frac{m}{Ce^{2\|a\|_\infty}} > 0,$$

and $(H1)$ is fulfilled.

Next we verify assumption $(H2)$. Recall that for all $\tilde{v}\in\omega(v)\setminus\{0\}$, there exist (t_n) with $t_n\to\infty$ and (x_n) such that $\tilde{v}(t,x) = \lim_{n\to\infty} v(t+t_n, x+x_n)$ where this limit holds locally uniformly for $(t,x)\in\mathbb{R}^2$. As in the proof of Claim 3.2.5, such a function $\tilde{v}$ satisfies

$$\partial_t\tilde{v}(t,x) \geq \int_{\mathbb{R}} k(y)\tilde{v}(t,x-y)\mathrm{d}y + \tilde{v}(t,x)(\tilde{\mu}(t) - \overline{K} - C\tilde{v}(t,x)), \ \forall (t,x)\in\mathbb{R}^2,$$

where $k(y)$ is defined in (3.1.5) and $\tilde{\mu}=\tilde{\mu}(t)\in L^\infty(\mathbb{R})$ is a weak star limit of some shifted function $\mu(t_n+\cdot)$. Similar to Definition 3.1.1 and Lemma 3.1.2, one can define the least mean of $\tilde{\mu}$ over $\mathbb{R}$ as

$$\lfloor\tilde{\mu}\rfloor = \lim_{T\to\infty}\inf_{s\in\mathbb{R}}\frac{1}{T}\int_0^T \tilde{\mu}(t+s)\mathrm{d}t.$$

Also, the least mean of $\tilde{\mu}$ satisfies

$$\lfloor\tilde{\mu}\rfloor = \sup_{a\in W^{1,\infty}(\mathbb{R})}\inf_{t\in\mathbb{R}}(a'+\tilde{\mu})(t).$$

Assumption 3.1.5 $(f4)$ implies that $\lfloor\tilde{\mu}(\cdot)\rfloor\geq\overline{K}$ and the same argument as above yields

$$\liminf_{t\to\infty}\tilde{v}(t,0) \geq \frac{m}{Ce^{2\|b\|_\infty}} > 0,$$

where $b\in W^{1,\infty}(\mathbb{R})$ such that $\tilde{\mu}(t)-\overline{K}+b'(t)\geq 0$ for all $t\in\mathbb{R}$. Hence the condition $(H2)$ is satisfied.

Before proving $(H3)$, we state a lemma related to a compactly supported sub-solution of (3.1.1). Since (3.1.12) is a special case of (3.1.1), one can construct the similar sub-solution of (3.1.12). The following lemma can be proved similarly to Lemma 6.1 in [58]. So that the proof is omitted.

Lemma 3.3.8. *Let Assumption 3.1.3, 3.1.5 and 3.1.9 be satisfied. Let $\gamma\in(0,\lambda_r^*)$ be given. Then there exist $B_0>0$ large enough and $\theta_0>0$ such that for all $B>B_0$ there exists $R_0=R_0(B)>0$ large enough enjoying the following properties: for all $B>B_0$ and $R>\max(R_0(B),B)$, there exists some function $a\in W^{1,\infty}(0,\infty)$ such that the function*

$$u_{R,B}(t,x) = \begin{cases} e^{a(t)}e^{-\gamma x}\cos(\frac{\pi x}{2R}) & \text{if } t\geq 0 \text{ and } x\in[-R,R],\\ 0 & \text{else,}\end{cases}$$

satisfies, for all $\theta \leq \theta_0$*, for all* $x \in [-R, R]$ *and for any* $t \geq 0$,

$$\partial_t u(t,x) - c_{R,B}(\gamma)\partial_x u(t,x) \leq \int_{\mathbb{R}} K(x-y)u(t,y)\mathrm{d}y + \left(\mu(t) - \theta - \overline{K}\right) u(t,x).$$

Herein the speed $c_{R,B}(\gamma)$ *is defined in (3.3.32). Furthermore, let*

$$\underline{u}(t,x) := \eta u_{R,B}(t, x - X(t)),$$

where $X(t) = c_{R,B}(\gamma)t$ *and* $\eta > 0$ *small enough, then* $\underline{u}(t,x)$ *is the sub-solution of (3.1.1).*

Now with the help of Lemma 3.3.8 and the comparison principle, one can choose $\eta > 0$ small enough such that $\underline{u}(0,x) \leq v_0(x)$ and therefore one has

$$\liminf_{t\to\infty} v(t, X(t)) \geq \liminf_{t\to\infty} \underline{u}(t, X(t)) = \liminf_{t\to\infty} \eta u_{R,B}(t, 0) > 0,$$

which ensures that $(H3)$ is satisfied.

As a conclusion all the conditions of Lemma 3.2.6 are satisfied and this yields

$$\liminf_{t\to\infty} \inf_{0\leq x\leq kX(t)} v(t,x) > 0.$$

So that

$$\liminf_{t\to\infty} \inf_{0\leq x\leq ct} v(t,x) > 0, \ \forall c \in (0, c_r^*). \tag{3.3.35}$$

Finally, let us prove that

$$\liminf_{t\to\infty} \inf_{0\leq x\leq ct} v(t,x) = 1, \ \forall c \in (0, c_r^*).$$

To do this, note that combining (3.3.34) and (3.3.35) yields

$$\liminf_{t\to\infty} \inf_{-c_1t\leq x\leq ct} v(t,x) > 0, \forall 0 < c_1 < \tilde{c}, \forall c \in (0, c_r^*).$$

By the similar analysis as proof of Lemma 3.3.6, one could show that the above limit is equal to 1. Hence the proof is completed. □

Next we prove Theorem 3.1.11 (ii). Firstly, we state a lemma about a sub-solution of (3.1.1), one can also construct the similar sub-solution for (3.1.12).

Lemma 3.3.9. *Let Assumption 3.1.3, 3.1.5 and 3.1.9 be satisfied, for each given* $\lambda \in (0, \lambda_r^*)$*, define that*

$$\varphi(t,x) = e^{-\lambda(x+a(t))} - e^{-\lambda a(t) + B_0(t) + B_1} e^{-(\lambda+h)x}, \ \ t \geq 0, x \in \mathbb{R}, \tag{3.3.36}$$

where $a, B_0 \in W^{1,\infty}(0,\infty)$*,* $B_1 > 0$ *and* $0 < h < \min\{\lambda, \sigma(K) - \lambda\}$*. Then*

$$\underline{\phi}(t,x) := \max\left\{0, \ \varphi\left(t, x - \int_0^t c_{\lambda,a}(s)\mathrm{d}s\right)\right\}$$

is the subsolution of (3.1.1).

Remark 3.3.10. *Note that* $\varphi(t,x)$ *is positive when*

$$x > \frac{\|B_0(t)\|_\infty + B_1}{h}.$$

We point out this lemma can be proved similarly as [58, Theorem 2.9]. So we omit the proof.

Proof of Theorem 3.1.11(ii). As proof of Theorem 3.1.11 (i), we can construct $v_0(x)$ alike in Lemma 3.3.4, through choosing proper parameter and spatial shifting (see Remark 3.3.5) such that $v_0(x) \leq u_0(x)$ for all $x \in \mathbb{R}$. Let $v(t,x)$ be the solution of (3.1.12) equipped with initial data v_0. Lemma 3.3.4 ensures that $v(t,x)$ is uniformly continuous for all $t \geq 0, x \in \mathbb{R}$.

Recalling (3.1.8) and (3.1.9), for each given $\lambda \in (0, \lambda_r^*)$ and for all $c < c' < \lfloor c(\lambda) \rfloor$, one can choose a proper function $a \in W^{1,\infty}(0,+\infty)$ such that

$$c' < c_{\lambda,a}(t), \ \forall t \geq 0.$$

Then we define

$$X(t) := \int_0^t c_{\lambda,a}(s)\mathrm{d}s + P,$$

where $P > \frac{\|B_0(t)\|_\infty + B_1}{h} > 0$ and $B_0(\cdot)$, B_1 and h are given in Lemma 3.3.9. Note that for all $\frac{c}{c'} < k < 1$,

$$ct \leq kX(t).$$

Next it is sufficiently to apply key Lemma 3.2.6 to show that

$$\liminf_{t\to\infty} \inf_{0\leq x\leq kX(t)} v(t,x) > 0.$$

Note that for exponential decay initial data v_0 on the right-hand side, that is $x \gg 1$, one can construct an initial data $\underline{v}_0$ alike in Lemma 3.3.1 with compact support such that $\underline{v}_0 \leq v_0$. Then comparison principle implies that $(H1)$ and $(H2)$ hold. To verify the condition $(H3)$, by Lemma 3.3.9 and comparison principle, one has

$$\liminf_{t\to\infty} v(t, X(t)) \geq \liminf_{t\to\infty} \underline{\phi}(t, X(t)) = \liminf_{t\to\infty} \varphi(t, P) > 0.$$

So $(H3)$ is satisfied. Hence the key Lemma 3.2.6 ensures that

$$\liminf_{t\to\infty} \inf_{0\leq x\leq kX(t)} v(t,x) > 0.$$

Then one has

$$\liminf_{t\to\infty} \inf_{0\leq x\leq ct} v(t,x) > 0, \ \forall 0 < c < \lfloor c(\lambda) \rfloor.$$

Similarly to the proof of Theorem 3.1.11 (i), one can show that

$$\lim_{t\to\infty} \sup_{x\in[0,ct]} |u(t,x) - 1| = 0, \ \forall 0 < c < \lfloor c(\lambda) \rfloor.$$

The proof is completed. □

Finally, we prove Corollary 3.1.12.

Proof of Corollary 3.1.12. Recalling $H > 0$ given in Remark 3.1.6, let us consider

$$\partial_t v(t,x) = \int_{\mathbb{R}} K(y)v(t,x-y)\mathrm{d}y - \overline{K}v(t,x) + \mu(t)v(t,x)\left(1 - Hv(t,x)\right), \ t \geq 0, x \in \mathbb{R}. \tag{3.3.37}$$

By the same analysis, one can obtain that the similar result for (3.3.37) as Theorem 3.1.11. For the reader convenience, we state it in the following.

Let $v = v(t,x)$ be the solution of (3.3.37) equipped with a continuous initial data u_0, with $0 \leq u_0 \leq 1$ and $u_0 \not\equiv 0$. Then the following inner spreading occurs:

(i) (fast exponential decay) If $u_0(x) = O(e^{-\lambda x})$ as $x \to \infty$ for some $\lambda \geq \lambda_r^*$ then one has

$$\lim_{t\to\infty} \sup_{x\in[0,ct]} \left| v(t,x) - \frac{1}{H} \right| = 0, \ \forall c \in (0, c_r^*);$$

(ii) (slow exponential decay) If $\liminf_{x\to\infty} e^{\lambda x} u_0(x) > 0$ for some $\lambda \in (0, \lambda_r^*)$ then

$$\lim_{t\to\infty} \sup_{x\in[0,ct]} \left| v(t,x) - \frac{1}{H} \right| = 0, \ \forall c \in (0, \lfloor c(\lambda) \rfloor).$$

Denote that $u(t,x)$ is a solution of (3.1.1) equipped with initial data u_0. Recall (3.1.6) that $v(t,x)$ is the sub-solution of (3.1.1). Then comparison principle implies that $u(t,x) \geq v(t,x)$ for all $t \geq 0, x \in \mathbb{R}$. Hence the conclusion is proved. $\square$

Chapter 4

Spreading speeds for time heterogeneous prey-predator systems with diffusion

This is a joint work with Arnaud Ducrot, submitted [60].

Abstract

We investigate the large time behaviour for two components reaction-diffusion systems of prey-predator type in a time varying environment. Here we assume that these variations in time exhibit an averaging property, which will be called mean value in this work. This framework includes in particular time periodicity, almost periodicity and unique ergodicity. We describe the spreading behaviour of the prey and the predator, wherein the two populations are able to co-invade the empty space. Our analysis is based on the parabolic strong maximum principle for scalar equation and on the derivation of local pointwise estimates that are used to compare the solutions of the prey-predator problem with those of a KPP scalar equation on suitable spatio-temporal domains.

4.1 Introduction

In this work, we investigate the spreading speed for a class of reaction-diffusion system of prey-predator type in a time heterogeneous environment. First, before introducing the general class of systems considered in this work, let us introduce a typical example. We consider the so-called diffusive Lotka-Volterra prey-predator model with time dependent coefficients, that reads as follows

$$\begin{cases} \partial_t u = d_u(t)\partial_{xx}u + r(t)u\,(1-u) - p(t)uv, \\ \partial_t v = d_v(t)\partial_{xx}v + q(t)uv - \nu(t)v. \end{cases} \tag{4.1.1}$$

This problem is set for time $t > 0$ and spatial location $x \in \mathbb{R}$ and it is supplemented with the continuous, non-negative and compactly supported initial data

$$u(0,x) = u_0(x) \text{ and } v(0,x) = v_0(x),\ x \in \mathbb{R}.$$

In the above system of equations, $u = u(t,x)$ and $v = v(t,x)$ denote the density of the prey and the predator, respectively. The functions r, p, q, ν are all positive and describe the growth rate of the prey, the predation rate, the conversion rate and the death rate of the predator, respectively; while d_u and d_v are positive functions which stand for the diffusion rates for the prey and the predator populations. Note that fluctuating environment modeled by time heterogeneities is important in ecology and in particular for prey-predator systems, see for instance [28, 45, 75] and the references cited therein. Various important factors vary with time as for instance climate variations (temperature, rainfall, wind...), seasonality, species mobility, the availability for food and so on.

As mentioned above, the goal of this work is to study the asymptotic speed of spread for a large class of diffusive prey-predator systems including (4.1.1) as a typical example. The notion of spreading speed was introduced by Aronson and Weinberger [8] in investigation of homogeneous scalar reaction-diffusion equations. As far as homogeneous reaction-diffusion systems are concerned, spreading speed has also received a lot of interests. For monotone systems, we refer the reader for instance to [164] for cooperation systems and to [99, 33, 77, 111, 112] for competition systems. We also refer the reader to Liang and Zhao [104, 105] for abstract monotone evolutionary system.

However, due to the asymmetry in prey-predator interactions, the prey-predator systems, as (4.1.1), are no longer monotone. Recently, spreading speed for some prey-predator systems (including the diffusive Lotka-Volterra system in a homogeneous environment) has been studied using ideas from dynamical system, see [55]. We also refer to [40] for the spreading speed of prey-predator systems with shifted habitat and to [53] for the study of the propagation phenomena arising in the interaction between two predators and one prey. We refer the reader to [37, 49, 51, 110, 159, 160] for the study of the large time propagation behaviour of other types of prey-predator systems, for instance when the predator has a positive intrinsic growth rate or when the prey is abundant.

In the last decades, the description of the spreading speed for non-autonomous scalar equations has attracted a lot of interests and has been widely studied. We refer the reader to Shen [140] (for time almost periodic and space periodic equation), Nadin and Rossi [124] (for general time dependence), Berestycki *et al.* [21, 23] (for general heterogeneities in time and space) and the references cited therein.

To deal with temporal heterogeneity, we recall the notion of mean value for bounded function which has been used in [124, 140]. We emphasize that periodic functions, almost periodic functions and uniquely ergodic functions have a mean value according to the next definition.

Definition 4.1.1. *A function $h \in L^\infty(0,\infty;\mathbb{R})$ is said to have a mean value if the following limit exists,*

$$\langle h\rangle := \lim_{T\to\infty}\frac{1}{T}\int_0^T h(t+s)\mathrm{d}t, \text{ uniformly for } s\geq 0.$$

In that case the quantity $\langle h\rangle$ is called the mean value of h.

An equivalent and useful characterization for a function h with a mean value $\langle h\rangle$ can be rewritten as follows,

$$\langle h\rangle = \sup_{a\in W^{1,\infty}(0,\infty)} \operatorname*{ess\,inf}_{t>0}\left(a'+h\right)(t) = \inf_{a\in W^{1,\infty}(0,\infty)} \operatorname*{ess\,sup}_{t>0}\left(a'+h\right)(t). \tag{4.1.2}$$

For $H \in L^\infty(\mathbb{R};\mathbb{R})$, we can also define the mean value of H and the similar reformulation also holds. We refer the reader to [124, 125] for more details about mean value, as well as for the definitions of the so-called least mean and upper mean to handle more general time heterogeneous medium.

Now observe that, when $v\equiv 0$, the u-equation in (4.1.1) becomes following KPP-type equation

$$\partial_t u = d_u(t)\partial_{xx}u + r(t)u\left(1-u\right),\ t>0, x\in\mathbb{R}.$$

Using the above definition, let us recall the spreading speed result for non-autonomous Fisher-KPP equation obtained in [124]. If d_u and r have mean value, then by setting

$$\tilde{c}_u^* = 2\sqrt{\langle d_u\rangle\langle r\rangle},$$

the following spreading property holds true: for nonnegative and nontrivial compactly supported initial data, the corresponding solution $u=u(t,x)$ satisfies

$$\begin{cases}\lim\limits_{t\to\infty}\sup\limits_{|x|\geq ct} u(t,x) = 0, & \forall c > \tilde{c}_u^*,\\ \lim\limits_{t\to\infty}\inf\limits_{|x|\leq ct} u(t,x) = 1, & \forall c \in [0,\tilde{c}_u^*).\end{cases}$$

However, to the best of our knowledge, the spreading behaviour for non-autonoumous prey-predator systems, such as the Lotka-Volterra system (4.1.1), remains at least theoretically unknown for general time variations and also in the periodic and the almost periodic cases. For the study of the spreading speed of monotone non-autonomous systems, we refer the reader to [103, 66] in periodic medium, to [12] for time almost periodic medium and the reference cited therein. We also mention that [157] show some upper and lower bounds of the spreading speeds for a time periodic prey-predator system where the predator has a positive intrinsic growth rate.

As already mentioned above, the spreading speed for prey-predator system in homogeneous medium, including Lotka-Volterra, has been studied in particular in [55]. While the method provided in this aforementioned paper could probably be extended to study the spreading speed for (4.1.1), here we provide a new approach that somehow allows comparison with Fisher-KPP scalar equation. Roughly speaking, using the strong maximum principle for scalar parabolic equations, we derive pointwise comparisons between $u(t,x)$ and $v(t,x)$ in suitable domains. These estimates ensure that one can compare the solution of prey-predator system with that of a KPP-type scalar equation on suitable spatio-temporal domains, typically where the prey has a low density and where the predator has a low density.

Let us explain the ideas of these estimates for (4.1.1). First the predator will starve without the prey. Hence if the prey has a small density, $u \sim 0$, then v becomes a solution of

$$\partial_t v = d_v(t)\partial_{xx} v - \nu(t)v,$$

and v decays exponentially to 0. This observation yields our first estimate: for all $\delta > 0$ small enough, one can find some constants $M_\delta > 0$ and $T_\delta > 0$ such that

$$v(t,x) \leq \delta + M_\delta u(t,x),\ \forall t \geq T_\delta, x \in \mathbb{R}.$$

Another important property of (4.1.1) is the following observation: when there is no predator, $v \equiv 0$, as noticed above, the density of the prey follows the Fisher-KPP equation and spread with the speed $\tilde{c}_u^*$. Through this fact, we show that for fixed $c \in (0, \tilde{c}_u^*)$, for all $\alpha > 0$, there exists some $M_\alpha > 0$ and $T_\alpha > 0$ such that

$$1 - u(t,x) \leq \alpha + M_\alpha v(t,x),\ \ \forall t \geq T_\alpha, \forall x \in [-ct, ct]\,.$$

These rough ideas can be applied to a large class of reaction-diffusion systems of prey-predator type, including (4.1.1) as a special case.

Rather similar estimates have been obtained and used by Wu in [165] to study the invasion of a single predator with two abundant preys in the case where the two prey species have the same diffusion coefficient. This analysis is based on the equation formed by the total density of the two preys coupled with refined estimates of the heat kernel. Here the situation is different since we study the co-invasion of the two species, the prey and the predator. We extend the analysis to handle time heterogeneities and propose a new methodology based on suitable applications of the strong comparison principle for scalar parabolic equations. This methodology is rather general and can extended to other problems. Indeed it can be extended to handle predator-prey systems on lattices (see [57]) or predator-prey systems in spatially heterogeneous habitats. This latter problem will be studied in a forthcoming work.

Hence, in this paper, we study the spreading speed for the following reaction-diffusion system:

$$\begin{cases} \partial_t u = d(t)\partial_{xx} u + uf\,(t,u,v) \\ \partial_t v = \partial_{xx} v + vg\,(t,u,v) \end{cases} \tag{4.1.3}$$

posed for time $t > 0$ and spatial $x \in \mathbb{R}$. This system is supplemented with suitable compactly supported initial data

$$u(0,x) = u_0(x) \text{ and } v(0,x) = v_0(x) \text{ for } x \in \mathbb{R}. \tag{4.1.4}$$

In (4.1.3), as for (4.1.1), $u = u(t,x)$ and $v = v(t,x)$ stand for the density of the prey and the predator. Here we assume, without loss of generality, that the diffusion rate for the predator equals one. This can be achieved with a suitable time rescaling $\tau(t) = \int_0^t d_v(s)\mathrm{d}s$, see for instance [4] for more details.

We now turn to the set of assumptions that will be needed along this work for the functions d, f and g arising in (4.1.3).

Assumption 4.1.2. *We assume that $d : [0,\infty) \to \mathbb{R}$ is a bounded and uniformly continuous function with a mean value $\langle d \rangle$ and $\inf_{t\geq 0} d(t) > 0$.*

Assumption 4.1.3. *The function $f : [0,\infty)^3 \to \mathbb{R}$ satisfies:*

(f1) For each given $u, v \geq 0$, the function $t \mapsto f(t,u,v)$ is bounded and uniformly continuous from $[0,\infty)$ to $\mathbb{R}$, and $t \mapsto f(t,u,v)$ has a mean value $\langle f(\cdot,u,v)\rangle$. The function $(u,v) \mapsto f(t,u,v)$ is Lipschitz continuous with respect to $u, v \geq 0$, uniformly for $t \geq 0$;

(f2) For all $t \geq 0$ and $u > 0$, the map $v \mapsto f(t,u,v)$ is strictly decreasing;

(f3) Assume $f(t,1,0) = 0$ for all $t \geq 0$ and

$$h(u) := \inf_{t\geq 0} f(t,u,0) > 0, \ \forall u \in [0,1);$$

(f4) For all $t \geq 0$ and $v \geq 0$, the map $u \mapsto f(t,u,v)$ is nonincreasing;

(f5) For all $v > 0$, the function f further satisfies $\sup_{t\geq 0} f(t,1,v) < 0$.

Assumption 4.1.4. *The function $g : [0,\infty)^3 \to \mathbb{R}$ satisfies:*

(g1) For each given $u, v \geq 0$, the function $t \mapsto g(t,u,v)$ is bounded and uniformly continuous from $[0,\infty)$ to $\mathbb{R}$, and $t \mapsto g(t,u,v)$ has a mean value $\langle g(\cdot,u,v)\rangle$, while the function $(u,v) \mapsto g(t,u,v)$ is Lipschitz continuous with respect to $u, v \geq 0$, uniformly with respect to $t \geq 0$;

(g2) For all $t \geq 0$ and $v \geq 0$, the map $u \mapsto g(t,u,v)$ is nondecreasing;

(g3) It satisfies $\inf_{t\geq 0} g(t,1,0) > 0$;

(g4) For all $t \geq 0$ and $u \geq 0$, the map $v \mapsto g(t,u,v)$ is nonincreasing;

(g5) Let the mean value of function $t \mapsto g(t,0,0)$ satisfy

$$\langle g(\cdot,0,0)\rangle < 0.$$

From now on and for notation simplicity, we set

$$r_1(t) := f(t,0,0) \text{ and } r_2(t) := g(t,1,0). \tag{4.1.5}$$

From the monotonicity and regularity of f and g, there exists some constant $L > 0$ such that for all $t \geq 0$, $u \in [0,1]$ and $v \geq 0$,

$$\begin{aligned} r_1(t)\,(1 - Lu - Lv) &\leq f(t,u,v) \leq r_1(t), \\ r_2(t)\big(1 - L(1-u) - Lv\big) &\leq g(t,u,v) \leq r_2(t). \end{aligned} \tag{4.1.6}$$

Now we explain Assumption 4.1.3 and 4.1.4 in the ecological context.

- As we mentioned before, the species typically live in a time varying environment. Thus we assume that f and g both depend on time.
- Assumptions $(f2)$ and $(g2)$ describe predatory behaviour. Condition $(f2)$ means that more predators reduce the prey density while $(g2)$ implies that more prey lead to an increase for the predator population. Due to this asymmetry, the comparison principle does not apply to (4.1.3).
- When there is no predator, $(f3)$ ensures that $u \equiv 1$ is the maximal environmental carrying capacity of the prey. $(g3)$ means that the predator density will increase when the prey is abundant.
- $(f4)$ and $(g4)$ imply that the growth rate of each species is maximal at low density. By analogy with the Fisher-KPP equation, this indicates that the propagation of two species is driven by the leading edge of the invasion.

- $(f5)$ is a technical assumption. Note also that $(f2)$ and $f(t,1,0)\equiv 0$ already ensure that $f(t,1,v)<0$ for all $t\geq 0$ and $v>0$. $(f5)$ implies that the prey cannot reach the environmental carrying capacity 1 as long as there exists the predator. $(g5)$ means that the predator cannot survive without the prey. The prey population is the only resource for the predator.

Coming back to (4.1.1), note that it corresponds to (4.1.3) with

$$\begin{aligned} f(t,u,v) &= r(t)\,(1-u)-p(t)v,\\ g(t,u,v) &= q(t)u-\nu(t). \end{aligned}$$

With the additional smoothness and sign conditions for the coefficients, it satisfies Assumption 4.1.3 and 4.1.4.

Next, to state our main results, we define two speed functions $\lambda\mapsto c_u(\lambda)$ and $\gamma\mapsto c_v(\gamma)$ from $(0,\infty)$ to $L^\infty(0,\infty)$ given by

$$c_u(\lambda)(t):=d(t)\lambda+\frac{r_1(t)}{\lambda}\ \text{ and }\ c_v(\gamma)(t):=\gamma+\frac{r_2(t)}{\gamma}, \tag{4.1.7}$$

for all $t\geq 0$, where r_1 and r_2 are defined in (4.1.5). These two functions corresponds to linear speeds for u and v respectively, around the stationary state $(0,0)$ (no species) and $(1,0)$ (predator free equilibrium) for solution with exponential decay λ and γ. We also introduce the quantities c_u^* and c_v^* given by

$$c_u^*:=\inf_{\lambda>0}\langle c_u(\lambda)\rangle\ \text{ and }\ c_v^*:=\inf_{\gamma>0}\langle c_v(\gamma)\rangle.$$

Setting

$$\lambda^*:=\sqrt{\frac{\langle r_1\rangle}{\langle d\rangle}}\ \text{and}\ \gamma^*:=\sqrt{\langle r_2\rangle}, \tag{4.1.8}$$

one has

$$c_u^*=\langle c_u(\lambda^*)\rangle=2\sqrt{\langle d\rangle\langle r_1\rangle}\ \text{ and }\ c_v^*=\langle c_v(\gamma^*)\rangle=2\sqrt{\langle r_2\rangle}. \tag{4.1.9}$$

Due to $(f1)$, $(f3)$ and $(f4)$, one can observe that for $v\equiv 0$, the system (4.1.3) degenerates to following Fisher-KPP type equation satisfied by u,

$$\partial_t u(t,x)=d(t)\partial_{xx}u(t,x)+u(t,x)f\left(t,u(t,x),0\right).$$

The quantity c_u^* is the spreading speed of above equation equipped with compactly supported initial data, we refer the reader to [21, 23, 124].

On the other hand, for $u\equiv 1$, the solution v of (4.1.3) satisfies following equation

$$\partial_t v(t,x)=\partial_{xx}v(t,x)+v(t,x)g\left(t,1,v(t,x)\right).$$

Note that we do not assume the existence of nontrivial stationary state solution in above equation. It is not a standard KPP-type equation. However, by the similar argument in [21, 124], one can show that c_v^* is the spreading speed of above equation equipped with compactly supported initial data. The main difference is that v may grow and become unbounded in the large time.

With above notations and assumptions, we state our main results.

Theorem 4.1.5 (Slow predator case)**.** *Let Assumption 4.1.2, 4.1.3 and 4.1.4 be satisfied. We assume that the predator is slower than the prey, in the sense that*

$$c_u^*>c_v^*.$$

Let u_0 and v_0 be two given bounded and continuous functions in $\mathbb{R}$ with compact support, and $0 \not\equiv\leq u_0 \leq 1$, $0 \not\equiv\leq v_0$. Let $(u,v) = (u(t,x), v(t,x))$ be the solution of (4.1.3) with initial data (u_0, v_0). Assume that (u,v) is bounded.
Then the function pair (u,v) satisfies:

(i) for all $c > c_u^*$, one has $\lim_{t\to\infty} \sup_{|x|\geq ct} u(t,x) = 0$;

(ii) for all $c_v^* < c_1 < c_2 < c_u^*$ and for all $c > c_v^*$ one has:

$$\lim_{t\to\infty} \sup_{c_1 t\leq |x|\leq c_2 t} |1 - u(t,x)| = 0 \text{ and } \lim_{t\to\infty} \sup_{|x|\geq ct} v(t,x) = 0;$$

(iii) for all $c \in [0, c_v^*)$ one has:

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} v(t,x) > 0,$$

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} u(t,x) > 0 \text{ and } \limsup_{t\to\infty} \sup_{|x|\leq ct} u(t,x) < 1.$$

Theorem 4.1.6 (Fast predator case)**.** *Let Assumption 4.1.2, 4.1.3 and 4.1.4 be satisfied and assume that the predator is faster than the prey, in the sense that*

$$c_u^* \leq c_v^*.$$

Let u_0 and v_0 be two given bounded and continuous functions in $\mathbb{R}$ with compact support, and $0 \not\equiv\leq u_0 \leq 1$, $0 \not\equiv\leq v_0$. Let $(u,v) = (u(t,x), v(t,x))$ be the solution of (4.1.3) with initial data (u_0, v_0). Assume that (u,v) is bounded.
Then (u,v) satisfies:

(i) for all $c > c_u^*$, one has $\lim_{t\to\infty} \sup_{|x|\geq ct} [u(t,x) + v(t,x)] = 0$;

(ii) for all $c \in [0, c_u^*)$ one has:

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} v(t,x) > 0,$$

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} u(t,x) > 0 \text{ and } \limsup_{t\to\infty} \sup_{|x|\leq ct} u(t,x) < 1.$$

Remark 4.1.7. *In the situation of all coefficients in (4.1.3) are independent of t, that is $d(t) \equiv d > 0$, $f(t,u,v) \equiv f(u,v)$ and $g(t,u,v) \equiv g(u,v)$, the above two theorems have been proved in Theorem 2.1 and 2.2 in [55]. In this note, we provide a new method that allows 1) to recover this result in the homogeneous case, 2) to extend them for non-autonomous prey-predator systems, 3) to provide a shorter proof as in [55] for the homogeneous problem.*

Remark 4.1.8. *Note that in above two theorems, we require that the solution (u,v) is bounded. We emphasize that this assumption is satisfied for a large classes of systems. Recall that the comparison principle does not hold for system (4.1.3). However we can apply it for each equation separately to obtain $0 \leq u(t,x) \leq 1$ and $v(t,x) \geq 0$ for all $t \geq 0$ and $x \in \mathbb{R}$. The boundedness of the solutions can be obtained if we assume that $\limsup_{t\to\infty} g(t,1,\infty) < 0$, which is satisfied for the prey-predator problems with intraspecific competition for the predator, for example $g(t,u,v) := q(t)u - v - \nu(t)$. When the above condition is not satisfied, that is when $\limsup_{t\to\infty} g(t,1,\infty) \geq 0$, the situation is more complicated and some results are discussed in the next proposition.*

Next we show that the solution (u, v) is bounded in case when $\inf_{t\geq 0} g(t, 0, \infty) > -\infty$, which includes the case of $\limsup_{t\to\infty} g(t, 1, \infty) \geq 0$. For technical requirement, we add the assumption that:

$$\text{there exists } M_0 > 0 \text{ such that the mean value } \langle f(\cdot, 0, M)\rangle < 0 \text{ for all } M \geq M_0. \tag{4.1.10}$$

Recall that $f(t, 0, v) \geq f(t, u, v)$ for all $u, v \geq 0$ and $t \geq 0$. Hence (4.1.10) means that even if the prey grows fast at low density, the sufficiently large density of the predator will cause reduction of the prey.

Proposition 4.1.9 (Boundedness). *Let Assumption 4.1.2, 4.1.3 and 4.1.4 be satisfied. Assume that (4.1.10) and* $\inf_{t\geq 0} g(t, 0, \infty) > -\infty$ *hold. Let* $(u, v) = (u, v)(t, x)$ *be the solution of (4.1.3) supplemented with nonnegative and uniformly continuous initial function* (u_0, v_0)*. If* $0 \leq u_0 \leq 1$ *and* $v_0 \geq 0$ *is bounded, then the function* $(u, v) = (u, v)(t, x)$ *is bounded on* $[0, \infty) \times \mathbb{R}$*.*

The rest of this paper is organized as follows. In Section 4.2, we construct proper super-solutions to obtain an upper estimate of the speed of propagation for each species. We show that the spreading speed of the prey cannot exceed c_u^* and the predator cannot spread faster than c_v^* and c_u^*. In Section 4.3.1, we discuss time and space shift argument of the equations that are used at several places in the sequel. Then we prove some key lemmas about our local pointwise estimates between u and v. With the help of the first key lemma (see Lemma 4.3.2), in Section 4.3.3 we prove the propagation in the intermediate zone in the case of slow predator. In Section 4.3.4, we use the other key lemma (see Lemma 4.3.4) to derive a Fisher-KPP type differential inequality satisfied by v in a moving domain. Through constructing a sub-solution with compact support, we obtain that v is persistent at $x = ct$ for suitable $c > 0$ and $t \gg 1$. Moreover, we complete the proof of Theorem 4.1.5 and 4.1.6. Lastly, for sake of completeness, we prove Proposition 4.1.9.

4.2 Upper estimates on the spreading speeds

In this section we prove Theorem 4.1.5 (i), half of Theorem 4.1.5 (ii) and Theorem 4.1.6 (i). In the proof we only focus on $x \geq 0$, for $x \leq 0$ which can be dealt with a similar symmetric argument.

Recalling (4.1.7), (4.1.8) and (4.1.9), the property of mean value (see (4.1.2)) ensures that for all $c > c' > c_u^*$, there exists a function $a \in W^{1,\infty}(0, \infty)$ such that for all $t > 0$

$$c' \geq d(t)\lambda^* + \frac{r_1(t)}{\lambda^*} + a'(t).$$

Then for $A > 0$ the function $\overline{u}$ given by

$$\overline{u}(t, x) := Ae^{-\lambda^* a(t)} e^{-\lambda^*(x - c't)}$$

satisfies

$$\partial_t \overline{u}(t, x) - d(t)\overline{u}_{xx}(t, x) - r_1(t)\overline{u}(t, x) = \overline{u}(t, x)\left(\lambda^* c' - \lambda^* a'(t) - d(t)(\lambda^*)^2 - r_1(t)\right) \geq 0.$$

Let $A > 0$ be large enough such that $\overline{u}(0, x) \geq u_0(x)$ for all $x \in \mathbb{R}$. Note that we have $f(t, u, v) \leq r_1(t)$ for all $t \geq 0$, $u \in [0, 1]$ and $v \geq 0$ from (4.1.6). Hence the comparison principle applies and yields for all $c > c' > c_u^*$,

$$\limsup_{t\to\infty} \sup_{x\geq ct} u(t, x) \leq \limsup_{t\to\infty} \sup_{x\geq ct} \overline{u}(t, x) \leq \lim_{t\to\infty} Ae^{-\lambda^* a(t)} e^{-\lambda^*(c - c')t} = 0.$$

Since u is nonnegative, this already proves statement (i) in Theorem 4.1.5 and the half of statement (i) in Theorem 4.1.6.

Similarly, for all $c > \tilde{c} > c_v^*$, there exists $\tilde{a} \in W^{1,\infty}(0,\infty)$ such that for all $t > 0$

$$\tilde{c} \geq \gamma^* + \frac{r_2(t)}{\gamma^*} + \tilde{a}'(t).$$

Then the function

$$\overline{v}_1(t,x) := Ae^{-\gamma^*\tilde{a}(t)}e^{-\gamma^*(x-\tilde{c}t)}$$

satisfies following differential inequality

$$\partial_t \overline{v}_1(t,x) - \partial_{xx}\overline{v}_1(t,x) - r_2(t)\overline{v}_1(t,x) \geq 0.$$

Choosing $A > 0$ large enough such that $\overline{v}_1(0,x) \geq v_0(x)$ for all $x \in \mathbb{R}$, through (4.1.6) and the comparison principle one obtains that for all $c > \tilde{c} > c_v^*$,

$$\lim_{t\to\infty} \sup_{x\geq ct} v(t,x) \leq \lim_{t\to\infty} \sup_{x\geq ct} \overline{v}_1(t,x) \leq \lim_{t\to\infty} Ae^{-\gamma^*\tilde{a}(t)}e^{-\gamma^*(c-\tilde{c})t} = 0.$$

Since v is nonnegative, then we have already proved the half of statement (ii) in Theorem 4.1.5.

Next we show that v cannot spread faster than c_u^*. Note that we have already obtained

$$\lim_{t\to\infty} \sup_{|x|\geq ct} u(t,x) = 0, \ \forall c > c_u^*.$$

Thus, fixing any $c > c_u^*$ and $\varepsilon > 0$ small enough, there exists $T > 0$ such that

$$\sup_{t\geq T} \sup_{|x|\geq ct} u(t,x) \leq \varepsilon.$$

Recalling that $\langle g(\cdot,0,0)\rangle < 0$ in Assumption 4.1.4 and observing that the map $u \mapsto \langle g(\cdot,u,0)\rangle$ is continuous, one has $\langle g(\cdot,\varepsilon,0)\rangle < 0$ for all $\varepsilon > 0$ sufficiently small. From (4.1.2), one can choose $b \in W^{1,\infty}(0,\infty)$ such that

$$\sup_{t>0} \{g(t,\varepsilon,0) + b'(t)\} < 0.$$

For some $B > 0$ and $\gamma' > 0$ which will be chosen below, for $c > c'' > c_u^*$, we define

$$\overline{v}_2(t,x) := Be^{-\gamma'(x-c''t)}e^{-b(t)}.$$

Now choose $\gamma' > 0$ small enough so that $\overline{v}_2(t,x)$ satisfies for all $t \geq 0$ and $x \in \mathbb{R}$,

$$\partial_t \overline{v}_2(t,x) - \partial_{xx}\overline{v}_2(t,x) - g(t,\varepsilon,0)\overline{v}_2(t,x) = \overline{v}_2(t,x)\left(-b'(t) + \gamma'c'' - (\gamma')^2 - g(t,\varepsilon,0)\right) \geq 0.$$

Since $g(t,u,v) \leq g(t,\varepsilon,0)$ for all $t \geq 0$, $v \geq 0$ and $0 \leq u \leq \varepsilon$, then $\overline{v}_2(t,x)$ is a super-solution of v-equation in (4.1.3) for all $t \geq T$ and $x \geq c''t$ with $c'' > c_u^*$.

Lastly, let us focus on the region $\{(t,x) : t \geq T, x \geq c''t\}$. Since v is assumed to be bounded, one can choose $B > 0$ large enough such that $Be^{-\|b\|_\infty} \geq v(t,x)$ for all $t \geq T$ and $x \in \mathbb{R}$. So for all $t \geq T$ and $x = c''t$, one has

$$v(t,c''t) \leq Be^{-\|b\|_\infty} \leq \overline{v}_2(t,c''t).$$

Recalling that $v(t,x) \leq \overline{v}_1(t,x)$ for all $t \geq 0$ and $x \in \mathbb{R}$ and that $\gamma' \in (0,\gamma^*)$ is sufficiently small, one can choose larger $B > 0$ if necessary such that at $t = T$ and for all $x \geq c''T > 0$,

$$v(T,x) \leq \overline{v}_1(T,x) = Ae^{-\gamma^*\tilde{a}(T)}e^{-\gamma^*(x-\tilde{c}T)} \leq Be^{-\gamma'(x-c''T)}e^{-b(T)} = \overline{v}_2(T,x).$$

Thus we obtain that $v(t,x) \leq \overline{v}_2(t,x)$ on the boundary set $\{(t,x) : t \geq T, x = c''t\}$ and $\{(t,x) : t = T, x \geq c''T\}$. Applying the comparison principle on domain

$$\{(t,x) : t \geq T, x \geq c''t\},$$

one has

$$\limsup_{t\to\infty} \sup_{x\geq ct} v(t,x) \leq \limsup_{t\to\infty} \sup_{x\geq ct} \overline{v}_2(t,x) \leq \lim_{t\to\infty} Be^{-\gamma'(c-c'')t}e^{-b(t)} = 0, \ \forall c > c'' > c_u^*.$$

This completes the proof of statement (i) in Theorem 4.1.6.

4.3 Lower estimates on the spreading speeds

In this section, we first introduce some notations and derive the equation satisfied by the limit of shifted solutions. Then we derive local estimates between u and v, that reflect the relationship between prey and predator. Lastly we apply these lemmas to complete the proof of Theorem 4.1.5 and 4.1.6. For brevity, troughout this section we assume that Asumption 4.1.2, 4.1.3 and 4.1.4 are satisfied. Let $(u,v) = (u(t,x), v(t,x))$ be the bounded solution of (4.1.3) with initial data (u_0, v_0) where u_0 and v_0 are bounded and continuous functions in $\mathbb{R}$ with compact support, as well as $0 \not\equiv\leq u_0 \leq 1$ and $0 \not\equiv\leq v_0$.

4.3.1 Limit problem

In this section we discuss time and space shift of the solution (u,v). Fix a sequence $(\tau_n)_{n\geq 0}$ such that $\tau_n \to \infty$. Then we claim that

Claim 4.3.1. *There exist $\tilde{f} : \mathbb{R}\times[0,\infty)^2 \to \mathbb{R}$ and $\tilde{g} : \mathbb{R}\times[0,\infty)^2 \to \mathbb{R}$ two bounded and uniformly continuous functions and a subsequence, still denoted (τ_n) such that*

$$f(t+\tau_n, u, v) \to \tilde{f}(t,u,v) \text{ and } g(t+\tau_n,u,v) \to \tilde{g}(t,u,v) \text{ as } n \to \infty,$$

locally uniformly for $t \in \mathbb{R}$ and $(u,v) \in [0,\infty)^2$. Note also that due to (f2), (f4) and (g2), (g4), $\tilde{f} = \tilde{f}(t,u,v)$ is nonincreasing in both u and v while $\tilde{g} = \tilde{g}(t,u,v)$ is nondeacreasing with respect to u and nonincreasing in the $v-$variable.

Assumptions 4.1.3 and 4.1.4 for f and g ensure that both functions are bounded and uniformly continuous on $[0,\infty)^3$ and the claim follows.

Now we define for $t \geq 0$:

$$\sigma(t) := \big(d(t), f(t,\cdot,\cdot), g(t,\cdot,\cdot)\big) \in \mathbb{R}\times \mathrm{BUC}\big([0,\infty)\times[0,\infty)\big)^2.$$

As d is uniformly continuous and using Claim 4.3.1, we define the set Σ as follows: $\tilde{\sigma} = (\tilde{d}, \tilde{f}, \tilde{g}) \in \Sigma$ if and only if there exist a sequence $\tau_n \geq 0$ such that

$$\big(d(t+\tau_n), f(t+\tau_n,\cdot,\cdot), g(t+\tau_n,\cdot,\cdot)\big) \to \Big(\tilde{d}(t), \tilde{f}(t,\cdot,\cdot), \tilde{g}(t,\cdot,\cdot)\Big),$$

locally uniformly for $t \in \mathbb{R}$, as $n \to \infty$.

Recall that $(u,v) = (u(t,x), v(t,x))$ denotes a bounded solution of (4.1.3). Define the set S by: $(\tilde{u}, \tilde{v}) \in S$ if there exist sequence $(t_n)_{n\geq 0} \subset [0,\infty)$ with $t_n \to \infty$ as $n \to \infty$ and $(x_n)_{n\geq 0} \subset \mathbb{R}$ such that

$$(u(t+t_n, x+x_n), v(t+t_n, x+x_n)) \to (\tilde{u}(t,x), \tilde{v}(t,x)),$$

locally uniformly for $(t,x)\in\mathbb{R}^2$, as $n\to\infty$.

Note that for each sequence of $(t_n,x_n)\in[0,\infty)\times\mathbb{R}$, the function pair $(u_n,v_n)(t,x):=(u,v)(t+t_n,x+x_n)$ defined for $t\geq -t_n$ and $x\in\mathbb{R}$ satisfies

$$\begin{cases}\partial_t u_n(t,x)=d(t+t_n)\partial_{xx}u_n(t,x)+u_n(t,x)f\left(t+t_n,u_n(t,x),v_n(t,x)\right),\\ \partial_t v_n(t,x)=\partial_{xx}v_n(t,x)+v_n(t,x)g\left(t+t_n,u_n(t,x),v_n(t,x)\right).\end{cases}\tag{4.3.11}$$

If $t_n\to\infty$, from parabolic regularity, up to a subsequence one has $(u_n,v_n)(t,x)\to(\tilde{u},\tilde{v})(t,x)$ locally uniformly for $(t,x)\in\mathbb{R}^2$ as $n\to\infty$ and there exists some $\tilde{\sigma}\in\Sigma$ with $\sigma(t+t_n)\to\tilde{\sigma}(t)$ locally uniformly for $t\in\mathbb{R}$ as $n\to\infty$. In addition, the function pair $(\tilde{u},\tilde{v})$ satisfies the system for $(t,x)\in\mathbb{R}^2$

$$(\mathbf{P}_{\tilde{\sigma}})\quad\begin{cases}\partial_t\tilde{u}(t,x)=\tilde{d}(t)\partial_{xx}\tilde{u}(t,x)+\tilde{u}(t,x)\tilde{f}\left(t,\tilde{u}(t,x),\tilde{v}(t,x)\right),\\ \partial_t\tilde{v}(t,x)=\partial_{xx}\tilde{v}(t,x)+\tilde{v}(t,x)\tilde{g}\left(t,\tilde{u}(t,x),\tilde{v}(t,x)\right).\end{cases}\tag{4.3.12}$$

4.3.2 Key lemmas

Now we construct some important lemmas which play a key role in proving Theorem 4.1.5 and 4.1.6. Roughly speaking, from Assumption 4.1.3 and 4.1.4, we have two important facts: the predator cannot survive without the prey and the prey asymptotically reach its carrying capacity without the predator. We transfer these facts into two important inequalities which are crucial to ensure that one can compare solutions of the system with those of a single KPP-type scalar equations on suitable moving domain.

Our first lemma reads as follows.

Lemma 4.3.2. *For all $\delta>0$, there exist $M_\delta>0$ and $T_\delta>0$ such that the following estimate holds true*

$$v(t,x)\leq\delta+M_\delta u(t,x),\ \forall t\geq T_\delta, x\in\mathbb{R}.$$

Proof. To prove the lemma we argue by contradiction by assuming that there exist $\delta_0>0$ and sequences $(t_n)_n$ and $(x_n)_n$ such that

$$t_n\to\infty \text{ as } n\to\infty \text{ and } v(t_n,x_n)>\delta_0+nu(t_n,x_n),\ \text{ for all } n\geq 0.\tag{4.3.13}$$

Set

$$u_n(t,x):=u(t+t_n,x+x_n) \text{ and } v_n(t,x):=v(t+t_n,x+x_n).$$

As we discussed in Section 4.3.1, due to parabolic regularity, there exist a subsequence, still denoted with the same indexes n, $(u_\infty,v_\infty)\in S$ and $\tilde{\sigma}\in\Sigma$ such that

$$\begin{aligned}&(u_n,v_n)(t,x)\to(u_\infty,v_\infty)(t,x) \text{ as } n\to\infty \text{ locally uniformly for } (t,x)\in\mathbb{R}^2,\\ &\text{the function pair } (u_\infty,v_\infty) \text{ satisfies } (\mathbf{P}_{\tilde{\sigma}}) \text{ (see (4.3.12))}.\end{aligned}$$

Due to the boundedness of v, (4.3.13) implies that $u(t_n,x_n)\to 0$ as $n\to\infty$, that is $u_\infty(0,0)=0$. The strong maximum principle for the u_∞-equation implies that $u_\infty\equiv 0$ and the limit function v_∞ satisfies

$$\partial_t v_\infty(t,x)=\partial_{xx}v_\infty(t,x)+v_\infty(t,x)\tilde{g}\left(t,0,v_\infty(t,x)\right)\text{for } (t,x)\in\mathbb{R}^2.\tag{4.3.14}$$

Since v_∞ is bounded, then one can choose $B>0$ large enough such that $B\geq v_\infty(t,x)$ for all $(t,x)\in\mathbb{R}^2$. For each $t_0<0$, we define

$$\overline{v}(t;t_0):=B\exp\left(\int_{t_0}^{t}\tilde{g}(s,0,0)\mathrm{d}s\right).$$

From Claim 4.3.1, one has $\tilde{g}(t,0,v) \leq \tilde{g}(t,0,0)$ for all $t \in \mathbb{R}$ and $v \geq 0$. Hence one can verify that $t \mapsto \overline{v}(t;t_0)$ is the super-solution of (4.3.14). The comparison principle implies that $v_\infty(t,x) \leq \overline{v}(t;t_0)$ for all $t_0 < 0$, $t \geq t_0$ and $x \in \mathbb{R}$. As a special case, letting $t = 0$, we obtain that

$$v_\infty(0,x) \leq \overline{v}(0;t_0) \text{ for all } x \in \mathbb{R} \text{ and } t_0 < 0.$$

Since $\langle g(\cdot,0,0)\rangle < 0$ (see (g5) in Assumption 4.1.4) and the definition of mean value, one has $\langle \tilde{g}(\cdot,0,0)\rangle < 0$. Let us observe that

$$\lim_{t_0\to-\infty} \int_{t_0}^{0} \tilde{g}(s,0,0)\mathrm{d}s = \lim_{t_0\to-\infty} (-t_0)\cdot\frac{1}{0-t_0}\int_{t_0}^{0} \tilde{g}(s,0,0)\mathrm{d}s = -\infty.$$

Hence, we conclude that for all $x \in \mathbb{R}$,

$$v_\infty(0,x) \leq \lim_{t_0\to-\infty} \overline{v}(0;t_0) = \lim_{t_0\to-\infty} B\exp\left(\int_{t_0}^{0} \tilde{g}(s,0,0)\mathrm{d}s\right) = 0.$$

This contradicts the property $v_\infty(0,0) \geq \delta_0$ that follows by passing to the limit $n \to \infty$ into the assumption $v(t_n,x_n) > \delta_0 > 0$ for all $n \geq 0$. The proof is completed. □

In the following proposition, we apply above lemma to show that u is persistent on the interval $[-ct,ct]$ with $t \gg 1$ for all $c \in (0,c_u^*)$.

Proposition 4.3.3. *For all $c \in [0,c_u^*)$, one has*

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} u(t,x) > 0.$$

Proof. Recalling (4.1.6) and Lemma 4.3.2, for each given $\delta > 0$, there exist $M_\delta > 0$ and $T_\delta > 0$ such that the solution $u(t,x)$ of (4.1.3) satisfies following differential inequality

$$\partial_t u(t,x) \geq d(t)\partial_{xx}u(t,x)+r_1(t)u(t,x)\Big(1-Lu(t,x)-L\big(\delta+M_\delta u(t,x)\big)\Big),\ \forall t\geq T_\delta,\ \forall x\in\mathbb{R}.$$

Let $\underline{u} = \underline{u}(t,x)$ be the solution of following equation for all $t > 0$ and $x \in \mathbb{R}$,

$$\partial_t\underline{u}(t,x) = d(t+T_\delta)\partial_{xx}\underline{u}(t,x) + r_1(t+T_\delta)\underline{u}(t,x)\Big(1-L\delta-L(1+M_\delta)\underline{u}(t,x)\Big), \quad (4.3.15)$$

equipped with a nontrivial continuous initial data $\underline{u}(0,\cdot)$ which is compactly supported, bounded and that satisfies $\underline{u}(0,\cdot) \leq u(T_\delta,\cdot)$. Then the comparison principle implies that

$$u(t+T_\delta,x) \geq \underline{u}(t,x),\ \forall t>0,\ \forall x\in\mathbb{R}.$$

Let us define the quantity $c_u^*(\delta)$ for all $\delta \in [0,\frac{1}{2L})$, regarded as the perturbation of c_u^*, given by

$$c_u^*(\delta) := 2\sqrt{\langle d\rangle\langle r_1\rangle(1-L\delta)}.$$

From the spreading speed results for scalar equation (4.3.15) (see [21, 23, 124]), one has for all $c \in [0,c_u^*(\delta))$,

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} u(t+T_\delta,x) \geq \lim_{t\to\infty} \inf_{|x|\leq ct} \underline{u}(t,x) = \frac{1-L\delta}{L+LM_\delta} > 0.$$

Due to the arbitrariness of $c \in [0,c_u^*(\delta))$, one can get rid of T_δ. So one has

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} u(t,x) > 0,\ \forall c\in[0,c_u^*(\delta)).$$

Note that $\delta \mapsto c_u^*(\delta)$ is a continuous and decreasing function defined in $\left[0, \frac{1}{2L}\right)$. Obviously one has $c_u^*(0) = c_u^*$ and $c_u^*(\delta) < c_u^*$ for all $\delta \in \left(0, \frac{1}{2L}\right)$. Thus, for $0 \leq \tilde{c} < c_u^*$, there exists $\delta' > 0$ small enough such that $\tilde{c} < c_u^*(\delta') < c_u^*$. Combining with the above limit, one has

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} u(t,x) > 0, \ \forall c \in [0, c_u^*).$$

The proof is completed. □

Next we prove another important inequality which will be used to compare the solution of the prey-predator system with the one of a suitable Fisher-KPP problem.

Lemma 4.3.4. *Fix $c \in [0, c_u^*)$. For each $\alpha > 0$, there exist $M_\alpha > 0$ and $T_\alpha > 0$ such that the following estimate holds true*

$$1 - u(t,x) \leq \alpha + M_\alpha v(t,x), \ \forall t \geq T_\alpha, |x| \leq ct.$$

Proof. By contradiction, we assume that there exist $\alpha_0 > 0$, sequences $(t_n)_n$ and $(x_n)_n$ such that

$$\begin{aligned} &|x_n| \leq ct_n, t_n \to \infty \text{ as } n \to \infty, \\ &\text{and } 1 - u(t_n, x_n) > \alpha_0 + nv(t_n, x_n), \ \forall n \geq 1. \end{aligned} \tag{4.3.16}$$

Set

$$u^n(t,x) := u(t + t_n, x + x_n) \text{ and } v^n(t,x) := v(t + t_n, x + x_n).$$

By parabolic estimates, one can extract the subsequence such that $u^n(t,x) \to u^\infty(t,x)$ and $v^n(t,x) \to v^\infty(t,x)$ as $n \to \infty$ locally uniformly for $(t,x) \in \mathbb{R}^2$ with $(u^\infty, v^\infty) \in S$. As discussed in Section 4.3.1, there exists $\tilde{\sigma} \in \Sigma$ such that (u^∞, v^∞) satisfies $(\mathbf{P}_{\tilde{\sigma}})$ (see (4.3.12)). Recalling $0 \leq u \leq 1$ note that assumption (4.3.16) implies that

$$v(t_n, x_n) \to 0 \text{ as } n \to \infty.$$

Hence we obtain that $v^\infty(0,0) = 0$ and the strong maximum principle for v^∞-equation implies that $v^\infty \equiv 0$. As a consequence $u^\infty = u^\infty(t,x)$ satisfies following Fisher-KPP equation

$$\partial_t u^\infty(t,x) = \tilde{d}(t)\partial_{xx} u^\infty(t,x) + u^\infty(t,x)\tilde{f}\left(t, u^\infty(t,x), 0\right), \ \forall (t,x) \in \mathbb{R}^2. \tag{4.3.17}$$

Next we claim that the following property holds.

Claim 4.3.5. *One has*

$$\inf_{(t,x)\in\mathbb{R}^2} u^\infty(t,x) > 0.$$

Proof of Claim 4.3.5. In Proposition 4.3.3, we obtain that

$$\liminf_{t\to\infty} \inf_{|x|\leq c't} u(t,x) > 0, \forall c' \in [0, c_u^*).$$

Fix $c' \in (c, c_u^*)$. Let (t_n, x_n) be the sequence (possibly sub-sequence) defined at the beginning of the proof for Lemma 4.3.4. Then there exists $T > 0$ large enough and $m > 0$ such that

$$\inf_{t\geq T} \inf_{|x|\leq c't} u(t,x) \geq m.$$

This rewrites as for all $n \geq 0$, $t \geq T - t_n$ and $|x + x_n| \leq c'(t + t_n)$,

$$u(t + t_n, x + x_n) \geq m.$$

Since $|x_n| \leq ct_n$, this also rewrites as for all $n \geq 0$, $t \geq T - t_n$ and $|x| \leq (c' - c)t_n + c't$:

$$u(t + t_n, x + x_n) \geq m.$$

Since $c' - c > 0$, passing to the limit $n \to \infty$ (possibly along a subsequence) we end-up with

$$u^\infty(t, x) \geq m, \ \forall (t, x) \in \mathbb{R}^2,$$

which completes the proof of Claim 4.3.5. □

We come back to the proof of Lemma 4.3.4. Recall that $f(t + t_n, u, v)$ converges to $\tilde{f}(t, u, v)$ locally uniformly for $t \in \mathbb{R}$ and $(u, v) \in [0, \infty)^2$. Since $f(t, 1, 0) \equiv 0$, one has $\tilde{f}(t, 1, 0) = 0$. Since $\inf_{t \geq 0} f(t, u, 0) > 0$ for each $u \in [0, 1)$, then $\inf_{t \in \mathbb{R}} \tilde{f}(t, u, 0) > 0$ for each $u \in [0, 1)$. Set

$$\Theta := \inf_{(t,x) \in \mathbb{R}^2} u^\infty(t, x) \text{ and } \tilde{h}(u) := \inf_{t \in \mathbb{R}} \tilde{f}(t, u, 0).$$

Note that $\Theta > 0$ and $\tilde{h}(u) > 0$ for all $u \in [0, 1)$. Next we consider $U(t)$, the solution of

$$U'(t) = U(t)\tilde{h}(U(t)), \ \ U(0) = \Theta.$$

Observe that it is a sub-solution of (4.3.17) and since $u^\infty(s, x) \geq \Theta$ for all $(s, x) \in \mathbb{R}^2$, then the comparison principle implies that

$$1 \geq u^\infty(t + s, x) \geq U(t), \ \ \forall t \geq 0, s \in \mathbb{R}, x \in \mathbb{R}.$$

Finally since $U(t) \to 1$ as $t \to \infty$, one has $u^\infty(0, 0) = 1$, which contradicts the property $1 - u^\infty(0, 0) \geq \alpha_0 > 0$ that follows by passing to the limit $n \to \infty$ into the assumption (4.3.16). The proof is completed. □

4.3.3 Proof of Theorem 4.1.5 (ii)

Now we complete the proof of Theorem 4.1.5 (*ii*).

Proof of Theorem 4.1.5 (*ii*). By contradiction, we fix $c_v^* < c_1 < c_2 < c_u^*$ and assume that there exist sequences $(t_n, x_n)_n$ such that

$$\begin{aligned} &t_n \to \infty \text{ as } n \to \infty, \\ &c_1 t_n \leq |x_n| \leq c_2 t_n, \ \forall n \geq 0 \\ &\text{and } \limsup_{n \to \infty} u(t_n, x_n) < 1. \end{aligned}$$

Set $u_n(t, x) := u(t + t_n, x + x_n)$ and $v_n(t, x) := v(t + t_n, x + x_n)$. As above (see Section 4.3.1) there exists $(u_\infty, v_\infty) \in S$ and $\tilde{\sigma} \in \Sigma$ such that $u_n(t, x) \to u_\infty(t, x)$ and $v_n(t, x) \to v_\infty(t, x)$ locally uniformly for $(t, x) \in \mathbb{R}^2$ as $n \to \infty$ while (u_∞, v_∞) satisfies $(\mathbf{P}_{\tilde{\sigma}})$(see (4.3.12)). Note also that one has $u_\infty(0, 0) < 1$.

Now observe that we have proved that for all $c_1' > c_v^*$,

$$\lim_{t \to \infty} \sup_{|x| \geq c_1' t} v(t, x) = 0.$$

This ensures that $v_\infty(0, 0) = 0$ and the strong maximum principle for v_∞-equation implies that $v_\infty \equiv 0$. Hence u_∞ becomes a solution of the problem

$$\partial_t u_\infty(t, x) = \tilde{d}(t)\partial_{xx} u_\infty(t, x) + u_\infty(t, x)\tilde{f}\left(t, u_\infty(t, x), 0\right).$$

Recalling Proposition 4.3.3, one also has, for any $0 < \varepsilon < \min\{c_u^* - c_2, c_1 - c_v^*\}$ small enough,

$$\liminf_{t\to\infty} \inf_{(c_1-\varepsilon)t\le|x|\le(c_2+\varepsilon)t} u(t,x) > 0.$$

Then one can proceed similarly as in the proof of Lemma 4.3.4 to obtain that $u_\infty(0,0) = 1$, a contradiction with $u_\infty(0,0) < 1$. The proof of Theorem 4.1.5 (ii) is over. □

4.3.4 Proof of Theorem 4.1.5 (iii) and Theorem 4.1.6 (ii)

In this subsection, we complete the proof of our inner spreading results. In order to prove Theorem 4.1.5 (iii) and Theorem 4.1.6 (ii) simultaneously, we define

$$c_* := \min\{c_u^*, c_v^*\}.$$

To prove our spreading result, let us first recall the following well known eigenvalue result. We omit the proof in here.

Lemma 4.3.6. *Let $c \in \mathbb{R}$ and $R > 0$ be given. Then the principal eigenvalue λ_R of the following Dirichlet elliptic problem*

$$\begin{cases} -\phi''(x) - c\phi'(x) = \lambda_R\phi(x), & x \in (-R,R), \\ \phi(\pm R) = 0 \text{ and } \phi > 0 \text{ in } (-R,R), \end{cases}$$

is given by

$$\lambda_R = \frac{c^2}{4} + \frac{\pi^2}{4R^2}.$$

Now observe that in Proposition 4.3.3 we already obtain that for all $c \in [0, c_u^*)$,

$$\liminf_{t\to\infty} \inf_{|x|\le ct} u(t,x) > 0.$$

Since $c_* \le c_u^*$, one has

$$\liminf_{t\to\infty} \inf_{|x|\le ct} u(t,x) > 0,\ \forall c \in [0, c_*).$$

Therefore it remains to show that

$$\liminf_{t\to\infty} \inf_{|x|\le ct} v(t,x) > 0 \text{ and } \limsup_{t\to\infty} \sup_{|x|\le ct} u(t,x) < 1, \quad \forall c \in [0, c_*).$$

The proof of these results shall make use of our key Lemma 4.3.4. Roughly speaking, we will derive a differential inequality satisfied by v and construct a suitable sub-solution to show that

$$\liminf_{t\to\infty} v(t, \pm ct) > 0 \text{ for all } c \in [0, c_*).$$

Finally we will make use a positive constant number as a sub-solution on a moving domain to conclude to our inner spreading result. Our first lemma reads as follows.

Lemma 4.3.7. *The solution v satisfies*

$$\liminf_{t\to\infty} v(t, \pm ct) > 0,\ \forall c \in [0, c_*).$$

Proof. We only show that $\liminf_{t\to\infty} v(t,ct) > 0$. The case of $\liminf_{t\to\infty} v(t,-ct) > 0$ can be proved similarly using a symmetrical argument. Fix $c \in [0, c_*)$ and let $c' \in (c, c_*)$ be given. Since $c_* \leq c_u^*$, Lemma 4.3.4 implies that, for any $\alpha > 0$, there exist $M_\alpha > 0$ and $T_\alpha > 0$ such that

$$1 - u(t,x) \leq \alpha + M_\alpha v(t,x), \ \forall t \geq T_\alpha, |x| \leq c't. \tag{4.3.18}$$

Combining this estimate with (4.1.6), one gets that $v = v(t,x)$ satisfies for all $t \geq T_\alpha$ and $x \in [-c't, c't]$,

$$\partial_t v(t,x) \geq \partial_{xx} v(t,x) + r_2(t)v(t,x)\big(1 - L\alpha - L(1 + M_\alpha)v(t,x)\big). \tag{4.3.19}$$

Let us consider $w(t,x) := v(t + T_\alpha, x)$ which satisfies for all $t \geq 0$ and $|x| \leq c'(t + T_\alpha)$ the problem

$$\partial_t w(t,x) \geq \partial_{xx} w(t,x) + r_2(t + T_\alpha)w(t,x)\big(1 - L\alpha - L(1 + M_\alpha)w(t,x)\big). \tag{4.3.20}$$

Next for each $\alpha \in [0, \frac{1}{2L}]$, we define $c_v^*(\alpha)$ given by

$$c_v^*(\alpha) := 2\sqrt{\langle r_2\rangle(1 - L\alpha)}.$$

One may observe that $\alpha \mapsto c_v^*(\alpha)$ is a continuous and decreasing function on $[0, \frac{1}{2L}]$ with $c_v^*(0) = c_v^*$. Recalling that $0 \leq c < c_* \leq c_v^*$, one can choose some $\alpha = \alpha_c > 0$ sufficiently small such that $c < c_v^*(\alpha) < c_v^*$. With such a choice, it rewrites as

$$c^2 < (c_v^*(\alpha))^2 = 4\langle r_2\rangle(1 - L\alpha).$$

The reformulation (4.1.2) ensures that one can choose some $a \in W^{1,\infty}(0,\infty)$ and some $\theta_0 > 0$ such that

$$\frac{c^2}{4} - r_2(t)(1 - L\alpha) + a'(t) \leq -2\theta_0, \ \ \forall t \geq 0.$$

Choose $R > 0$ large enough such that

$$\frac{\pi^2}{4R^2} + \frac{c^2}{4} - r_2(t)(1 - L\alpha) + a'(t) \leq -\theta_0, \ \ \forall t \geq 0. \tag{4.3.21}$$

Now for $\eta > 0$ to be chosen latter, let ϕ be the eigenfunction corresponding to the principal eigenvalue λ_R (see Lemma 4.3.6). Then we define

$$\underline{v}(t,x) := \begin{cases} \eta\phi(x - ct)e^{a(t+T_\alpha)}, & t \geq 0, x \in [-R + ct, R + ct], \\ 0, & \text{else.} \end{cases}$$

where T_α is defined above for $\alpha = \alpha_c$. Note that one can choose T_α large enough such that $c'T_\alpha > R$. So one has

$$[-R + ct, R + ct] \subset [-c'(t + T_\alpha), c'(t + T_\alpha)] \text{ for all } t \geq 0.$$

In addition note that (4.3.21) implies

$$\frac{\pi^2}{4R^2} + \frac{c^2}{4} - r_2(t + T_\alpha)(1 - L\alpha) + a'(t + T_\alpha) \leq -\theta_0, \ \ \forall t \geq 0. \tag{4.3.22}$$

On the other hand, straightforward computations yield for all $t \geq 0$ and $x \in [-R+ct, R+ct]$,

$$\begin{aligned}
&\partial_t \underline{v}(t,x) - \partial_{xx}\underline{v}(t,x) - r_2(t+T_\alpha)(1-L\alpha)\underline{v}(t,x) \\
&\quad = a'(t+T_\alpha)\underline{v}(t,x) - c\eta\phi'(x-ct)e^{a(t+T_\alpha)} - \eta\phi''(x-ct)e^{a(t+T_\alpha)} \\
&\qquad - r_2(t+T_\alpha)(1-L\alpha)\underline{v}(t,x) \\
&\quad = \left(a'(t+T_\alpha) + \frac{c^2}{4} + \frac{\pi^2}{4R^2} - r_2(t+T_\alpha)(1-L\alpha)\right)\underline{v}(t,x) \\
&\quad \leq -\theta_0 \underline{v}(t,x).
\end{aligned}$$

Let us choose $\eta > 0$ small enough such that

$$r_2(t+T_\alpha)L(1+M_\alpha)\underline{v}(t,x) \leq \eta \|r_2\|_\infty L(1+M_\alpha)\|\phi\|_\infty e^{\|a\|_\infty} < \theta_0.$$

So that for all $t \geq 0$ and $x \in [-R+ct, R+ct]$, one has

$$-\theta_0 \underline{v}(t,x) \leq -r_2(t+T_\alpha)L(1+M_\alpha)\underline{v}^2(t,x).$$

Hence $\underline{v}(t,x)$ is the sub-solution of (4.3.20). One can furthermore choose $\eta > 0$ small enough such that $w(0,x) = v(T_\alpha, x) \geq \underline{v}(0,x)$ for all $x \in [-R, R]$. Since $\underline{v}(t, \pm R + ct) = 0$ for all $t \geq 0$, the comparison principle on domain $\{(t,x) : t \geq 0, x \in [-R+ct, R+ct]\}$ applies and ensures that

$$w(t,x) = v(t+T_\alpha, x) \geq \underline{v}(t,x), \ \forall t \geq 0, x \in [-R+ct, R+ct].$$

Thus, we have obtained that

$$\liminf_{t\to\infty} v\left(t+T_\alpha, ct\right) \geq \liminf_{t\to\infty} \underline{v}(t,ct) > 0.$$

Due to the arbitrariness of $c \in [0, c_*)$, we obtain that

$$\liminf_{t\to\infty} v(t,ct) > 0, \ \forall c \in [0, c_*).$$

The proof is completed. □

Next we complete the proof of Theorem 4.1.5 and 4.1.6.

Lemma 4.3.8. *The solution of (4.1.3) satisfies:*

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} v(t,x) > 0, \ \forall c \in [0, c_*),$$

and

$$\limsup_{t\to\infty} \sup_{|x|\leq ct} u(t,x) < 1, \ \forall c \in [0, c_*).$$

Proof. Fix $c \in [0, c_*)$ and let $c' \in (c, c_*)$ be given. As in the proof of Lemma 4.3.7, the function v satisfies (4.3.19) which is written as follows

$$\partial_t v(t,x) \geq \partial_{xx} v(t,x) + r_2(t)v(t,x)\big(1 - L\alpha - L(1+M_\alpha)v(t,x)\big), \text{ for } (t,x) \in \Omega.$$

Wherein we have set

$$\Omega := \left\{(t,x) \in \mathbb{R}^2 : t \geq T_\alpha \text{ and } |x| \leq c't\right\}.$$

For some $\tilde{\eta} > 0$ small enough, we define $\underline{v}(t,x) := \tilde{\eta}$ for all $(t,x) \in \Omega$. One can verify that $\underline{v}$ is the sub-solution of (4.3.19) provided that $\tilde{\eta} < \frac{1-L\alpha}{2L(1+M_\alpha)}$.

Since $v_0 \not\equiv 0$, then the strong maximum principle for v-equation implies that $v(t,x) > 0$ for $t > 0$ and $x \in \mathbb{R}$. One can choose $\tilde{\eta} > 0$ smaller if necessary such that

$$\underline{v}(T_\alpha, x) = \tilde{\eta} \leq v(T_\alpha, x), \ \forall x \in [-c'T_\alpha, c'T_\alpha].$$

On the other hand, since $c' < c_*$, then Lemma 4.3.7 tells that choosing $\tilde{\eta} > 0$ even smaller such that

$$v(t, \pm c't) \geq \tilde{\eta} > 0, \ \forall t \geq T_\alpha.$$

Hence one can apply the comparison principle on the moving domain Ω to obtain that $v(t,x) \geq \underline{v}(t,x) = \tilde{\eta} > 0$ for all $(t,x) \in \Omega$. Thus, one has

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} v(t,x) > 0, \ \forall c \in [0, c_*).$$

Finally, let us consider the $u-$component and show that

$$\limsup_{t\to\infty} \sup_{|x|\leq ct} u(t,x) < 1, \ \forall c \in [0, c_*).$$

To that aim we proceed by contradiction again. Assume that there exist $c \in [0, c_*)$ and sequences $(y_n)_n$ and $(\tau_n)_n$ such that

$$\begin{aligned}&|y_n| \leq c\tau_n, \ \forall n \geq 0,\\ &\tau_n \to \infty \text{ and } u(\tau_n, y_n) \to 1, \quad \text{as } n \to \infty.\end{aligned}$$

Through extracting subsequence, one can obtain that $(u(t+\tau_n, x+y_n), v(t+\tau_n, x+y_n))$ converges locally uniformly to $(u_\infty, v_\infty) = (u_\infty(t,x), v_\infty(t,x)) \in S$ which is the entire solution of $(\mathbf{P}_{\tilde{\sigma}})$ (see (4.3.12)) with suitable $\tilde{\sigma} \in \Sigma$. Since $u_\infty(0,0) = 1$ and $0 \leq u_\infty \leq 1$, then the strong maximum principle for $u_\infty-$equation implies that $u_\infty \equiv 1$. Hence the first equation in $(\mathbf{P}_{\tilde{\sigma}})$ yields

$$\tilde{f}\,(t, 1, v_\infty(t,x)) = 0, \quad \forall (t,x) \in \mathbb{R}^2.$$

On the other hand, we have already proved

$$\liminf_{t\to\infty} \inf_{|x|\leq c't} v(t,x) > 0, \ \forall c' \in [0, c_*).$$

So that using similar arguments as in the proof of Claim 4.3.5, one has

$$\inf_{(t,x)\in\mathbb{R}^2} v_\infty(t,x) > 0.$$

Recalling Assumption 4.1.3 $(f5)$ which ensures that $\sup_{t\in\mathbb{R}} \tilde{f}(t,1,v) < 0$ for all $v > 0$, we have reach a contradiction which completes the proof of the result. □

4.4 Proof of Proposition 4.1.9

From Remark 4.1.8 and Proposition 4.1.9, one can show that the solutions are bounded for large classes of systems. In this last section, for sake of completeness, we prove Proposition 4.1.9 in detail. The proof is close to some ideas derived in [4, 55].

Proof of Proposition 4.1.9. As already noticed, one knows that

$$0 \leq u \leq 1 \text{ and } v \geq 0.$$

Therefore to prove the proposition, it is sufficient to check that v is bounded. To do so, first note that the function $(t,x) \mapsto \kappa e^{\|r_2\|_\infty t}$ is a super-solution of v-equation, with $\kappa > 0$ sufficiently large so that $v_0 \leq \kappa$. As a consequence, one has $v(t,\cdot) \in L^\infty(\mathbb{R})$ for all $t \geq 0$. In order to prove the proposition, we argue by contradiction by assuming that

$$\limsup_{t\to\infty} \|v(t,\cdot)\|_{L^\infty(\mathbb{R})} = \infty.$$

Next for each $n \geq 0$ large enough, we choose $t_n > 0$ such that

$$t_n := \min\left\{t > 0 : \ \|v(t,\cdot)\|_\infty = n\right\}.$$

Observe that $t_n \to \infty$ as $n \to \infty$. Next pick a sequence $(x_n)_n$ such that for all n (large enough)

$$v(t_n, x_n) \in \left[\frac{n}{2}, n\right],$$

and let us define the sequence of function

$$\tilde{v}_n(t,x) = \frac{v(t+t_n, x+x_n)}{v(t_n,x_n)}.$$

Note that the function $(t,x) \mapsto \tilde{v}_n(t,x)$ satisfies for $t \geq -t_n$ and $x \in \mathbb{R}$ the equation

$$\partial_t \tilde{v}_n(t,x) = \partial_{xx}\tilde{v}_n(t,x) + \tilde{v}_n(t,x) g\left(t+t_n, u\left(t+t_n, x+x_n\right), v\left(t+t_n, x+x_n\right)\right), \quad (4.4.23)$$

together with the normalization condition $\tilde{v}_n(0,0) = 1$.

We claim that sequence $\tilde{v}_n$ is locally uniformly bounded. Indeed, for $t \leq 0$, from the construction of t_n, one has for all $n \geq 1$:

$$\tilde{v}_n(t,x) \leq \|v(t+t_n,\cdot)\|_\infty \cdot \frac{2}{n} \leq 2, \ \text{ for } t \in [-t_n, 0], \ x \in \mathbb{R}.$$

On the other hand for $t > 0$, the local uniform boundedness follows from the super-solution $(t,x) \mapsto 2e^{\|r_2\|_\infty t}$. Then by parabolic estimates, one can extract a converging sub-sequence, still denoted with the same index, such that

$$\tilde{v}_n(t,x) \to \tilde{v}_\infty(t,x), \text{ locally uniformly for } (t,x) \in \mathbb{R}^2, \text{ as } n \to \infty.$$

Note that $\|r_2\|_\infty \geq g(t,u,v) \geq g(t,u,\infty) \geq \inf_{t\geq 0} g(t,0,\infty) > -\infty$ for all $t \geq 0$, $u \in [0,1]$ and $v \geq 0$. Hence one has $\tilde{v}_\infty$ satisfies

$$\partial_t \tilde{v}_\infty(t,x) = \partial_{xx}\tilde{v}_\infty(t,x) + \tilde{v}_\infty(t,x)\tilde{g}_\infty(t,x), \quad (4.4.24)$$

where $\tilde{g}_\infty$ is the $L^\infty_{\text{loc}}(\mathbb{R}^2)$ weak-$\star$ limit of $g_n(t,x) := g(t+t_n, u(t+t_n, x+x_n), v(t+t_n, x+x_n))$.

From the construction, one has $\tilde{v}_\infty(0,0) = 1$. The strong maximum principle implies that $\tilde{v}_\infty(t,x) > 0$ for all $(t,x) \in \mathbb{R}^2$. We can conclude that

$$v(t+t_n, x+x_n) \to \infty, \ \text{ locally uniformly for } (t,x) \in \mathbb{R}^2 \text{ as } n \to \infty. \quad (4.4.25)$$

Next we claim that u satisfies the following limit.

Claim 4.4.1. *One has*

$$\lim_{n\to\infty} u(t+t_n, x+x_n) = 0, \text{ locally uniformly for } (t,x) \in \mathbb{R}^2. \tag{4.4.26}$$

Proof of claim 4.4.1 . To prove this claim, let $T > 0$ and $R > 0$ be given. Recall that $\langle f(\cdot, 0, M)\rangle < 0$ for all $M \geq M_0$ (see assumption (4.1.10)). Let $M > M_0$ be given. We consider $u^n_{T,R} = u^n_{T,R}(t,x)$ which satisfies

$$\begin{cases} \partial_t u^n_{T,R} = d(t+t_n)\partial_{xx} u^n_{T,R} + u^n_{T,R} f\left(t+t_n, u^n_{T,R}, M\right), & |t| < T, |x| \leq R, \\ u^n_{T,R}(-T,x) = 1, & |x| \leq R, \\ u^n_{T,R}(t, \pm R) = 1, & |t| < T. \end{cases}$$

From (4.4.25), one can choose $N_0 > 0$ (which may depend on T, R and M) large enough such that

$$v(t+t_n, x+x_n) \geq M, \text{ for } |t| \leq T, |x| \leq R \text{ and } n \geq N_0.$$

Since $v \mapsto f(\cdot,\cdot,v)$ is decreasing (see Assumption 4.1.3), then the comparison principle implies that

$$u(t+t_n, x+x_n) \leq u^n_{T,R}(t,x), \quad \text{for } |t| \leq T, |x| \leq R \text{ and } n \geq N_0.$$

For $B > 0$, $T > 0$, $|t| \leq T$ and $n \geq N_0$, we define

$$\overline{u^n}(t;-T) := B \exp\left\{\int_{-T}^{t} f(s+t_n, 0, M) \mathrm{d}s\right\},$$

Since $f(t,u,v) \leq f(t,0,v)$ for all $t \geq 0$, $u \in [0,1]$ and $v \geq 0$, then one can verify that $\overline{u^n}(t;-T)$ satisfies

$$\partial_t \overline{u^n}(t;-T) \geq d(t+t_n)\partial_{xx}\overline{u^n}(t;-T) + \overline{u^n}(t;-T) f\left(t+t_n, \overline{u^n}(t;-T), M\right).$$

Let $B > \max\left\{1, e^{2T\|f(\cdot,0,M)\|_\infty}\right\}$ be chosen. Therefore, one gets $\overline{u^n}(-T;-T) \geq u^n_{T,R}(-T,x)$ for all $|x| \leq R$ and $\overline{u^n}(t;-T) \geq u^n_{T,R}(t,\pm R)$ for all $|t| < T$. Then the comparison principle implies that

$$u^n_{T,R}(t,x) \leq \overline{u^n}(t;-T), \quad \text{for } |t| \leq T, |x| \leq R \text{ and } n \geq N_0.$$

Due to $\langle f(\cdot,0,M)\rangle < 0$, one has for each $|t| < T$ and $n \geq N_0$,

$$\lim_{T\to\infty} \int_{-T}^{t} f(s+t_n,0,M)\mathrm{d}s = \lim_{T\to\infty} (t+T)\cdot \frac{1}{t+T}\int_{-T}^{t} f(s+t_n,0,M)\mathrm{d}s = -\infty.$$

So that for all $n \geq N_0$, one has

$$\limsup_{T\to\infty, R\to\infty} u^n_{T,R}(t,x) \leq \limsup_{T\to\infty} B\exp\left\{\int_{-T}^{t} f(s+t_n,0,M)\mathrm{d}s\right\} = 0,$$

locally uniformly for $(t,x) \in \mathbb{R}^2$. We end-up with

$$\lim_{n\to\infty} u(t+t_n, x+x_n) = 0, \text{ locally uniformly for } (t,x) \in \mathbb{R}^2,$$

and the claim is proved. □

Now we come back to (4.4.23). Recalling Assumption 4.1.4, one can note that for all $v \geq 0$ and $t \geq 0$,

$$g(t, 0, v) \leq g(t, 0, 0).$$

Recalling (4.4.24) and (4.4.26), one has

$$\tilde{g}_\infty(t, x) = \lim_{n\to\infty} g(t + t_n, 0, v(t + t_n, x + x_n)) \text{ in } L^\infty_{\text{loc}}(\mathbb{R}^2) \text{ weak-}\star \text{ topology}.$$

Since one has

$$g(t + t_n, 0, v(t + t_n, x + x_n)) \leq g(t + t_n, 0, 0),$$

then for all $t \in \mathbb{R}$, one gets

$$\sup_{x\in\mathbb{R}} \tilde{g}_\infty(t, x) \leq \tilde{g}(t, 0, 0),$$

where $\tilde{g}(t, 0, 0)$ is the limit of $g(t + t_n, 0, 0)$ in local uniform topology. So one has

$$\langle \sup_{x\in\mathbb{R}} \tilde{g}_\infty(\cdot, x)\rangle \leq \langle \tilde{g}(\cdot, 0, 0)\rangle = \langle g(\cdot, 0, 0)\rangle,$$

and one can choose $a \in W^{1,\infty}(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}} \tilde{g}_\infty(t, x) + a'(t) \leq \langle g(\cdot, 0, 0)\rangle\,, \ \forall t \in \mathbb{R}.$$

Then one can check that for all $t_0 > 0$ and $b > 0$, the function

$$\overline{\overline{v}}(t; -t_0) := b \exp\left\{\langle g(\cdot, 0, 0)\rangle(t + t_0) - a(t)\right\}$$

is a super-solution of (4.4.24). Since $\tilde{v}_\infty(t, x) \leq 2$ for all $t \leq 0$ and $x \in \mathbb{R}$, then one can choose $b > 0$ large enough such that for all $t_0 > 0$ and $x \in \mathbb{R}$,

$$\tilde{v}_\infty(-t_0, x) \leq 2 \leq be^{-\|a\|_\infty} \leq \overline{\overline{v}}(-t_0; -t_0).$$

The comparison principle implies that

$$\tilde{v}_\infty(0, 0) \leq \overline{\overline{v}}(0; -t_0) = b \exp\left\{\langle g(\cdot, 0, 0)\rangle t_0 - a(0)\right\}.$$

Letting $t_0 \to \infty$, one has $\tilde{v}_\infty(0, 0) = 0$ due to $\langle g(\cdot, 0, 0)\rangle < 0$. This is in contradiction with $\tilde{v}_\infty(0, 0) = 1$ and completes the proof of the proposition. □

Chapter 5

Spreading speeds for time heterogeneous prey-predator systems with nonlocal diffusion on lattice

This is a joint work with Arnaud Ducrot, in preparation.

Abstract

We investigate the spreading behaviour of solutions to a class prey-predator system in lattice with time heterogeneities. These time variations are assumed to enjoy an averaging property including periodicity, almost periodicity and unique ergoidicty as special cases. The spatial motion of individuals from one site to another site is modeled by a discrete convolution operator. In order to take into account exterior fluctuations such as seasonality, daily variation and so on, the convolution kernels and reaction terms are considered to vary with time. In this work, the prey and the predator are assumed to be able to co-invade the empty environment. We prove that the solutions with suitable initial data exhibit definite spreading speed by comparing these solutions with a non-autonomous KPP scalar equation on the lattice.

5.1 Introduction

In this paper, we study the large time behaviour for solutions of the following lattice differential system

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t,i) = \sum_{j\in\mathbb{Z}} J_1(t,j)\left[u(t,i-j)-u(t,i)\right] + u(t,i)f\left(t,u(t,i),v(t,i)\right),\ t>0, i\in\mathbb{Z}, \\ \frac{\mathrm{d}}{\mathrm{d}t}v(t,i) = \sum_{j\in\mathbb{Z}} J_2(t,j)\left[v(t,i-j)-v(t,i)\right] + v(t,i)g\left(t,u(t,i),v(t,i)\right),\ t>0, i\in\mathbb{Z}, \end{cases} \tag{5.1.1}$$

which is supplemented with bounded and nonnegative initial data

$$u(0,i)=u_0(i) \text{ and } v(0,i)=v_0(i),\ i\in\mathbb{Z}.$$

Herein the two sets $\{i\in\mathbb{Z};\ u_0(i)\neq 0\}\neq\emptyset$ and $\{i\in\mathbb{Z};\ v_0(i)\neq 0\}\neq\emptyset$ have finite elements.

On one hand, lattice differential equations/systems arise in several different contexts, for instance modeling species grow over patchy environments, we refer the reader to [16, 96, 115] and to [47] for a list of ecological scenarios with patched environments. Lattice equations can also be used to describe phase transition, see [14]. On the other hand, lattice equations/systems can also be regarded as a discretization of differential equations in which the spatial variable are continuous. The propagation phenomena in lattice single equations and systems have attracted a lot of interest. For travelling wave solutions, we refer the reader to [174, 32, 35, 79, 81] and references cited therein. For spreading speed results, we refer to [31, 65, 115, 139, 24] and references cited therein. In this work, we are interested in the asymptotic speed of spread for solutions to (5.1.1) which stands for a nonlocal diffusion lattice system of prey-predator type. To the best of our knowledge, the spreading speed for a single KPP type equation with nonlocal diffusion in general time heterogeneous environment is still unknown before this work.

For a better exposition of our work, the detailed assumptions of (5.1.1) are postponed in the next section. Let us first introduce a typical example of prey-predator system that will be considered in this work: the Lotka-Volterra prey-predator system with nonlocal diffusion on lattice $\mathbb{Z}$. It reads as follows:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t,i) = \sum_{j\in\mathbb{Z}} J_1(t,j)\left[u(t,i-j)-u(t,i)\right] + u(t,i)\left(1-u(t,i)\right) - p(t)u(t,i)v(t,i), \\ \frac{\mathrm{d}}{\mathrm{d}t}v(t,i) = \sum_{j\in\mathbb{Z}} J_2(t,j)\left[v(t,i-j)-v(t,i)\right] + q(t)u(t,i)v(t,i) - \nu(t)v(t,i). \end{cases} \tag{5.1.2}$$

From the point view of biological, in (5.1.2), $u(t,i)$ and $v(t,i)$ denote the density of the prey and the predator at time t and location i respectively. The functions p, q, ν are positive and describe the predation rate, the conversion rate and the death rate of the predator, respectively.

Herein the kernel functions $J_1 = J_1(t,j)$ and $J_2 = J_2(t,j)$ are nonnegative and depend on time. The quantities $J_k(t,i-j)$, $(k=1,2)$ describe the probability of a species to jump from point j to i at time t. In (5.1.2), both the prey and the predator can exhibit long distance dispersal. In the last decades, most work have focused on the time independent diffusion kernel, that is $J_k(t,j)\equiv J_k(j)$. For travelling waves and spreading speed results of such lattice equations, we refer the reader to [14, 32, 115] and references cited therein. Since the seasonality and external influences varying with time, we consider time dependent dispersal kernel functions in this work. Note that the diffusion operator $\phi \mapsto \sum_{j\in\mathbb{Z}} J(t,j)[\phi(i-j)-\phi(i)]$ considered in here can be seen as a discretization of following convolution operator with spatial variable in continuous space $\phi \mapsto \int_{\mathbb{R}} J_k(t,y)[\phi(\cdot-y)-\phi(\cdot)]\mathrm{d}y$. The (generalized) travelling wave solution for KPP equations with this time dependent convolution operator has been investigated in [58].

Species usually live in fluctuating environment [93, 173]. Both the biotic factors (for instance the growth rate, the availability for food and the dispersion ability...) and the abiotic factors (such as temperature, wind, rainfall...) are varying with time. In particular, non-autonomous Lotka-Voterra prey-predator systems have attracted a lot of attention, see for example [45, 75].

As we mentioned at the beginning, the goal of this work is to investigate the asymptotic speed of spread for (5.1.1). The notion of spreading speed was introduced by Aronson and Weinberger [8]. For the case of discrete scalar equations, we refer to [162]. The spreading speed for KPP equations with nonlocal diffusion also have been studied in [114, 167]. In the last decades, spreading speed for homogeneous systems has drawn a lot of attention. We refer the reader to [164] for cooperative system. For competitive system, we refer to [99, 33, 77] for random diffusion case and to [169] for nonlocal diffusion case. We also refer the reader to Liang and Zhao [104, 105] for abstract monotone evolutionary system.

However, the prey-predator systems as in (5.1.2), are no longer monotone since this type interaction is not symmetric. Recently, spreading speed for some prey-predator systems in the homogeneous environment including a typical example as follows

$$\begin{cases} u_t = d_1 u_{xx} + u(1-u) - puv, \\ v_t = d_2 v_{xx} + quv - \nu v, \end{cases} \tag{5.1.3}$$

has been well studied using ideas from dynamical system, see [55]. Note that if we let J_k in (5.1.2) be given by

$$J_k(t,j) = \begin{cases} d_k, & j = \pm 1, \\ 0, & j \neq \pm 1, \end{cases} \quad (k = 1, 2),$$

and all parameters in (5.1.2) be constants, then (5.1.2) can be regarded as a spatially discrete approximation of (5.1.3). We also refer to [40] for the persistence of species in a prey-predator system with climate change and either local diffusion or nonlocal diffusion. The spreading speed for a two predators and one prey system was studied in [53]. For the large time behaviour of other types of prey-predator systems with random diffusion, we refer the reader to [37, 51, 110], for instance when the predator has a positive intrinsic growth rate or when the prey is abundant. For nonlocal dispersal prey-predator systems with continuous spatial variable, we refer the reader to [56], wherein spreading speed for the predator invading into the habitat of the aborigine prey has been studied, and to [172] for a study of the prey and the predator co-invading an empty space with large dispersal rate.

In the last decades, spreading speed for non-autonomous scalar equations with local diffusion and nonlocal diffusion in either continuous space or lattice, has drawn a lot of attention and been widely studied. For the case of local diffusion with continuous spatial variable, Nadin and Rossi [124] studied general time dependence case, Shen [140] considered time almost periodic and space periodic coefficients, Berestycki and his collaborators [21, 23] investigated the case of general heterogeneities in both time and space. For the case of nonlocal diffusion in continuous space, we refer to Jin *et al.* [93, 94] (for time periodic) and the references cited therein. For the case of spatially discrete equations, we refer the reader to Shen [139] (for time recurrent) and to Liang and Zhou [106] (for spatial heterogeneous). However, it seems that spreading speed result is still unknown for the general time dependent nonlocal diffusion lattice equation such as the following one which is derived from (5.1.2) when $v \equiv 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t,i) = \sum_{j\in\mathbb{Z}} J_1(t,j)\left[u(t,i-j) - u(t,i)\right] + u(t,i)\left(1 - u(t,i)\right). \tag{5.1.4}$$

As a by-product, in this work we also provide the spreading speed for (5.1.4).

To overcome the difficulty brought by general time heterogeneity, throughout this work, we assume that these time variations exhibit an averaging property. More precisely, we recall the notion of mean value for bounded functions which has been used in [124, 140]. We point out that this framework includes in particular time periodicity, almost periodicity and unique ergodicity according to the next definition.

Definition 5.1.1. *A function $h \in L^\infty(0,\infty;\mathbb{R})$ is said to have a mean value if the following limit exists,*

$$\langle h\rangle := \lim_{T\to\infty}\frac{1}{T}\int_0^T h(t+s)\mathrm{d}t, \ \textit{uniformly for } s \geq 0.$$

In that case the quantity $\langle h\rangle$ is called the mean value of h.

With the help of this notion, we can apply similar ideas as developed in [59] to derive spreading property for scalar non-autonomous lattice KPP equations with nonlocal disperse (see Section 5.3.4 and 5.3.5 for more details).

However, to the best of our knowledge, there are few results about spreading speed for non-autonomous prey-predator systems such as the Lotka-Volterra system (5.1.2), with general time variations, neither time periodic nor almost periodic. In non-autonomous monotone systems, the periodic case was studied in [66, 103] and the case of time almost periodic coefficients was considered in [12]. We also mention that [157] show some estimates for the spreading speeds of a time periodic prey-predator system where the predator has a positive intrinsic growth rate. Recently, the authors of this paper have obtained the exact spreading speed for a class non-autonomous prey-predator systems with local diffusion in [60].

In this work, we provide a new approach to study spreading speed for prey-predator system. Through some new local and pointwise estimates between $u(t,i)$ and $v(t,i)$, these somehow allow us to compare solutions of systems with those of Fisher-KPP scalar equations on suitable domains. Let us use (5.1.2) to explain the ideas of these estimates. Firstly, one can observe that the predator will starve in the absence of the prey. Hence if there is no prey, $u \equiv 0$, then v will degenerate to a solution of following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t,i) = \sum_{j\in\mathbb{Z}} J_2(t,j)\left[v(t,i-j) - v(t,i)\right] - \nu(t)v(t,i),$$

and v decays exponentially to 0. This observation yields our first estimate: for all $\delta > 0$ small enough, one can find some constants $M_\delta > 0$ and $T_\delta > 0$ such that

$$v(t,i) \leq \delta + M_\delta u(t,i), \ \forall t \geq T_\delta, i \in \mathbb{Z}.$$

Another important observation in (5.1.2) is following: when there is no predator, $v \equiv 0$, as noticed above, the density of the prey satisfies (5.1.4). If u invades successfully with some speed $c > 0$, then we can show that for all $\alpha > 0$, there exist some $M_\alpha > 0$ and $T_\alpha > 0$ such that

$$1 - u(t,i) \leq \alpha + M_\alpha v(t,i), \ \ \forall t \geq T_\alpha, \forall i \in [-ct, ct]\,.$$

With these estimates, we can analysis the spreading speed for (5.1.2).

In fact, we can extend these ideas to study the general system (5.1.1) with suitable shape of functions f and g. In the next section, we turn to the precise assumptions that will be needed to deal with (5.1.1) along this work and state our main spreading speed results for this system.

5.2 Assumptions and main results

In order to state our assumptions of the kernel function $J_k = J_k(t,i)$ for $(k = 1, 2)$, let us introduce the following definition, that will be referred along this work as the abscissa of convergence.

Definition 5.2.1. *Let $(X, \|\cdot\|_X)$ be a Banach space and $f \in l^1(\mathbb{Z}; X)$. We define the quantity, denoted by* $\text{abs}(f)$ *and called the abscissa of convergence of f, as follows*

$$\text{abs}(f) = \sup\left\{\lambda \geq 0 : \text{ the series } \sum_{j=-\infty}^{\infty} e^{\lambda j} f(j) \text{ converges in } X\right\}.$$

With the above notations, we first give the assumption of nonlocal diffusion kernel function $J_k = J_k(t,i)$, $(k = 1, 2)$.

Assumption 5.2.2 (Kernel $J_k = J_k(t,i)$)**.** *The kernel function $J_k : [0,\infty) \times \mathbb{Z} \to [0,\infty)$ for each $k = 1, 2$ satisfies the following set of assumptions:*

(J1) The function J_k is nonnegative and $J_k(\cdot, i) \in L^\infty(0,\infty)$ has a mean value for each $i \in \mathbb{Z}$;

(J2) The function $\hat{J}_k : i \mapsto J_k(\cdot, i)$ from $\mathbb{Z}$ to $L^\infty(0,\infty)$ whose series is absolutely convergent, that is $\hat{J}_k \in l^1(\mathbb{Z}, L^\infty(0,\infty))$. And we assume that its abscissa of convergence satisfies

$$\text{abs}(\hat{J}_k) > 0.$$

In the following, for simplicity of notation, we use $\text{abs}(J_k)$ *instead of* $\text{abs}(\hat{J}_k)$*;*

(J3) Assume that $J_k(\cdot, i) = J_k(\cdot, -i)$ for all $i \in \mathbb{Z}$ (symmetric);

(J4) The function J_k satisfies $\inf_{t \geq 0} J_k(t, \pm 1) > 0$;

(J5) Let the following limits hold true

$$\limsup_{\lambda \to \text{abs}(J_1)^-} \lambda^{-1}\left(\sum_{j\in\mathbb{Z}} \langle J_1(\cdot, j)\rangle\, e^{\lambda j}\right) = \limsup_{\gamma \to \text{abs}(J_2)^-} \gamma^{-1}\left(\sum_{j\in\mathbb{Z}} \langle J_2(\cdot, j)\rangle\, e^{\gamma j}\right) = \infty, \tag{5.2.5}$$

where $\langle J_k(\cdot, j)\rangle$, $(k = 1, 2)$ is the mean value of function $t \mapsto J_k(t, j)$, $(k = 1, 2)$ for each $j \in \mathbb{Z}$.

Next we state the assumptions for reaction terms f and g.

Assumption 5.2.3. *The function $f : [0,\infty)^3 \to \mathbb{R}$ satisfies:*

(f1) For each given $u, v \geq 0$, the function $t \mapsto f(t, u, v)$ is bounded and uniformly continuous from $[0,\infty)$ to $\mathbb{R}$, and $t \mapsto f(t, u, v)$ has a mean value $\langle f(\cdot, u, v)\rangle$, while the function $(u, v) \mapsto f(t, u, v)$ is Lipschitz continuous with respect to $u, v \geq 0$, uniformly for $t \geq 0$;

(f2) *For all $t \geq 0$ and $u > 0$, the map $v \mapsto f(t,u,v)$ is strictly decreasing;*

(f3) *Assume $f(t,0,0) = 1$ and $f(t,1,0) = 0$ for all $t \geq 0$ and*

$$h(u) := \inf_{t\geq 0} f(t,u,0) > 0 \text{ for all } u \in [0,1);$$

(f4) *For all $t \geq 0$ and $v \geq 0$, the map $u \mapsto f(t,u,v)$ is nonincreasing;*

(f5) *For all $v > 0$, it further satisfies $\sup_{t\geq 0} f(t,1,v) < 0$.*

Assumption 5.2.4. *The function $g : [0,\infty)^3 \to \mathbb{R}$ satisfies:*

(g1) *For each given $u, v \geq 0$, the function $t \mapsto g(t,u,v)$ is bounded and uniformly continuous from $[0,\infty)$ to $\mathbb{R}$, and $t \mapsto g(t,u,v)$ has a mean value $\langle g(\cdot,u,v)\rangle$. The function $(u,v) \mapsto g(t,u,v)$ is Lipschitz continuous with respect to $u, v \geq 0$, uniformly for $t \geq 0$;*

(g2) *For all $t \geq 0$ and $v \geq 0$, the map $u \mapsto g(t,u,v)$ is nondecreasing;*

(g3) *Set $r(t) := g(t,1,0)$. It satisfies $\inf_{t\geq 0} r(t) > 0$;*

(g4) *For all $t \geq 0$ and $u \geq 0$, the map $v \mapsto g(t,u,v)$ is nonincreasing;*

(g5) *Assume that the mean value of function $t \mapsto g(t,0,0)$ satisfies*

$$\langle g(\cdot,0,0)\rangle < 0.$$

Remark 5.2.5. *Note that in Assumption 5.2.3 $(f3)$, for simplicity, we assume that $f(t,0,0) \equiv 1$. Indeed, this can be relaxed through the time variable transformation to consider a more general assumption $f(t,0,0) = m(t)$ for $t \geq 0$. We refer the reader to [58, Remark 2.5] for more details about the transformation.*

Remark 5.2.6. *From the monotonicity and Lipschitz regularity of f and g, recalling that $(f3)$ and $(g3)$, one can choose some constant $L > 0$ such that for all $t \geq 0$, $u \in [0,1]$ and $v \geq 0$,*

$$\begin{aligned} &1 - Lu - Lv \leq f(t,u,v) \leq 1, \\ &r(t)\big(1 - L(1-u) - Lv\big) \leq g(t,u,v) \leq r(t). \end{aligned} \tag{5.2.6}$$

As explained for the typical example (5.1.2) in previous, here the function u and v stand for the prey and the predator density, respectively. Note that the Lotka-Volterra model (5.1.2) corresponds to (5.1.1) with

$$\begin{aligned} f(t,u,v) &= 1 - u - p(t)v, \\ g(t,u,v) &= q(t)u - \nu(t). \end{aligned}$$

It satisfies Assumption 5.2.3 and 5.2.4 provided satisfying additional smoothness and sign conditions.

In order to state our main results, we introduce some notations. Define two functions $c_u : (0, \mathrm{abs}(J_1)) \to L^\infty(0,\infty)$ and $c_v : (0, \mathrm{abs}(J_2)) \to L^\infty(0,\infty)$ by

$$\begin{aligned} c_u(\lambda)(\cdot) &:= \lambda^{-1}\left(\sum_{j\in\mathbb{Z}} J_1(\cdot,j)[e^{\lambda j} - 1] + 1\right), \ \forall \lambda \in (0, \mathrm{abs}(J_1)), \\ c_v(\gamma)(\cdot) &:= \gamma^{-1}\left(\sum_{j\in\mathbb{Z}} J_2(\cdot,j)[e^{\gamma j} - 1] + r(\cdot)\right), \ \forall \gamma \in (0, \mathrm{abs}(J_2)). \end{aligned} \tag{5.2.7}$$

Herein J_1 and J_2 satisfy Assumption 5.2.2 and r is given in Assumption 5.2.4 ($g3$). For each $\lambda \in (0, \text{abs}(J_1))$, $\gamma \in (0, \text{abs}(J_2))$ and $a \in W^{1,\infty}(0,\infty)$, we define $c_{u,a}(\lambda) \in L^\infty(0,\infty)$ and $c_{v,a}(\gamma) \in L^\infty(0,\infty)$ respectively by

$$c_{u,a}(\lambda)(\cdot) := c_u(\lambda)(\cdot) + a'(\cdot) \text{ and } c_{v,a}(\gamma)(\cdot) := c_v(\gamma)(\cdot) + a'(\cdot).$$

From the definition of mean value, one can observe that

$$\langle c_u \rangle = \langle c_{u,a} \rangle \text{ and } \langle c_v \rangle = \langle c_{v,a} \rangle.$$

The following two important quantities c_u^* and c_v^* related to speed, are defined by

$$c_u^* := \inf_{\lambda \in (0, \text{abs}(J_1))} \langle c_u(\lambda)(\cdot) \rangle \text{ and } c_v^* := \inf_{\gamma \in (0, \text{abs}(J_2))} \langle c_v(\gamma)(\cdot) \rangle. \tag{5.2.8}$$

Next, we state some properties of the speed functions $c_u(\lambda)$ and $c_v(\gamma)$ as follows.

Proposition 5.2.7. *Let Assumption 5.2.2 be satisfied and assume function $r \in L^\infty(0,\infty)$ has a mean value. Then the following properties hold:*

(i) Two functions $\lambda \mapsto \langle c_u(\lambda)(\cdot) \rangle$ from $(0, \text{abs}(J_1))$ to $\mathbb{R}$ and $\gamma \mapsto \langle c_v(\gamma)(\cdot) \rangle$ from $(0, \text{abs}(J_2))$ to $\mathbb{R}$ are of class C^1.

(ii) There exist $\lambda^* \in (0, \text{abs}(J_1))$ and $\gamma^* \in (0, \text{abs}(J_2))$ such that

$$\langle c_u(\lambda^*)(\cdot) \rangle = c_u^* \text{ and } \langle c_v(\gamma^*)(\cdot) \rangle = c_v^*.$$

Moreover, the map $\lambda \mapsto \langle c_u(\lambda)(\cdot) \rangle$ is decreasing on $(0, \lambda^*)$ and the map $\gamma \mapsto \langle c_v(\gamma)(\cdot) \rangle$ is decreasing on $(0, \gamma^*)$.

(iii) One has

$$\left.\frac{\mathrm{d}\langle c_u(\lambda) \rangle}{\mathrm{d}\lambda}\right|_{\lambda=\lambda^*} = 0 \text{ and } \left.\frac{\mathrm{d}\langle c_v(\gamma) \rangle}{\mathrm{d}\gamma}\right|_{\gamma=\gamma^*} = 0. \tag{5.2.9}$$

Moreover,

$$c_u^* = \sum_{j \in \mathbb{Z}} \langle J_1(\cdot, j) \rangle \, e^{\lambda^* j} j \text{ and } c_v^* = \sum_{j \in \mathbb{Z}} \langle J_2(\cdot, j) \rangle \, e^{\gamma^* j} j.$$

(iv) One has $c_u^* > 0$ and $c_v^* > 0$.

Remark 5.2.8. *From the definition of mean value, one can observe that*

$$\langle c_u(\lambda)(\cdot) \rangle = \lambda^{-1} \left(\sum_{j \in \mathbb{Z}} \langle J_1(\cdot, j) \rangle \left[e^{\lambda j} - 1 \right] + 1 \right),$$

$$\langle c_v(\gamma)(\cdot) \rangle = \gamma^{-1} \left(\sum_{j \in \mathbb{Z}} \langle J_2(\cdot, j) \rangle \left[e^{\gamma j} - 1 \right] + \langle r(\cdot) \rangle \right).$$

Note that

$$\langle c_u(\lambda)(\cdot) \rangle \sim \frac{1}{\lambda} \to \infty \text{ as } \lambda \to 0^+ \text{ and } \langle c_v(\gamma)(\cdot) \rangle \sim \frac{\langle r \rangle}{\gamma} \to \infty \text{ as } \gamma \to 0^+.$$

The Assumption 5.2.2 ($J5$) yields

$$\langle c_u(\lambda)(\cdot) \rangle \to \infty \text{ as } \lambda \to \text{abs}(J_1)^- \text{ and } \langle c_v(\gamma)(\cdot) \rangle \to \infty \text{ as } \gamma \to \text{abs}(J_2)^-.$$

From the above limits, one can rewrite (5.2.8) as

$$c_u^* := \min_{\lambda \in (0, \text{abs}(J_1))} \langle c_u(\lambda)(\cdot) \rangle \text{ and } c_v^* := \min_{\gamma \in (0, \text{abs}(J_2))} \langle c_v(\gamma)(\cdot) \rangle.$$

With these observations, one can verify Proposition 5.2.7 (i)-(iii) directly. Proposition 5.2.7 (iv) follows from the symmetry of the kernel functions J_k, $(k = 1, 2)$. Hence we omit the detail of proof.

In order to prove the hair trigger effect of scalar KPP lattice equations with nonlocal diffusion, for some technical reasons (see Section 5.3.4 for more details), we impose following assumption.

Assumption 5.2.9. *Set $\overline{J}_k(t) = \sum_{j \in \mathbb{Z}} J_k(t, j)$ for $k = 1, 2$. Assume that*

$$\langle f(t, 0, 0) \rangle > \langle \overline{J}_1(t) \rangle \text{ and } \langle g(t, 1, 0) \rangle > \langle \overline{J}_2(t) \rangle.$$

With the above notations and assumptions, we now describe the prey and the predator propagation behaviours in which two populations co-invade an empty space.

Theorem 5.2.10 (Slow predator)**.** *Let Assumption 5.2.2, 5.2.3, 5.2.4 and 5.2.9 be satisfied. Assume that the predator is slower than the prey, in the sense that*

$$c_v^* < c_u^*.$$

Let $1 \geq u_0 \geq 0$ and $v_0 \geq 0$ be two given bounded functions in $\mathbb{Z}$. Assume that two sets $\{i \in \mathbb{Z}; u_0(i) \neq 0\} \neq \emptyset$ and $\{i \in \mathbb{Z}; v_0(i) \neq 0\} \neq \emptyset$ have finite elements. Let $(u, v) = (u(t, i), v(t, i))$ be the solution of (5.1.1) with initial data (u_0, v_0). Assume that (u, v) is bounded. Then the function pair (u, v) satisfies:

(i) for all $c > c_u^*$, one has $\lim_{t \to \infty} \sup_{|i| \geq ct} u(t, i) = 0$;

(ii) for all $c_v^* < c_1 < c_2 < c_u^*$ and for all $c > c_v^*$, one has

$$\lim_{t \to \infty} \sup_{c_1 t \leq |i| \leq c_2 t} |1 - u(t, i)| = 0 \text{ and } \lim_{t \to \infty} \sup_{|i| \geq ct} v(t, i) = 0;$$

(iii) for all $c \in [0, c_v^*)$, one has

$$\liminf_{t \to \infty} \inf_{|i| \leq ct} v(t, i) > 0,$$

$$\liminf_{t \to \infty} \inf_{|i| \leq ct} u(t, i) > 0 \text{ and } \limsup_{t \to \infty} \sup_{|i| \leq ct} u(t, i) < 1.$$

In the next theorem, we consider the case of the predator faster than the prey.

Theorem 5.2.11 (Fast predator)**.** *Let Assumption 5.2.2, 5.2.3, 5.2.4 and 5.2.9 be satisfied. Assume that the predator is faster than the prey, in the sense that*

$$c_v^* \geq c_u^*.$$

Let $1 \geq u_0 \geq 0$ and $v_0 \geq 0$ be two given bounded functions in $\mathbb{Z}$. Assume that two sets $\{i \in \mathbb{Z}; u_0(i) \neq 0\} \neq \emptyset$ and $\{i \in \mathbb{Z}; v_0(i) \neq 0\} \neq \emptyset$ have finite elements. Let $(u, v) = (u(t, i), v(t, i))$ be the solution of (5.1.1) with initial data (u_0, v_0). Assume that (u, v) is bounded. Then the function pair (u, v) satisfies:

(i) for all $c > c_u^*$*, one has* $\lim_{t\to\infty} \sup_{|i|\geq ct} [u(t,i) + v(t,i)] = 0$*;*

(ii) for all $c \in [0, c_u^*)$*, one has*

$$\liminf_{t\to\infty} \inf_{|i|\leq ct} v(t,i) > 0,$$

$$\liminf_{t\to\infty} \inf_{|i|\leq ct} u(t,i) > 0 \text{ and } \limsup_{t\to\infty} \sup_{|i|\leq ct} u(t,i) < 1.$$

Remark 5.2.12. *In the above two theorems, we require that the solution* (u,v) *is bounded. This assumption can be satisfied for some systems under certain additional conditions. We will show this in the next proposition.*

To ensure the boundedness of solutions, we impose following assumption.

Assumption 5.2.13. *Assume that there exist some constants* $\varepsilon > 0$*,* $\eta > 0$ *and* $\mathcal{M} > 0$ *such that for all* $0 \leq u \leq 1$*,* $v \geq 0$ *and* $t \geq 0$*,*

$$uf(t,u,v) + \varepsilon v g(t,u,v) \leq \mathcal{M} - \eta v.$$

Remark 5.2.14. *Let us show that the typical example (5.1.2) satisfies Assumption 5.2.13. Let us choose* $0 < \varepsilon < \inf_{t\geq 0} p(t)/\sup_{t\geq 0} q(t)$*. Assume that* $\inf_{t\geq 0} \nu(t) > 0$*. Note that for all* $0 \leq u \leq 1$*,* $v \geq 0$ *and* $t \geq 0$*,*

$$\begin{aligned} uf(t,u,v) + \varepsilon v g(t,u,v) &= u(1-u) - p(t)uv + \varepsilon q(t)uv - \varepsilon \nu(t)v \\ &\leq 1 - \varepsilon \inf_{t\geq 0} \nu(t)v. \end{aligned}$$

Hence (5.1.2) satisfies Assumption 5.2.13 with some given $0 < \varepsilon < \inf_{t\geq 0} p(t)/\sup_{t\geq 0} q(t)$*,* $\mathcal{M} = 1$ *and* $\eta = \varepsilon \inf_{t\geq 0} \nu(t)$*.*

Let $\varepsilon > 0$, $\eta > 0$ and $\mathcal{M} > 0$ be given in Assumption 5.2.13. Set $\overline{J}_k(\cdot) = \sum_{j\in\mathbb{Z}} J_k(\cdot, j)$ for $k = 1,2$. Note that $\overline{J}_k \in L^\infty(0,\infty)$ for $k = 1,2$.

Proposition 5.2.15. *Let Assumption 5.2.2, 5.2.3, 5.2.4 and 5.2.13 be satisfied. Let* $(u,v) = (u,v)(t,i)$ *be the solution of (5.1.1) supplemented with initial data* (u_0, v_0)*. If* $0 \leq u_0 \leq 1$ *and* $v_0 \geq 0$ *is bounded, then the solution* (u,v) *is bounded.*

Remark 5.2.16. *Note that* $u \equiv 0$ *is a solution of* u*-equation in (5.1.1) and* $v \equiv 0$ *is a solution of* v*-equation in (5.1.1). Assumption 5.2.3 yields that* $f(t,1,v) \leq f(t,1,0) \equiv 0$ *for all* $t \geq 0$ *and* $v \geq 0$*. Hence,* $u \equiv 1$ *is the super-solution of* u*-equation with* $v \equiv 0$*. Since initial data satisfies* $0 \leq u_0 \leq 1$ *and* $v_0 \geq 0$*, then the partial comparison principle (which will be shown in Proposition 5.3.3) applies and ensures that* $0 \leq u(t,i) \leq 1$ *and* $v(t,i) \geq 0$ *for all* $t \geq 0$ *and* $i \in \mathbb{Z}$*.*

The rest of this paper is organized as follows. In Section 5.3, we state some propositions which will be used often in proving our main results, for instance maximum principles and spreading property for scalar lattice KPP equations. In Section 5.4, we construct proper super-solutions to obtain upper estimates for the speed of propagation for each species. In the main part of this work, Section 5.5, we first prove two key lemmas about our local pointwise estimates between u and v. Then, with the help of two key lemmas, we compare the solutions of systems with those of a scalar KPP type equations in a suitable domain. Combining some dynamical system ideas, we complete the proof of Theorem 5.2.10 and 5.2.11. In the last section, we show that the solution (u,v) is bounded under certain conditions. For the sake of completeness, the proof of some preliminary results in Section 5.3 are given in Appendix Section 5.7 and 5.8.

5.3 Preliminary

In this preliminary section, we first recall the property of mean value. Then we show the comparison principles and strong maximum principles for scalar nonlocal dispersal lattice equations. Next, we discuss the time and space shift argument of the equations that are used throughout this paper. Lastly, the spreading speed for scalar nonlocal diffusion KPP equations in a lattice is shown. For independent interest, we also state a persistence lemma which plays a key role to prove spreading speed for the scalar nonlocal diffusion equations.

5.3.1 Property of mean value

Let us first recall the property of mean value (see Definition 5.1.1) in the following lemma which has been proved in [124, 125]. Hence we omit the proof of following lemma.

Lemma 5.3.1. *Let $h \in L^\infty(0,\infty;\mathbb{R})$ be given. Then h has a mean value if and only if one has*

$$\sup_{a\in W^{1,\infty}(0,\infty)} \operatorname*{ess\,inf}_{t\geq 0} (a'+h)(t) = \inf_{a\in W^{1,\infty}(0,\infty)} \operatorname*{ess\,sup}_{t\geq 0} (a'+h)(t),$$

and in that case, the mean value corresponds to this common value. In other words the mean value is given by

$$\langle h\rangle = \sup_{a\in W^{1,\infty}(0,\infty)} \operatorname*{ess\,inf}_{t\geq 0} (a'+h)(t) = \inf_{a\in W^{1,\infty}(0,\infty)} \operatorname*{ess\,sup}_{t\geq 0} (a'+h)(t). \tag{5.3.10}$$

Remark 5.3.2. *For $H \in L^\infty(\mathbb{R};\mathbb{R})$, the mean value of H can be defined similarly to Definition 5.1.1 except herein required the limit exists uniformly for $s \in \mathbb{R}$. And the quantity $\langle H\rangle$ also has an equivalent characterization as follows*

$$\langle H\rangle = \sup_{a\in W^{1,\infty}(\mathbb{R})} \operatorname*{ess\,inf}_{t\in\mathbb{R}} (a'+H)(t) = \inf_{a\in W^{1,\infty}(\mathbb{R})} \operatorname*{ess\,sup}_{t\in\mathbb{R}} (a'+H)(t).$$

We refer the reader to [124, 125] for more details about mean value, as well as for the definitions of the so-called least mean and upper mean to solve more general time heterogeneous media.

5.3.2 Maximum principles

Now, we state various maximum principles for scalar lattice equations. For the paper readable, we postpone the proof of Proposition 5.3.3, 5.3.5 and 5.3.7 to Appendix section 5.7.

Let us first show the maximum principle of scalar equation with more general assumptions on the whole space, that is $i \in \mathbb{Z}$. The proof is close to Proposition 3.1 in [58] and Proposition 2.1 in [146] for nonlocal diffusion equations in continuous spatial space.

Proposition 5.3.3. *For any $t_0 \in \mathbb{R}$, let $T > t_0$ be given. Let $J : [t_0,T]\times\mathbb{Z} \mapsto [0,\infty)$ be the kernel function with $\sum_{j\in\mathbb{Z}} \|J(\cdot,j)\|_\infty < \infty$. Assume that $a = a(t,i)$ is a bounded function defined in $[t_0,T]\times\mathbb{Z}$. Let u be a bounded function in $[t_0,T]\times\mathbb{Z}$ and $u(\cdot,i) \in W^{1,1}(t_0,T)$ for each $i \in \mathbb{Z}$. If u satisfies*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}u(t,i) \geq \displaystyle\sum_{j\in\mathbb{Z}} J(t,j)[u(t,i-j)-u(t,i)] + a(t,i)u(t,i), & \forall t\in(t_0,T],\ \forall i\in\mathbb{Z},\\ u(t_0,i)\geq 0, & \forall i\in\mathbb{Z}, \end{cases}$$

then $u(t,i) \geq 0$ for all $t \in [t_0,T]$ and $i \in \mathbb{Z}$.

As a consequence of above proposition, one has following comparison principle.

Corollary 5.3.4 (Comparison principle)**.** *For any $t_0 \in \mathbb{R}$, let $T > t_0$ be given. Let $J : [t_0, T] \times \mathbb{Z} \mapsto [0, \infty)$ be the kernel function with $\sum_{j\in\mathbb{Z}} \|J(\cdot, j)\|_\infty < \infty$. Let $M > 0$ be given. Let $F = F(t, u)$ be a function defined in $[0, \infty) \times [0, M]$ which is Lipschitz continuous with respect to $u \in [0, M]$, uniformly for $t \geq 0$. Let $\overline{u}$ and $\underline{u}$ be two functions defined from $[t_0, T] \times \mathbb{Z}$ to $[0, M]$ and $\overline{u}(\cdot, i), \underline{u}(\cdot, i) \in W^{1,1}(t_0, T)$ for each $i \in \mathbb{Z}$. If $\overline{u}$ and $\underline{u}$ satisfy*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\overline{u}(t,i) \geq \displaystyle\sum_{j\in\mathbb{Z}} J(t,j)[\overline{u}(t,i-j) - \overline{u}(t,i)] + F(t, \overline{u}(t,i)), & \forall t \in (t_0, T],\ \forall i \in \mathbb{Z}, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\underline{u}(t,i) \leq \displaystyle\sum_{j\in\mathbb{Z}} J(t,j)[\underline{u}(t,i-j) - \underline{u}(t,i)] + F(t, \underline{u}(t,i)), & \forall t \in (t_0, T],\ \forall i \in \mathbb{Z}, \\ \overline{u}(t_0, i) \geq \underline{u}(t_0, i), & \forall i \in \mathbb{Z}. \end{cases}$$

Then $\overline{u}(t, i) \geq \underline{u}(t, i)$ for all $t \in [t_0, T]$ and $i \in \mathbb{Z}$.

Proof. Let us set $v := \overline{u} - \underline{u}$. Note that the Lipschitz continuity of F ensures that there exists a bounded function $a = a(t, i)$ such that $F(t, \overline{u}(t, i)) - F(t, \underline{u}(t, i)) = a(t, i)v(t, i)$ for all $i \in \mathbb{Z}$ and $t \in (t_0, T]$. One can apply Proposition 5.3.3 for equation satisfied by v. The proof is completed. □

Next by a slight modification, we state the maximum principle on the moving domain as follows. This is inspired by [1, 145, 169].

Proposition 5.3.5. *For any $t_0 \in \mathbb{R}$, let $T > t_0$ be given. Let $J : [t_0, T] \times \mathbb{Z} \mapsto [0, \infty)$ be the kernel function with $\sum_{j\in\mathbb{Z}} \|J(\cdot, j)\|_\infty < \infty$. Assume that I_1 and I_2 are two continuous functions defined in $[t_0, T]$, satisfying $[I_2(t)] - I_1(t) \geq 2$ where $[I_2(t)]$ is the maximal integer less than $I_2(t)$ for all $t \in [t_0, T]$. We define the set Ω_T given by*

$$\Omega_T := \{(t, i) \in \mathbb{R} \times \mathbb{Z} : t \in (t_0, T],\ I_1(t) < i < I_2(t)\}.$$

As well as, for each given $i \in \mathbb{Z}$, we define $A_T(i)$ by

$$A_T(i) := \left\{t \in [t_0, T] : (t, i) \in \overline{\Omega}_T\right\}.$$

Let $a = a(t, i)$ be a bounded function defined in $\overline{\Omega}_T$. Let u be a bounded function in $[t_0, T] \times \mathbb{Z}$. For $(t, i) \in \Omega_T$, we assume that the map $t \mapsto u(t, i) \in W^{1,1}(A_T(i))$. If u satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}u(t,i) \geq \displaystyle\sum_{j\in\mathbb{Z}} J(t,j)[u(t,i-j) - u(t,i)] + a(t,i)u(t,i), & \forall (t,i) \in \Omega_T, \\ u(t,i) \geq 0, & \forall (t,i) \in \{(t_0, T] \times \mathbb{Z}\} \setminus \Omega_T, \\ u(t_0, i) \geq 0, & \forall i \in \left[I_1(t_0), I_2(t_0)\right] \cap \mathbb{Z}. \end{cases} \tag{5.3.11}$$

Then $u(t, i) \geq 0$ for all $(t, i) \in \Omega_T$.

Remark 5.3.6. *In fact, if we modify $I_2(t) = \infty$ (resp. $I_1(t) = -\infty$) in above proposition, then we can also prove similarly to obtain that $u(t, i) \geq 0$ for all $t \in [t_0, T]$ and $i \in [I_1(t), \infty) \cap \mathbb{Z}$ (resp. $i \in (-\infty, I_2(t)] \cap \mathbb{Z}$). Hence we omit the detail in here.*

Similar to Corollary 5.3.4, Proposition 5.3.5 yields the comparison principle on moving domain Ω_T. Details are omitted.

Next we state the following strong maximum principle.

Proposition 5.3.7 (Strong maximum principle). *For any given $t_0 \in \mathbb{R}$, let $J : [t_0,\infty) \times \mathbb{Z} \mapsto [0,\infty)$ be the kernel function satisfying $\sum_{j\in\mathbb{Z}} \|J(\cdot,j)\|_\infty < \infty$ and $\inf_{t\geq t_0} J(t,\pm 1) > 0$. Assume that $a = a(t,i)$ is a bounded function defined in $[t_0,\infty)\times\mathbb{Z}$. Let u be a bounded function in $[t_0,\infty)\times\mathbb{Z}$ and $u(\cdot,i) \in W^{1,1}(t_0,\infty)$ for each $i \in \mathbb{Z}$. If u satisfies*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}u(t,i) \geq \displaystyle\sum_{j\in\mathbb{Z}} J(t,j)[u(t,i-j)-u(t,i)] + a(t,i)u(t,i), & \forall t \in (t_0,\infty),\ \forall i \in \mathbb{Z}, \\ u(t_0,i) \geq 0 \text{ and } \not\equiv 0, & \forall i \in \mathbb{Z}, \end{cases}$$

then $u(t,i) > 0$ for all $t \in (t_0,\infty)$ and $i \in \mathbb{Z}$.

5.3.3 Limit problem

In this subsection, we discuss time and space shift for bounded solution (u,v) to (5.1.1). Choose sequence $(\tau_n)_{n\geq 0} \subset [0,\infty)$ such that $\tau_n \to \infty$ as $n \to \infty$. We can claim as follows.

Claim 5.3.8. *There exist two bounded and uniformly continuous functions $\tilde{f} : \mathbb{R}\times[0,\infty)^2 \to \mathbb{R}$ and $\tilde{g} : \mathbb{R}\times[0,\infty)^2 \to \mathbb{R}$, two function sequences $j \mapsto \tilde{J}_1(t,j) \in l^1(\mathbb{Z}, L^\infty(\mathbb{R}))$ and $j \mapsto \tilde{J}_2(t,j) \in l^1(\mathbb{Z}, L^\infty(\mathbb{R}))$ and a subsequence $(\tau_n)_n$ (still denoted by the same index) such that*

$$\left(f(t+\tau_n,u,v), g(t+\tau_n,u,v)\right) \to \left(\tilde{f}(t,u,u), \tilde{g}(t,u,v)\right), \tag{5.3.12}$$

in local uniform topology as $n \to \infty$. Also, one has

$$(J_1(t+\tau_n,j), J_2(t+\tau_n,j)) \to \left(\tilde{J}_1(t,j), \tilde{J}_2(t,j)\right), \tag{5.3.13}$$

in weak-$\star$ topology of $L^\infty_{\mathrm{loc}}(\mathbb{R})^2$ as $n \to \infty$, for all $j \in \mathbb{Z}$. Moreover, due to Assumption 5.2.3 $(f2)$, $(f4)$ and Assumption 5.2.4 $(g2)$, $(g4)$, the function $\tilde{f} = \tilde{f}(t,u,v)$ is nonincreasing in both u and v while $\tilde{g} = \tilde{g}(t,u,v)$ is nondecreasing in u and nonincreasing in v.

Assumption 5.2.3 and 5.2.4 ensure that f and g are bounded and uniformly continuous functions on $[0,\infty)^3$ while Assumption 5.2.2 implies that $\sum_{j\in\mathbb{Z}} \|J_k(\cdot,j)\|_\infty < \infty$, $(k = 1,2)$. The above claim holds.

Now, let $\mathrm{BUC}([0,\infty)\times[0,\infty))$ denote the Banach space of bounded and uniformly continuous functions on $[0,\infty)\times[0,\infty)$. For $t \in [0,\infty)$, we define

$$\sigma(t) := \left(f(t,\cdot,\cdot), g(t,\cdot,\cdot), J_1(t,\cdot), J_2(t,\cdot)\right) \in \mathrm{BUC}([0,\infty)\times[0,\infty))^2 \times \left(l^1(\mathbb{Z})\right)^2.$$

According to Claim 5.3.8, we define the limit set Σ as follows: $\tilde{\sigma} = \left(\tilde{f}, \tilde{g}, \tilde{J}_1, \tilde{J}_2\right) \in \Sigma$, there exists sequence $\tau_n \to \infty$ as $n \to \infty$ such that (5.3.12) and (5.3.13) hold.

Recall that $(u,v) = (u(t,i), v(t,i))$ is a bounded solution to (5.1.1). Let us define the set S by: $(\tilde{u},\tilde{v}) \in S$ if there exist sequence $(t_n)_n \subset [0,\infty)$ and $(i_n)_n \subset \mathbb{Z}$ such that

$$(u(t+t_n, i+i_n), v(t+t_n, i+i_n)) \to (\tilde{u}(t,i), \tilde{v}(t,i)),$$

locally uniformly for $(t,i) \in \mathbb{R}\times\mathbb{Z}$, as $n \to \infty$. Note that for each pair of $t_n \geq 0$ and $i_n \in \mathbb{Z}$, the function pair $(u_n,v_n)(t,i) := (u,v)(t+t_n, i+i_n)$ defined for $t \geq -t_n$ and $i \in \mathbb{Z}$

satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}u_n(t,i) = \sum\limits_{j\in\mathbb{Z}} J_1(t+t_n,j)\left[u_n(t,i-j)-u_n(t,i)\right] + u_n(t,i)f\left(t+t_n,u_n(t,i),v_n(t,i)\right), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}v_n(t,i) = \sum\limits_{j\in\mathbb{Z}} J_2(t+t_n,j)\left[v_n(t,i-j)-v_n(t,i)\right] + v_n(t,i)g\left(t+t_n,u_n(t,i),v_n(t,i)\right). \end{cases}$$

Due to (u,v) is assumed to be bounded, there exists some constant $C>0$ such that for all $i\in\mathbb{Z}$

$$\begin{aligned} \left\|\frac{\mathrm{d}}{\mathrm{d}t}u_n(\cdot,i)\right\|_{L^\infty(\mathbb{R})} &\le 2\sum_{j\in\mathbb{Z}}\|J_1(\cdot,j)\|_\infty + 1 < \infty, \\ \left\|\frac{\mathrm{d}}{\mathrm{d}t}v_n(\cdot,i)\right\|_{L^\infty(\mathbb{R})} &\le C\left(2\sum_{j\in\mathbb{Z}}\|J_2(\cdot,j)\|_\infty + \|g(\cdot,1,0)\|_\infty\right) < \infty. \end{aligned} \tag{5.3.14}$$

Hence, one can choose subsequence (still denote with the same index) such that

$$u_n(t,i)\to\tilde{u}(t,i) \text{ and } v_n(t,i)\to\tilde{v}(t,i),$$

locally uniformly for $(t,i)\in\mathbb{R}\times\mathbb{Z}$ as $n\to\infty$

Next, we derive the system satisfied by $(\tilde{u},\tilde{v})$. Let us first observe that

$$\sum_{j\in\mathbb{Z}} J_k(t+\tau_n,j) \to \sum_{j\in\mathbb{Z}}\tilde{J}_k(t,j),\ (k=1,2), \tag{5.3.15}$$

in weak-$\star$ topology of $L^\infty_{\mathrm{loc}}(\mathbb{R})$, as $n\to\infty$. That means for all $T>0$ and $\phi\in L^1(-T,T)$,

$$\lim_{n\to\infty}\int_{-T}^{T}\sum_{j\in\mathbb{Z}}\left|J_k(t+\tau_n,j)\phi(t)\right|\mathrm{d}t = \int_{-T}^{T}\sum_{j\in\mathbb{Z}}\left|\tilde{J}_k(t,j)\phi(t)\right|\mathrm{d}t,\ (k=1,2).$$

To show this property, note that $\sum\limits_{j\in\mathbb{Z}}\|J_k(\cdot,j)\|_\infty<\infty$ since $i\mapsto J_k(\cdot,i)\in l^1(\mathbb{Z},L^\infty(\mathbb{R}))$. Next, for each $j\in\mathbb{Z}$, we have

$$\int_{-T}^{T}\left|J_k(t+\tau_n,j)\phi(t)\right|\mathrm{d}t \le \|J_k(\cdot,j)\|_\infty\|\phi\|_1, (k=1,2),$$

and

$$\sum_{j\in\mathbb{Z}}\|J_k(\cdot,j)\|_\infty\|\phi\|_1<\infty,\ (k=1,2).$$

Hence, the Lebesgue dominated convergence theorem ensures that

$$\begin{aligned} \lim_{n\to\infty}\sum_{j\in\mathbb{Z}}\int_{-T}^{T}\left|J_k(t+\tau_n,j)\phi(t)\right|\mathrm{d}t &= \sum_{j\in\mathbb{Z}}\lim_{n\to\infty}\int_{-T}^{T}\left|J_k(t+\tau_n,j)\phi(t)\right|\mathrm{d}t \\ &= \sum_{j\in\mathbb{Z}}\int_{-T}^{T}\left|\tilde{J}_k(t,j)\phi(t)\right|\mathrm{d}t, \quad (k=1,2). \end{aligned}$$

From Fubini-Tonelli theorem and the above equality, one can observe that (5.3.15) holds.

Combining with (5.3.15) and $(\tilde{u},\tilde{v})\in S$, one has for all $i\in\mathbb{Z}$,

$$\begin{cases} u_n(t,i)\sum\limits_{j\in\mathbb{Z}}J_1(t+t_n,j)\to\tilde{u}(t,i)\sum\limits_{j\in\mathbb{Z}}\tilde{J}_1(t,j), \\ v_n(t,i)\sum\limits_{j\in\mathbb{Z}}J_2(t+t_n,j)\to\tilde{v}(t,i)\sum\limits_{j\in\mathbb{Z}}\tilde{J}_2(t,j), \end{cases} \tag{5.3.16}$$

in the weak-$\star$ topology of $L^\infty_{\mathrm{loc}}(\mathbb{R})$ as $n\to\infty$. Then let us prove following claim.

Claim 5.3.9. *One has for all $i \in \mathbb{Z}$,*

$$\begin{cases} \sum_{j\in\mathbb{Z}} J_1(t+t_n,j)u_n(t,i-j) \to \sum_{j\in\mathbb{Z}} \tilde{J}_1(t,j)\tilde{u}(t,i-j), \\ \sum_{j\in\mathbb{Z}} J_2(t+t_n,j)v_n(t,i-j) \to \sum_{j\in\mathbb{Z}} \tilde{J}_2(t,j)\tilde{v}(t,i-j), \end{cases} \tag{5.3.17}$$

in the weak-$\star$ topology of $L^\infty_{\text{loc}}(\mathbb{R})$ as $n \to \infty$.

Proof. We only show the first convergence result. The second one can be proved similarly. Note that we have

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} J_1(t+t_n,j)u_n(t,i-j) - \sum_{j\in\mathbb{Z}} \tilde{J}_1(t,j)\tilde{u}(t,i-j) \\ &= \sum_{j\in\mathbb{Z}} J_1(t+t_n,j)\left[u_n(t,i-j) - \tilde{u}(t,i-j)\right] + \sum_{j\in\mathbb{Z}} \left[J_1(t+t_n,j) - \tilde{J}_1(t,j)\right]\tilde{u}(t,i-j). \end{aligned}$$

Recalling (5.3.15) and $0 \le \tilde{u} \le 1$, it is sufficiently to show that for all $i \in \mathbb{Z}$,

$$\sum_{j\in\mathbb{Z}} J_1(t+t_n,j)\left[u_n(t,i-j) - \tilde{u}(t,i-j)\right] \to 0,$$

in the weak-$\star$ topology of $L^\infty_{\text{loc}}(\mathbb{R})$ as $n \to \infty$. To do this, let us observe that for any $B > 0$, for each $t \ge -t_n$ and $i \in \mathbb{Z}$,

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} J_1(t+t_n,j)\left[u_n(t,i-j) - \tilde{u}(t,i-j)\right] \\ &= \sum_{|j|\ge B} J_1(t+t_n,j)\left[u_n(t,i-j) - \tilde{u}(t,i-j)\right] + \sum_{|j|\le B} J_1(t+t_n,j)\left[u_n(t,i-j) - \tilde{u}(t,i-j)\right] \\ &\le 2\sum_{|j|\ge B} \|J_1(\cdot,j)\|_\infty + \sum_{|j|\le B} \|J_1(\cdot,j)\|_\infty \sup_{|j|\le B} |u_n(t,i-j) - \tilde{u}(t,i-j)|. \end{aligned}$$

Next since $u_n(t,i) \to \tilde{u}(t,i)$ locally uniformly for $(t,i) \in \mathbb{R}\times\mathbb{Z}$ as $n \to \infty$, then for all $A > 0$ and $B > 0$,

$$\lim_{n\to\infty} \sup_{\substack{t\in[-A,A] \\ i\in[-A,A]\cap\mathbb{Z}}} \left|\sum_{j\in\mathbb{Z}} J_1(t+t_n,j)\left[u_n(t,i-j) - \tilde{u}(t,i-j)\right]\right| \le 2\sum_{|j|\ge B} \|J_1(\cdot,j)\|_\infty.$$

Due to $j \mapsto J_1(\cdot,j) \in l^1(\mathbb{Z}, L^\infty(0,\infty))$, letting $B \to \infty$, we obtain that the first convergence result in (5.3.17). The second one in (5.3.17) can be proved similarly. The proof of this claim is completed. □

Note that

$$\begin{aligned} \left|f(t+t_n,u_n,v_n) - \tilde{f}(t,\tilde{u},\tilde{v})\right| \le& \left|f(t+t_n,u_n,v_n) - f(t+t_n,\tilde{u},\tilde{v})\right| \\ &+ \left|f(t+t_n,\tilde{u},\tilde{v}) - \tilde{f}(t,\tilde{u},\tilde{v})\right|. \end{aligned}$$

Recalling (5.3.12) and Assumption 5.2.3 $(f1)$, one can observe that

$$\lim_{n\to\infty} f(t+t_n,u_n,v_n) = \tilde{f}(t,\tilde{u},\tilde{v}), \text{ in local uniform topology.}$$

Similarly, one has

$$\lim_{n\to\infty} g(t+t_n, u_n, v_n) = \tilde{g}(t,\tilde{u},\tilde{v}), \text{ in local uniform topology.}$$

From (5.3.14), (5.3.16), (5.3.17) and the above two limits, one can obtain that there exists some $\tilde{\sigma} = (\tilde{f}, \tilde{g}, \tilde{J}_1, \tilde{J}_2) \in \Sigma$ such that $(\tilde{u},\tilde{v})$ satisfies for all $(t,i) \in \mathbb{R}\times\mathbb{Z}$,

$$(\mathbf{P}_{\tilde{\sigma}}) \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{u}(t,i) = \displaystyle\sum_{j\in\mathbb{Z}} \tilde{J}_1(t,j)\left[\tilde{u}(t,i-j) - \tilde{u}(t,i)\right] + \tilde{u}(t,i)\tilde{f}\left(t,\tilde{u}(t,i),\tilde{v}(t,i)\right), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{v}(t,i) = \displaystyle\sum_{j\in\mathbb{Z}} \tilde{J}_2(t,j)\left[\tilde{v}(t,i-j) - \tilde{v}(t,i)\right] + \tilde{v}(t,i)\tilde{g}\left(t,\tilde{u}(t,i),\tilde{v}(t,i)\right). \end{cases} \tag{5.3.18}$$

5.3.4 Spreading speed for scalar Fisher-KPP equation

Now we consider the following Cauchy problem of scalar KPP type non-autonomous lattice equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}w(t,i) = \displaystyle\sum_{j\in\mathbb{Z}} J(t,j)[w(t,i-j)-w(t,i)] + m(t)w(t,i)\big(1-lw(t,i)\big), & t\geq 0, i\in\mathbb{Z}, \\ w(0,i) = w_0(i), & i\in\mathbb{Z}, \end{cases} \tag{5.3.19}$$

where the constant $l > 0$.

Let us first show the hair trigger effect property in (5.3.19).

Lemma 5.3.10 (Hair trigger effect)**.** *Assume that the kernel function $J = J(t,j)$ is nonnegative and $i \mapsto J(\cdot,i) \in l^1(\mathbb{Z}, L^\infty(0,\infty))$. Let $\inf_{t\geq 0} J(t,\pm 1) > 0$ be satisfied. Assume that $m : [0,\infty) \to \mathbb{R}$ is a bounded and uniformly continuous function with $\inf_{t\geq 0} m(t) > 0$. Set $\overline{J}(t) := \sum_{j\in\mathbb{Z}} J(t,j)$. Assume that m and $\overline{J}$ have mean value, denoted by $\langle m\rangle$ and $\langle\overline{J}\rangle$ respectively, which are satisfying $\langle m\rangle > \langle\overline{J}\rangle$. Let $w(t,i)$ be the solution of (5.3.19). If $w_0 \geq 0$ and $w_0 \not\equiv 0$. Then there exists a constant $\tilde{\varepsilon}_0 > 0$ which is independent of w_0 such that*

$$\liminf_{t\to\infty} w(t,0) \geq \tilde{\varepsilon}_0.$$

Remark 5.3.11. *Note that due to the technical reason, the condition $\langle m\rangle > \langle\overline{J}\rangle$ is required. This lemma will be used to prove spreading speed for scalar KPP equation with nonlocal diffusion.*

Remark 5.3.12. *From the proof below, one can notice that $\tilde{\varepsilon}_0$ is independent of the time shift limit functions of J and m. The similar idea can also be used to analyze the time-space shift limit equation.*

Proof. Let $w = w(t,i)$ be the solution of (5.3.19) which is equipped with initial data $w_0 \geq 0$ and $w_0 \not\equiv 0$. Let us set $d := \min\{\inf_{t\geq 0} J(t,1), \inf_{t\geq 0} J(t,-1)\}$. Since $J = J(t,j)$ is nonnegative and $\inf_{t\geq 0} J(t,\pm 1) > 0$, then one has

$$\sum_{j\in\mathbb{Z}} J(t,j)w(t,i-j) \geq d\left[w(t,i-1)+w(t,i+1)\right], \ \forall t\in\mathbb{R}, \forall i\in\mathbb{Z}.$$

Note that $\langle m(\cdot)\rangle > \langle\overline{J}(\cdot)\rangle$ is imposed. The property of mean value ensures that there exists $A \in W^{1,\infty}(0,\infty)$ such that

$$m(t) - \overline{J}(t) + A'(t) > 0, \ \forall t \geq 0.$$

Let us define $U(t,i)=e^{A(t)}w(t,i)$. Note that $U=U(t,i)$ satisfies for all $t\geq 0$ and $i\in\mathbb{Z}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}U(t,i)\geq{}& d\big[U(t,i+1)+U(t,i-1)-2U(t,i)\big]+2dU(t,i)-\overline{J}(t)U(t,i)+A'(t)U(t,i)\\ &+m(t)U(t,i)\big(1-le^{-A(t)}U(t,i)\big)\\ \geq{}& d[U(t,i+1)+U(t,i-1)-2U(t,i)]+U(t,i)\big(\beta-l\|m\|_\infty e^{\|A\|_\infty}U(t,i)\big),\end{aligned}$$

where

$$\beta:=2d+\inf_{t\geq 0}\big\{m(t)-\overline{J}(t)+A'(t)\big\}>0.$$

Let us define $\underline{U}=\underline{U}(t,i)$ as the solution of following Cauchy problem for $t>0$ and $i\in\mathbb{Z}$,

$$\partial_t\underline{U}(t,i)=d[\underline{U}(t,i+1)+\underline{U}(t,i-1)-2\underline{U}(t,i)]+\underline{U}(t,i)\big(\beta-l\|m\|_\infty e^{\|A\|_\infty}\underline{U}(t,i)\big),\quad (5.3.20)$$

supplemented with initial data $\underline{U}(0,i)=e^{-\|A\|_\infty}w(0,i)$ for $i\in\mathbb{Z}$. Since $w(0,i)\not\equiv 0$, then one has $\underline{U}(0,i)\not\equiv 0$. The spreading speed results for (5.3.20) (see [162]) implies that there exists some speed $c_0>0$ such that

$$\lim_{t\to\infty}\inf_{|i|\leq ct}\underline{U}(t,i)=\frac{\beta}{l\|m\|_\infty e^{\|A\|_\infty}},\ \forall c\in[0,c_0).$$

Note that U is the super-solution of (5.3.20). The comparison principle implies that for all $t>0$ and $i\in\mathbb{Z}$,

$$U(t,i)\geq\underline{U}(t,i).$$

Hence, one has

$$\lim_{t\to\infty}U(t,0)\geq\frac{\beta}{l\|m\|_\infty e^{\|A\|_\infty}}.$$

Moreover, one has

$$\lim_{t\to\infty}w(t,0)\geq\lim_{t\to\infty}e^{-\|A\|_\infty}U(t,0)\geq\frac{\beta}{l\|m\|_\infty e^{2\|A\|_\infty}}\geq\frac{2d}{l\|m\|_\infty e^{2\|A\|_\infty}}=:\tilde{\varepsilon}_0.$$

The proof of this lemma is completed.

□

In order to study spreading speed for (5.3.19), more conditions on the kernel function J should be given, for instance, J is exponentially bounded. We state assumptions satisfied by J and m as follows:

Assumption 5.3.13. *The kernel function J satisfies Assumption 5.2.2. Assume the function $m:[0,\infty)\to\mathbb{R}$ is bounded and uniformly continuous with $\inf_{t\geq 0}m(t)>0$. Assume that m has a mean value, denoted by $\langle m\rangle$, which satisfies $\langle m\rangle>\langle\overline{J}\rangle$.*

Before stating the spreading speed result for (5.3.19), let us introduce the speed function $\mu\mapsto c_w(\mu)$ defined in $(0,\mathrm{abs}(J))$ given by

$$c_w(\mu)(\cdot):=\mu^{-1}\Bigg(\sum_{j\in\mathbb{Z}}J(\cdot,j)[e^{\mu j}-1]+m(\cdot)\Bigg).$$

Define c_w^* as

$$c_w^*:=\inf_{0<\mu<\mathrm{abs}(J)}\langle c_w(\mu)\rangle=\inf_{0<\mu<\mathrm{abs}(J)}\mu^{-1}\Bigg(\sum_{j\in\mathbb{Z}}\langle J(\cdot,j)\rangle\,[e^{\mu j}-1]+\langle m(\cdot)\rangle\Bigg).\qquad(5.3.21)$$

Similarly to Proposition 5.2.7, one can observe that $\mu\mapsto\langle c_w(\mu)\rangle$ is of class C^1 in $(0,\mathrm{abs}(J))$ and there exists $\mu^*\in(0,\mathrm{abs}(J))$ such that $\langle c_w(\mu^*)\rangle=c_w^*$.

With above notations, we state the following proposition.

Proposition 5.3.14 (Spreading speed for Fisher-KPP equations)**.** *Let Assumption 5.3.13 be satisfied. Let initial data $0 \le w_0 \le \frac{1}{l}$ be given. Assume that the set $\{i \in \mathbb{Z} : w_0(i) \neq 0\} \neq \emptyset$ has finite elements. Then the solution $w = w(t,i)$ of (5.3.19) satisfies:*

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{|i|\ge ct} w(t,i) = 0, & \forall c > c_w^*, \\ \lim\limits_{t\to\infty} \inf\limits_{|i|\le ct} w(t,i) = \frac{1}{l}, & \forall c \in [0, c_w^*), \end{cases}$$

where c_w^* is defined in (5.3.21).

Remark 5.3.15. *Note that here we only consider the logistic growth term. It is sufficiently for this paper. In fact, our method for proving Proposition 5.3.14 is also valid for more general KPP-type reaction term $F(t,u)$ satisfying $\langle F_u'(\cdot,0)\rangle > \langle \overline{J}\rangle$. The proof of this theorem makes use of the similar idea in [59] which studied the nonlocal dispersal KPP equation with continuous spatial variable. We postpone the proof in Subsection 5.8.1.*

5.3.5 Key persistence lemma

For independent interest, we state following persistence lemma which plays a crucial role to prove Proposition 5.3.14. In the following, we will also apply a similar idea to prove the persistence lemma. Let us introduce some notations.

Definition 5.3.16 (Limit orbits set)**.** *For each $i \in \mathbb{Z}$, let $w(t,i)$ be the solution of (5.3.19) equipped with initial data w_0, where $0 \le w_0 \le \frac{1}{l}$ and the set $\{i \in \mathbb{Z} : w_0(i) \neq 0\} \neq \emptyset$ has finite elements. We define $\mathcal{H}(w)$ as: $\tilde{w} \in \mathcal{H}(w)$ if there exist a sequence $(t_n)_n \subset [0,\infty)$ and $(i_n)_n \subset \mathbb{Z}$ such that $t_n \to \infty$ and*

$$\tilde{w}(t,i) = \lim_{n\to\infty} w(t+t_n, i+i_n), \text{ locally uniformly for } (t,i) \in \mathbb{R}\times\mathbb{Z}.$$

One can observe that $w_n(t,i) := w(t+t_n, i+i_n)$ defined in $t \ge -t_n$ and $i \in \mathbb{Z}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} w_n(t,i) = \sum_{j\in\mathbb{Z}} J(t+t_n,j)[w_n(t,i-j) - w_n(t,i)] + m(t+t_n)w_n(t,i)\big(1 - l w_n(t,i)\big).$$

By the same analysis in Section 5.3.3, for a given $\tilde{w} \in \mathcal{H}(w)$, one can derive that $\tilde{w}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{w}(t,i) = \sum_{j\in\mathbb{Z}} \tilde{J}(t,j)[\tilde{w}(t,i-j) - \tilde{w}(t,i)] + \tilde{m}(t)\tilde{w}(t,i)\big(1 - l\tilde{w}(t,i)\big), \ \forall (t,i) \in \mathbb{R}\times\mathbb{Z}, \tag{5.3.22}$$

where

$$J(t+t_n,j) \to \tilde{J}(t,j) \text{ in weak-}\star \text{ topology of } L^\infty_{\mathrm{loc}}(\mathbb{R}) \text{ as } n\to\infty, \text{ for all } j \in \mathbb{Z},$$
$$m(t+t_n) \to \tilde{m}(t) \text{ in local uniform topology of } C(\mathbb{R};\mathbb{R}) \text{ as } n \to \infty.$$

From the strong maximum principle, we can claim that the set $\mathcal{H}(w)$ enjoys following property:

Claim 5.3.17. *Let $\tilde{w} \in \mathcal{H}(w)$ be given. Then one has:*

$$\textit{Either } \tilde{w}(t,i) > 0 \textit{ for all } (t,i) \in \mathbb{R}\times\mathbb{Z} \textit{ or } \tilde{w}(t,i) \equiv 0 \textit{ on } \mathbb{R}\times\mathbb{Z}.$$

Lemma 5.3.18 (Persistence lemma). *Let w and $\mathcal{H}(w)$ be defined as in Definition 5.3.16. Let $t \mapsto X(t)$ from $[0,\infty)$ to $[0,\infty)$ be a given continuous function. Assume that the following set of hypothesis is satisfied,*

(H1) there exists $\varepsilon_1 > 0$ such that

$$\liminf_{t\to\infty} w(t,0) \geq \varepsilon_1; \tag{5.3.23}$$

(H2) there exists $\varepsilon_2 > 0$ such that for all $\tilde{w} \in \mathcal{H}(w) \setminus \{0\}$ one has

$$\liminf_{t\to\infty} \tilde{w}(t,0) \geq \varepsilon_2; \tag{5.3.24}$$

(H3) there exists $\varepsilon_3 > 0$ such that

$$\liminf_{t\to\infty} w(t,[X(t)]) \geq \varepsilon_3, \tag{5.3.25}$$

where $[X(t)]$ means taking maximal integer less than or equal to $X(t)$ for each $t \geq 0$.

Then for any $k \in (0,1)$, one has

$$\liminf_{t\to\infty} \inf_{i\in[0,[kX(t)]]\cap\mathbb{Z}} w(t,i) > 0.$$

The proof of this lemma is given in Subsection 5.8.

5.4 Upper estimates on the spreading speeds

Now we give the proof of Theorem 5.2.10 (i), half of Theorem 5.2.10 (ii) and Theorem 5.2.11 (i). In the proof, we only focus on $i \geq 0$, for $i \leq 0$ which can be dealt similarly by a symmetric argument.

Recalling that the definition of $c_u(\lambda)$ in (5.2.7) and $c_u^* = \langle c_u(\lambda^*)\rangle$ in Proposition 5.2.7, the property of mean value (see (5.3.10)) ensures that there exists $a \in W^{1,\infty}(0,\infty)$ such that for all $c > c' > c_u^*$ and for all $t \geq 0$,

$$c' \geq \frac{\sum_{j\in\mathbb{Z}} J_1(t,j)[e^{\lambda^* j} - 1] + 1}{\lambda^*} + a'(t).$$

Then for all $A > 0$, the function $\overline{u}$ given by

$$\overline{u}(t,i) := Ae^{-\lambda^* a(t)} e^{-\lambda^*(i-c't)},$$

satisfies for all $i \in \mathbb{Z}$ and $t \geq 0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\overline{u}(t,i) &- \sum_{j\in\mathbb{Z}} J_1(t,j)[\overline{u}(t,i-j) - \overline{u}(t,i)] - \overline{u}(t,i)\\ &= \overline{u}(t,i)\left(\lambda^* c' - \lambda^* a'(t) - \sum_{j\in\mathbb{Z}} J_1(t,j)\left[e^{\lambda^* j} - 1\right] - 1\right) \geq 0.\end{aligned}$$

Let $A > 0$ be large enough such that $\overline{u}(0,i) \geq u_0(i)$ for all $i \in \mathbb{Z}$. Note that $f(t,u,v) \leq 1$ for all $t \geq 0$, $u \in [0,1]$ and $v \geq 0$ from (5.2.6). The comparison principle applies and ensures that for all $c > c' > c_u^*$,

$$\limsup_{t\to\infty} \sup_{i\geq ct} u(t,i) \leq \limsup_{t\to\infty} \sup_{i\geq ct} \overline{u}(t,i) \leq \lim_{t\to\infty} Ae^{-\lambda^* a(t)} e^{-\lambda^*(c-c')t} = 0.$$

Since u is nonnegative, then we obtain the statement (i) in Theorem 5.2.10 and the half of statement (i) in Theorem 5.2.11.

Similarly, for all $c > \tilde{c} > c_v^*$, there exists $\tilde{a} \in W^{1,\infty}(0,\infty)$ such that for all $t \geq 0$,

$$\tilde{c} \geq \frac{\sum_{j\in\mathbb{Z}} J_2(t,j)[e^{\gamma^* j} - 1] + r(t)}{\gamma^*} + \tilde{a}'(t).$$

Then the function

$$\overline{v}_1(t,i) := \tilde{A} e^{-\gamma^* \tilde{a}(t)} e^{-\gamma^*(i-\tilde{c}t)},$$

satisfies following differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{v}_1(t,i) - \sum_{j\in\mathbb{Z}} J_2(t,j)[\overline{v}_1(t,i-j) - \overline{v}_1(t,i)] - r(t)\overline{v}_1(t,i) \geq 0.$$

Choosing $\tilde{A} > 0$ large enough such that $\overline{v}_1(0,i) \geq v_0(i)$ for all $i \in \mathbb{Z}$, from (5.2.6) and comparison principle, one obtains that for all $c > \tilde{c} > c_v^*$,

$$\limsup_{t\to\infty} \sup_{i\geq ct} v(t,i) \leq \limsup_{t\to\infty} \sup_{i\geq ct} \overline{v}_1(t,i) \leq \lim_{t\to\infty} \tilde{A} e^{-\gamma^* \tilde{a}(t)} e^{-\gamma^*(c-\tilde{c})t} = 0.$$

Since v is nonnegative, then this proves the half of statement (ii) in Theorem 5.2.10.

Next we show that v cannot spread faster than c_u^*. Note that we have already obtained

$$\lim_{t\to\infty} \sup_{|i|\geq ct} u(t,i) = 0, \ \forall c > c_u^*.$$

Thus, fixing any $c > c_u^*$ and $\varepsilon > 0$ small enough, there exists $T > 0$ such that

$$\sup_{t\geq T} \sup_{|i|\geq ct} u(t,i) \leq \varepsilon.$$

Note that the map $u \mapsto \langle g(\cdot,u,0)\rangle$ is continuous and recall that $\langle g(\cdot,0,0)\rangle < 0$ in Assumption 5.2.4. Hence, one has $\langle g(\cdot,\varepsilon,0)\rangle < 0$ for all $\varepsilon > 0$ sufficiently small. From the property of mean value, one can choose $b \in W^{1,\infty}(0,\infty)$ such that

$$\sup_{t\geq 0} \{g(t,\varepsilon,0) + b'(t)\} < 0.$$

On the other hand, since $\sum_{j\in\mathbb{Z}} \|J_2(\cdot,j)\|_\infty e^{\gamma j} < \infty$ for all $\gamma \in (0, \mathrm{abs}(J_2))$, then for all $c'' \in (c_u^*, c)$, one has

$$\lim_{\gamma\to 0^+} \left\{\gamma c'' - \sum_{j\in\mathbb{Z}} J_2(t,j)[e^{\gamma j} - 1]\right\} = 0, \text{ uniformly for } t \geq 0.$$

Thus one can choose some $\gamma' > 0$ small enough such that

$$\gamma' c'' - \sum_{j\in\mathbb{Z}} J_2(t,j)[e^{\gamma' j} - 1] - g(t,\varepsilon,0) - b'(t) \geq 0, \forall t \geq 0.$$

Note that solution (u,v) is assumed to be bounded. One can choose $B > 0$ large enough such that $Be^{-\|b\|_\infty} \geq v(t,i)$ for all $t \geq 0$ and $i \in \mathbb{Z}$. For $c'' \in (c_u^*, c)$, we define

$$\overline{v}_2(t,i) := Be^{-\gamma'(i-c''t)} e^{-b(t)}, \ \forall t \geq 0, i \in \mathbb{Z}.$$

From the chosen of γ', B and $b(t)$ above, one can verify that $\overline{v}_2(t,i)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{v}_2(t,i) - \sum_{j\in\mathbb{Z}} J_2(t,j)[\overline{v}_2(t,i-j) - \overline{v}_2(t,i)] - g(t,\varepsilon,0)\overline{v}_2(t,i) \geq 0.$$

Next we define the set

$$\Omega := \{(t,i) \in (T,\infty)\times\mathbb{Z};\ i > c''t\},$$

Recalling the definition of $\overline{v}_2$, one has

$$\overline{v}_2(t,i) \geq Be^{-\|b\|_\infty} \geq v(t,i), \quad \text{for all } t\in[T,\infty) \text{ and } i\in\mathbb{Z}\cap(-\infty,c''t]. \tag{5.4.26}$$

Let us observe that $v(T,i)\leq \overline{v}_2(T,i)$ for all $i\in\mathbb{Z}\cap(c''T,\infty)$. Indeed, since $v(t,i)\leq\overline{v}_1(t,i)$ for all $t\geq 0$ and $i\in\mathbb{Z}$ which can notice from the above analysis, and $\gamma'\in(0,\gamma^*)$, then one can choose $B>0$ larger such that for all $i>c''T>0$,

$$v(T,i)\leq\overline{v}_1(T,i) = \tilde{A}e^{-\gamma^*(i-\tilde{c}T)}e^{-\gamma^*\tilde{a}(T)} \leq Be^{-\gamma'(i-c''T)}e^{-b(T)} = \overline{v}_2(T,i).$$

(5.4.26) yields that $\overline{v}_2(T,i)\geq v(T,i)$ for all $i\leq c''T$. Thus combing with above inequality, one has $\overline{v}_2(T,i)\geq v(T,i)$ for all $i\in\mathbb{Z}$.

Now we can conclude that

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\overline{v}_2(t,i) \geq \sum_{j\in\mathbb{Z}} J_2(t,j)[\overline{v}_2(t,i-j)-\overline{v}_2(t,i)] + g(t,\varepsilon,0)\overline{v}_2(t,i), & (t,i)\in\Omega,\\ \frac{\mathrm{d}}{\mathrm{d}t}v(t,i) = \sum_{j\in\mathbb{Z}} J_2(t,j)[v(t,i-j)-v(t,i)] + g(t,u,v)v(t,i), & (t,i)\in\Omega,\\ \overline{v}_2(T,i)\geq v(T,i), & i\in\mathbb{Z},\\ \overline{v}_2(t,i)\geq v(t,i), & (t,i)\in\{(T,\infty)\times\mathbb{Z}\}\setminus\Omega.\end{cases}$$

Recall that $0\leq u(t,i)\leq\varepsilon$ for $(t,i)\in\Omega$. Assumption 5.2.4 ensures that $g(t,u,v)\leq g(t,\varepsilon,0)$ for $t\geq 0$, $u\in[0,\varepsilon]$ and $v\geq 0$. Applying the comparison principle (see Remark 5.3.6) on the moving domain Ω, one obtains

$$v(t,i)\leq\overline{v}_2(t,i), \forall (t,i)\in\Omega.$$

For all $c''\in(c_u^*,c)$, one has

$$0\leq \limsup_{t\to\infty}\sup_{i\geq ct} v(t,i) \leq \limsup_{t\to\infty}\sup_{i\geq ct}\overline{v}_2(t,i) \leq \lim_{t\to\infty} Be^{-\gamma'(c-c'')t}e^{-b(t)} = 0.$$

This completes the proof of statement (i) in Theorem 5.2.11.

5.5 Lower estimates on the spreading speeds

In this section, we first show some key lemmas about the local pointwise estimates between u and v. Then with the help of these key lemmas, we can compare the solution of system with those of a KPP scalar equations defined in a suitable domain. Lastly, through constructing proper sub-solutions and using some dynamical system arguments, we complete the proof Theorem 5.2.10 and 5.2.11. For brevity, throughout this section we let Assumption 5.2.2, 5.2.3, 5.2.4 and 5.2.9 be satisfied. Let $1\geq u_0\geq 0$ and $v_0\geq 0$ be two given bounded initial data. Assume that two sets $\{i\in\mathbb{Z}; u_0(i)\neq 0\}\neq\emptyset$ and $\{i\in\mathbb{Z}; v_0(i)\neq 0\}\neq\emptyset$ have finite elements. Let $(u,v)=(u(t,i),v(t,i))$ be the bounded solution of (5.1.1) equipped with initial data (u_0,v_0).

5.5.1 Key lemmas

Now we state our key lemmas. Roughly speaking, from Assumption 5.2.3 and 5.2.4, one has two important facts: the predator will starve in the absence of the prey and the prey asymptotically reach its maximal environmental carrying capacity without the predator.

Our first key lemma reads as follows.

Lemma 5.5.1. *For all $\delta > 0$, there exist $M_\delta > 0$ and $T_\delta > 0$ such that the following estimate holds true*

$$v(t,i) \le \delta + M_\delta u(t,i),\ \forall t \ge T_\delta, i \in \mathbb{Z}. \tag{5.5.27}$$

Proof. By contradiction argument, we assume that there exist $\delta_0 > 0$ and sequences $(t_n)_n \subset [0,\infty)$ and $(i_n)_n \subset \mathbb{Z}$ such that

$$t_n \to \infty \text{ as } n \to \infty \text{ and } v(t_n,i_n) > \delta_0 + nu(t_n,i_n). \tag{5.5.28}$$

Set

$$u_n(t,i) := u(t+t_n, i+i_n) \text{ and } v_n(t,i) := v(t+t_n, i+i_n).$$

As we discussed in Section 5.3.3, one can choose a subsequence $(u_n,v_n)_n$ (still denoted with same index), $(u_\infty,v_\infty) \in S$ and $\tilde{\sigma} \in \Sigma$ such that

$u_n(t,i) \to u_\infty(t,i)$ and $v_n(t,i) \to v_\infty(t,i)$ as $n \to \infty$ locally uniformly for $(t,i) \in \mathbb{R}\times\mathbb{Z}$, and the function pair (u_∞,v_∞) satisfies $(\mathbf{P}_{\tilde{\sigma}})$ (see (5.3.18)).

Since v is bounded, then assumption (5.5.28) implies that $u(t_n,i_n) \to 0$ as $n \to \infty$, that is $u_\infty(0,0) = 0$. The strong maximum principle for $u_\infty-$equation ensures that $u_\infty \equiv 0$. Hence v_∞ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} v_\infty(t,i) = \sum_{j\in\mathbb{Z}} \tilde{J}_2(t,j)\left[v_\infty(t,i-j) - v_\infty(t,i)\right] + v_\infty(t,i)\tilde{g}\left(t,0,v_\infty(t,i)\right). \tag{5.5.29}$$

Claim 5.3.8 tells that $\tilde{g}(t,0,v) \le \tilde{g}(t,0,0)$ for all $v \ge 0$ and $t \in \mathbb{R}$. Due to the boundedness of v_∞, one can choose some $B > 0$ large enough such that $B \ge v_\infty(t,i)$ for all $(t,i) \in [0,\infty)\times\mathbb{Z}$. For each $t_0 < 0$, we define

$$\overline{v}(t;t_0) := B\exp\left\{\int_{t_0}^t \tilde{g}(s,0,0)\mathrm{d}s\right\}.$$

One can verify that $\overline{v}(t;t_0)$ is the super-solution of (5.5.29). The comparison principle implies that $v_\infty(t,i) \le \overline{v}(t;t_0)$ for all $t_0 < 0$, $t \ge t_0$ and $i \in \mathbb{Z}$. As a special case, letting $t = 0$, one has

$$v_\infty(0,i) \le \overline{v}(0;t_0) \text{ for all } i \in \mathbb{Z} \text{ and } t_0 < 0.$$

Recalling $\langle g(t,0,0)\rangle < 0$ in Assumption 5.2.4 and the definition of mean value, one obtains that $\langle \tilde{g}(t,0,0)\rangle < 0$. Let us observe that

$$\lim_{t_0\to-\infty}\int_{t_0}^0 \tilde{g}(s,0,0)\mathrm{d}s = \lim_{t_0\to-\infty}(-t_0)\cdot\frac{1}{0-t_0}\int_{t_0}^0 \tilde{g}(s,0,0)\mathrm{d}s = -\infty.$$

Hence we obtain that

$$v_\infty(0,i) \le \overline{v}(0;t_0) = \lim_{t_0\to-\infty} B\exp\left\{\int_{t_0}^0 \tilde{g}(s,0,0)\mathrm{d}s\right\} = 0.$$

This contradicts $v_\infty(0,0) > \delta_0 > 0$ which is obtained by passing to the limit $n \to \infty$ in (5.5.28). The proof is completed. □

To state the next proposition, for each $\delta \in [0, \frac{1}{2L})$, let us define $c_u^*(\delta)$ given by

$$c_u^*(\delta) := \inf_{0<\lambda<\mathrm{abs}(J_1)} \lambda^{-1}\bigg(\sum_{j\in\mathbb{Z}} \langle J_1(\cdot, j)\rangle \left[e^{\lambda j} - 1\right] + 1 - L\delta\bigg),$$

where L is Lipschitz constant defined in Remark 5.2.6. Same as Remark 5.2.8, one can rewrite $c_u^*(\delta)$ as

$$c_u^*(\delta) = \min_{0<\lambda<\mathrm{abs}(J_1)} \lambda^{-1}\bigg(\sum_{j\in\mathbb{Z}} \langle J_1(\cdot, j)\rangle \left[e^{\lambda j} - 1\right] + 1 - L\delta\bigg). \tag{5.5.30}$$

Next we apply the above key lemma to prove that u is persistent on the interval $[-ct, ct]$ with $t \gg 1$ for all $c \in (0, c_u^*)$.

Proposition 5.5.2. *For all $\tilde{c} \in [0, c_u^*)$, one has*

$$\liminf_{t\to\infty} \inf_{|i|\le \tilde{c}t} u(t, i) > 0.$$

Proof. Recalling (5.2.6) and Lemma 5.5.1, for each given $\delta > 0$, there exist $M_\delta > 0$ and $T_\delta > 0$ such that the solution $u(t, i)$ of (5.1.1) satisfies following differential inequality, for all $t \ge T_\delta$ and $i \in \mathbb{Z}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t, i) \ge \sum_{j\in\mathbb{Z}} J_1(t, j) \left[u(t, i-j) - u(t, i)\right] + u(t, i)\Big(1 - Lu(t, i) - L\big(\delta + M_\delta u(t, i)\big)\Big).$$

Let $\underline{u}(t, i)$ be the solution of following equation for all $t \ge 0$ and $i \in \mathbb{Z}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \underline{u}(t, i) = \sum_{j\in\mathbb{Z}} J_1(t + T_\delta, j) \left[\underline{u}(t, i-j) - \underline{u}(t, i)\right] + \underline{u}(t, i)\Big(1 - L\delta - L\left(1 + M_\delta\right) \underline{u}(t, i)\Big). \tag{5.5.31}$$

equipped with nontrivial initial data $0 \le \underline{u}(0, \cdot) \le \frac{1-L\delta}{L(1+M_\delta)}$ which satisfies $\underline{u}(0, \cdot) \le u(T_\delta, \cdot)$ and the set $\{i \in \mathbb{Z} : \ \underline{u}(0, i) \ne 0\}$ has finite elements. Then the comparison principle implies that

$$u(t + T_\delta, i) \ge \underline{u}(t, i), \ \forall t \ge 0, \forall i \in \mathbb{Z}.$$

From the spreading speeds result for scalar equation (5.5.31) (see Proposition 5.3.14), one has for all $c \in [0, c_u^*(\delta))$,

$$\liminf_{t\to\infty} \inf_{|i|\le ct} u(t + T_\delta, i) \ge \lim_{t\to\infty} \inf_{|i|\le ct} \underline{u}(t, i) = \frac{1 - L\delta}{L + LM_\delta} > 0.$$

Since $c \in [0, c_u^*(\delta))$ is arbitrary, then one can get rid of T_δ and obtain that

$$\liminf_{t\to\infty} \inf_{|i|\le ct} u(t, i) > 0, \ \forall c \in [0, c_u^*(\delta)).$$

Note that $\delta \mapsto c_u^*(\delta)$ is continuous on the interval $\left[0, \frac{1}{2L}\right)$ with $c_u^*(0) = c_u^*$ and $c_u^*(\delta) < c_u^*$ for $\delta \in \left(0, \frac{1}{2L}\right)$. Thus, for all $\tilde{c} \in [0, c_u^*)$, there exists some $\delta' > 0$ small enough such that $c_u^*(\delta') \in (\tilde{c}, c_u^*)$. Combining with the above limit, one has

$$\liminf_{t\to\infty} \inf_{|i|\le \tilde{c}t} u(t, i) > 0, \ \forall \tilde{c} \in [0, c_u^*).$$

The proof is completed. □

Next we show another important estimate.

Lemma 5.5.3. *Let $c \in [0, c_u^*)$ be given. For all $\alpha > 0$, there exist $M_\alpha > 0$ and $T_\alpha > 0$ such that the following estimate holds true*

$$1 - u(t,i) \leq \alpha + M_\alpha v(t,i), \ \forall t \geq T_\alpha, |i| \leq ct.$$

Proof. By contradiction, we assume that there exist $\alpha_0 > 0$ and sequences $(t_n)_n$ and $(i_n)_n$ such that

$$\begin{aligned} &|i_n| \leq ct_n, \ t_n \to \infty \text{ as } n \to \infty, \\ &\text{and } 1 - u(t_n, i_n) > \alpha_0 + nv(t_n, i_n), \ \forall n \geq 1. \end{aligned} \tag{5.5.32}$$

Set

$$u^n(t,i) := u(t + t_n, i + i_n) \text{ and } v^n(t,i) := v(t + t_n, i + i_n).$$

By the same analysis in Section 5.3.3, one can extract subsequence such that $u^n(t,i) \to u^\infty(t,i)$ and $v^n(t,i) \to v^\infty(t,i)$ as $n \to \infty$ locally uniformly for $(t,i) \in \mathbb{R} \times \mathbb{Z}$. As well as, there exists $\tilde{\sigma} \in \Sigma$ such that (u^∞, v^∞) satisfies $(\mathbf{P}_{\tilde{\sigma}})$ (see (5.3.18)). Recalling $0 \leq u \leq 1$, the assumption (5.5.32) yields that

$$v(t_n, i_n) \to 0 \text{ as } n \to \infty.$$

Hence one has $v^\infty(0,0) = 0$ and the strong maximum principle for v^∞-equation in $(\mathbf{P}_{\tilde{\sigma}})$ implies that $v^\infty \equiv 0$. Therefore $u^\infty = u^\infty(t,i)$ satisfies following Fisher-KPP equation

$$\frac{\mathrm{d}}{\mathrm{d}t} u^\infty(t,i) = \sum_{j \in \mathbb{Z}} \tilde{J}_1(t,j) \left[u^\infty(t, i-j) - u^\infty(t,i)\right] + u^\infty(t,i) \tilde{f}\left(t, u^\infty(t,i), 0\right). \tag{5.5.33}$$

Next, we claim that following property holds true.

Claim 5.5.4. *One has*

$$\inf_{(t,i) \in \mathbb{R} \times \mathbb{Z}} u^\infty(t,i) > 0.$$

Proof of Claim 5.5.4. Recalling Proposition 5.5.2, one has

$$\liminf_{t \to \infty} \inf_{|i| \leq \tilde{c}t} u(t,i) > 0, \ \forall \tilde{c} \in [0, c_u^*).$$

Fix $\tilde{c} \in (c, c_u^*)$. Let $(t_n)_n$ and $(i_n)_n$ be the same sequences (possibly up to sub-sequence) chosen in (5.5.32). Then there exist $T > 0$ large enough and some constant $m > 0$ such that

$$\inf_{t \geq T} \inf_{|i| \leq \tilde{c}t} u(t,i) \geq m.$$

This can be rewritten as for all $n \geq 0$, $t \geq T - t_n$ and $|i + i_n| \leq \tilde{c}(t + t_n)$,

$$u(t + t_n, i + i_n) \geq m.$$

Due to $|i_n| \leq ct_n$, it can rewrite as for all $n \geq 0$, $t \geq T - t_n$ and $|i| \leq \tilde{c}t + (\tilde{c} - c)t_n$,

$$u(t + t_n, i + i_n) \geq m.$$

Since $\tilde{c} - c > 0$, letting $n \to \infty$, then one has

$$u^\infty(t,i) \geq m, \forall (t,i) \in \mathbb{R} \times \mathbb{Z}.$$

This completes the proof of Claim 5.5.4. □

We come back to the proof of Lemma 5.5.3. From (5.3.12) and $f(t,1,0)\equiv 0$, one has $\tilde{f}(t,1,0)=0$. Since $\inf_{t\geq 0} f(t,u,0)>0$ for each $u\in[0,1)$, then $\inf_{t\in\mathbb{R}}\tilde{f}(t,u,0)>0$ for each $u\in[0,1)$. Set

$$\Theta := \inf_{(t,i)\in\mathbb{R}\times\mathbb{Z}} u^\infty(t,i) \text{ and } \tilde{h}(u) := \inf_{t\in\mathbb{R}} \tilde{f}(t,u,0).$$

Note that $\Theta>0$ and $\tilde{h}(u)>0$ for all $u\in[0,1)$. Next we consider $U(t)$ which is the solution of

$$U'(t)=U(t)\tilde{h}(U(t)),\ \ U(0)=\Theta.$$

Thus $U(t)$ is a sub-solution of (5.5.33). Since $u^\infty(s,i)\geq\Theta$ for all $(s,i)\in\mathbb{R}\times\mathbb{Z}$, then the comparison principle implies that

$$1\geq u^\infty(t+s,i)\geq U(t),\ \ \forall t\geq 0, s\in\mathbb{R}, i\in\mathbb{Z}.$$

Observe that $U(t)\to 1$ as $t\to\infty$. Thus, one has $u^\infty(0,0)=1$. This contradicts with the property $1-u^\infty(0,0)\geq\alpha_0>0$ that follows by passing to the limit $n\to\infty$ into (5.5.32). The proof is completed.

□

5.5.2 Proof of Theorem 5.2.10 (ii)

Now we apply Proposition 5.5.2 to complete the proof of Theorem 5.2.10 (*ii*).

Proof of Theorem 5.2.10 (*ii*). By contradiction argument, let us fix $c_v^*<c_1<c_2<c_u^*$ and assume that there exist sequences $(t_n)_n$ and $(i_n)_n$ such that

$$\begin{aligned}&t_n\to\infty \text{ as } n\to\infty,\\ &c_1t_n\leq |i_n|\leq c_2t_n,\\ &\text{and } \limsup_{n\to\infty} u(t_n,i_n)<1.\end{aligned}$$

Set $u_n(t,i):=u(t+t_n,i+i_n)$ and $v_n(t,i):=v(t+t_n,i+i_n)$. As discussed in Section 5.3.3, there exists $(u_\infty,v_\infty)\in S$ and $\tilde{\sigma}\in\Sigma$ such that $u_n(t,i)\to u_\infty(t,i)$ and $v_n(t,i)\to v_\infty(t,i)$ as $n\to\infty$ locally uniformly for $(t,i)\in\mathbb{R}\times\mathbb{Z}$. And (u_∞,v_∞) satisfies $(\mathbf{P}_{\tilde{\sigma}})$(see (5.3.18)). Note that $u_\infty(0,0)<1$.

Recall that we have proved that for all $c_1'>c_v^*$,

$$\lim_{t\to\infty}\sup_{|i|\geq c_1't} v(t,i)=0.$$

This yields that $v_\infty(0,0)=0$. The strong maximum principle for v_∞-equation implies that $v_\infty\equiv 0$. Hence u_∞ satisfies the following problem

$$\frac{\mathrm{d}}{\mathrm{d}t}u_\infty(t,i)=\sum_{j\in\mathbb{Z}}J_1(t,j)\left[u_\infty(t,i-j)-u_\infty(t,i)\right]+u_\infty(t,i)\tilde{f}\left(t,u_\infty(t,i),0\right).$$

On the other hand, from Proposition 5.5.2, one has, for all $0<\varepsilon<\min\{c_u^*-c_2,c_1-c_v^*\}$ small enough,

$$\liminf_{t\to\infty}\inf_{(c_1-\varepsilon)t\leq|i|\leq(c_2+\varepsilon)t} u(t,i)>0.$$

Next one can proceed similarly to the proof of Lemma 5.5.3 to obtain $u_\infty\equiv 1$. This is contradicted with assumption $u_\infty(0,0)<1$. We complete the proof of Theorem 5.2.10 (*ii*). □

5.5.3 Proof of Theorem 5.2.10 (iii) and Theorem 5.2.11 (ii)

In this subsection, we complete the proof of our inner spreading results. In order to prove Theorem 5.2.10 (*iii*) and Theorem 5.2.11 (*ii*) simultaneously, we define

$$c_* := \min\{c_u^*, c_v^*\}. \tag{5.5.34}$$

Recalling Proposition 5.5.2, note that

$$\liminf_{t\to\infty} \inf_{|i|\le ct} u(t,i) > 0, \ \forall c \in [0, c_u^*).$$

Since $c_* \le c_u^*$, then one has

$$\liminf_{t\to\infty} \inf_{|i|\le ct} u(t,i) > 0, \ \forall c \in [0, c_*).$$

Hence, it remains to show that

$$\liminf_{t\to\infty} \inf_{|i|\le ct} v(t,i) > 0 \text{ and } \limsup_{t\to\infty} \sup_{|i|\le ct} u(t,i) < 1, \quad \forall c \in [0, c_*).$$

To do this, we will use the key Lemma 5.5.3 to derive a differential inequality satisfied by v. With this help, we construct proper sub-solutions to show that v does not converge to 0 at some points (see **Step 1** and **Step 2** in below). Then we use some ideas in uniform persistence theory to show that the spreading is in fact uniformly on the whole interval (see **Step 3** and **Step 4**), which somehow close to those developed in [53, 55, 59]. Finally, we show that the limit of u is strictly less than 1 (see **Step 5**).

Now we prove Theorem 5.2.10 (*iii*) and Theorem 5.2.11 (*ii*).

Proof of Theorem 5.2.10 (*iii*) *and Theorem 5.2.11* (*ii*). We split into five steps to prove that

$$\liminf_{t\to\infty} \inf_{|i|\le ct} v(t,i) > 0 \text{ and } \limsup_{t\to\infty} \sup_{|i|\le ct} u(t,i) < 1, \quad \forall c \in [0, c_*).$$

Fix $c \in [0, c_*)$ and let $c' \in (c, c_*)$ be given. Recalling Lemma 5.5.3 and estimate (5.2.6), due to $c' < c_* \le c_u^*$, one can choose some $M_\alpha > 0$ and $T_\alpha > 0$ (large enough) such that $v(t,i)$ for all $t \ge T_\alpha$ and $|i| \le c't$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t,i) \ge \sum_{j\in\mathbb{Z}} J_2(t,j)\,[v(t,i-j) - v(t,i)] + r(t)v(t,i)\big(1 - L\alpha - L(1+M_\alpha)v(t,i)\big). \tag{5.5.35}$$

Hence $v(t+T_\alpha, i)$ satisfies for all $t \ge 0$ and $|i| \le c'(t+T_\alpha)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} v(t+T_\alpha,i) \ge &\sum_{j\in\mathbb{Z}} J_2(t+T_\alpha,j)\,[v(t+T_\alpha,i-j) - v(t+T_\alpha,i)] \\ &+ r(t+T_\alpha)v(t+T_\alpha,i)\big(1 - L\alpha - L(1+M_\alpha)v(t+T_\alpha,i)\big).\end{aligned} \tag{5.5.36}$$

Step 1: Prove that there exists $\varepsilon_0 > 0$ such that

$$\liminf_{t\to\infty} v(t,0) > \varepsilon_0.$$

To do this, we construct a continuous sub-solution of (5.5.36) in the following lemma. For better exposition, we postpone the proof of the following lemma.

Lemma 5.5.5. *For some $B_0 > 0$ large enough, for all $B > B_0$, there exists $R_0(B) > 0$ large enough enjoying the following properties:*
For all $B > B_0$ and $R > \max(R_0(B), B+1)$, for some $\eta > 0$ small enough, we define

$$\underline{v}_1(t,x) := \begin{cases} \eta \cos \frac{\pi x}{2R}, & x \in [-R,R], t \geq 0, \\ 0, & \textit{else}. \end{cases} \tag{5.5.37}$$

Then one can choose $T_\alpha > 0$ large enough such that $[-R,R] \subset [-c'(t+T_\alpha), c'(t+T_\alpha)]$ and $\underline{v}_1$ is the sub-solution of (5.5.36).

Let $\underline{v}_1$ be defined in above lemma. Let $\eta > 0$ be sufficiently small such that

$$v(T_\alpha, i) - \sup_{x\in[-R,R]} \underline{v}_1(0,x) \geq 0, \quad \forall i \in [-R,R] \cap \mathbb{Z}.$$

From Lemma 5.5.5 and the maximum principle (see Proposition 5.3.5), one can obtain that

$$v(t+T_\alpha, i) - \underline{v}_1(t,x) \geq 0, \ \forall t \geq 0, x \in [-R,R], i \in [-R,R] \cap \mathbb{Z}.$$

This implies that

$$v(t+T_\alpha, i) \geq \underline{v}_1(t,i), \ \forall t \geq 0, i \in [-R,R] \cap \mathbb{Z}.$$

Recalling the definition of $\underline{v}_1$, there exists some $\varepsilon_0 > 0$ such that

$$\liminf_{t\to\infty} v\left(t+T_\alpha, 0\right) \geq \liminf_{t\to\infty} \underline{v}_1(t,0) \geq 2\varepsilon_0 > 0.$$

Since $t \to \infty$, then one can get rid of T_α in the above limit. Thus, one obtains that

$$\liminf_{t\to\infty} v\left(t, 0\right) > \varepsilon_0.$$

The proof of **Step 1** is completed.

Before stating **Step 2**, we introduce some notations. For each given $B > 0$, set $J_{2,B} = J_{2,B}(t,i)$ by

$$J_{2,B}(t,i) := \begin{cases} J_2(t,i), & \forall t \geq 0, i \in [-B,B] \cap \mathbb{Z}, \\ 0, & \textit{else}. \end{cases} \tag{5.5.38}$$

For each given $B > 0$, $R > 0$ and $\gamma \geq 0$, we define function $t \mapsto c_{R,B}(\gamma)(t) \in L^\infty(0,\infty)$ given by

$$c_{R,B}(\gamma)(t) := \frac{2R}{\pi} \sum_{j\in\mathbb{Z}} J_{2,B}(t,j) e^{\gamma j} \sin \frac{\pi j}{2R}. \tag{5.5.39}$$

Let $C_\gamma > 0$ (which will be chosen in **Step 4**) and $T_\alpha > 0$ be given. We define

$$I(t) := \int_0^t c_{R,B}(\gamma)(s+T_\alpha) \mathrm{d}s + C_\gamma. \tag{5.5.40}$$

Next we claim that following property for above notations holds true.

Claim 5.5.6. *Fix $c \in [0, c_*)$ and $c' \in (c, c_*)$. Let $B_0 > 1$ and $R_0 > 0$ large enough be given. For all $B > B_0$, for all $R > \max(R_0, B)$, one can choose some $\gamma = \hat{\gamma} \in (0, \gamma^*)$ and $T_\alpha > 0$ large enough such that*

$$\left[-R + I(t), R + I(t)\right] \subset \left[-c'(t+T_\alpha), c'(t+T_\alpha)\right].$$

Herein γ^* is given in Proposition 5.2.7 and $I(t)$ is defined in (5.5.40) with choosing $\gamma = \hat{\gamma}$.

Proof of Claim 5.5.6. Recalling (5.5.39) and $j \mapsto J_2(\cdot, j) \in l^1(\mathbb{Z}, L^\infty(0,\infty))$, the Lebesgue dominated convergence theorem ensures that for each given $\gamma \geq 0$,

$$\lim_{\substack{B\to\infty\\R\to\infty}} c_{R,B}(\gamma)(t) = \sum_{j\in\mathbb{Z}} J_2(t,j)e^{\gamma j}j, \text{ uniformly for } t \geq 0. \tag{5.5.41}$$

Observe that the function $\gamma \mapsto \sum_{j\in\mathbb{Z}}\langle J_2(\cdot,j)\rangle e^{\gamma j}j$ is continuous and increasing for $\gamma \in [0,\infty)$. Herein $\langle J_2(\cdot,j)\rangle$ is the mean value of $J_2(\cdot,j)$. The symmetric of J_2 implies that

$$\sum_{j\in\mathbb{Z}}\langle J_2(\cdot,j)\rangle j = 0.$$

Since $c_v^* = \sum_{j\in\mathbb{Z}} \langle J_2(\cdot,j)\rangle\, e^{\gamma^* j}j$ from Proposition 5.2.7, then for all $\gamma \in [0,\gamma^*)$ one has

$$\sum_{j\in\mathbb{Z}} \langle J_2(\cdot,j)\rangle\, e^{\gamma j}j < c_v^*.$$

For some constant $m_0 > 0$ small enough, one can choose some $\hat{\gamma} \in (0,\gamma^*)$ such that

$$c' - \sum_{j\in\mathbb{Z}} \langle J_2(\cdot,j)\rangle\, e^{\hat{\gamma} j}j \geq 2m_0. \tag{5.5.42}$$

Recalling the limit in (5.5.41), the above estimate ensures that for some $R_0 > 0$ and $B_0 > 0$ large enough, for all $B > B_0$ and $R > \max(R_0, B)$, one has

$$c' - \langle c_{R,B}(\hat{\gamma})(\cdot)\rangle \geq m_0 > 0.$$

Due to the property of mean value, there exists some $b \in W^{1,\infty}(0,\infty)$ such that

$$c_{R,B}(\hat{\gamma})(t) + b'(t) < c', \ \forall t > 0.$$

Thus, it can be rewritten as

$$c_{R,B}(\hat{\gamma})(s+T_\alpha) + b'(s+T_\alpha) < c', \ \forall s > -T_\alpha.$$

This implies that

$$\int_0^t c_{R,B}(\hat{\gamma})(s+T_\alpha) + b'(s+T_\alpha)\mathrm{d}s < c't.$$

One can also observe that $\inf_{t\geq 0} c_{R,B}(\hat{\gamma})(t) > 0$. This is due to Assumption 5.2.2 $(J4)$ and $R > B$. Recalling the definition of $I(t)$ in (5.5.40), one can choose $C_{\hat{\gamma}} > 0$ such that $I(t) > 0$ for all $t > 0$. Note that

$$I(t) \leq c't + 2\|b\|_\infty + C_{\hat{\gamma}}, \ \forall t > 0.$$

One can choose $T_\alpha > 0$ large enough such that $c'T_\alpha > R + 2\|b\|_\infty + C_{\hat{\gamma}}$. Hence, we obtain that

$$[-R + I(t), R + I(t)] \subset [-c'(t+T_\alpha), c'(t+T_\alpha)]\,.$$

The claim is proved. □

Step 2: Let $I(t)$ be chosen in Claim 5.5.6. Prove that there exists $\varepsilon_1 > 0$ such that

$$\liminf_{t\to\infty} v(t,[I(t)]) > \varepsilon_1,$$

where $[I(t)]$ is the maximal integer not larger than $I(t)$ for $t > 0$.

To prove this, similarly as the proof of **Step 1**, we first construct a proper sub-solution of (5.5.36) in the following lemma. For a better exposition, we also postpone its proof.

Lemma 5.5.7. *For all $c \in [0, c_*)$, for the given $c' \in (c, c_*)$. For some $B_0 > 0$ large enough, for all $B > B_0$, there exists $R_0(B) > 0$ large enough enjoying following properties:*
For all $B > B_0$ and $R > \max(R_0(B), B+1)$, let $\hat{\gamma} \in (0, \gamma^*)$ be chosen in Claim 5.5.6. Let $T_\alpha > 0$ large enough and $I(t)$ be chosen in Claim 5.5.6. For some $\eta > 0$ small enough, for some $a \in W^{1,\infty}(0, \infty)$, we define the function $\underline{v}_2$ by

$$\underline{v}_2(t,x) := \begin{cases} \eta e^{a(t)} e^{-\hat{\gamma}(x-I(t))} \cos \frac{\pi(x-I(t))}{2R}, & x \in [-R+I(t), R+I(t)], t \geq 0, \\ 0, & else. \end{cases} \tag{5.5.43}$$

Then $\underline{v}_2$ is the sub-solution of (5.5.36).

Let $\underline{v}_2$ be defined in the above lemma. Let $\eta > 0$ be sufficiently small such that

$$v(T_\alpha, i) - \sup_{x \in [-R,R]} \underline{v}_2(0,x) \geq 0, \ \forall i \in [-R, R] \cap \mathbb{Z}.$$

From Lemma 5.5.7 and the maximum principle (see Proposition 5.3.5), one can obtain that

$$v(t+T_\alpha, i) - \underline{v}_2(t,x) \geq 0, \ \forall t \geq 0, x \in [-R+I(t), R+I(t)], i \in [-R+I(t), R+I(t)] \cap \mathbb{Z}.$$

This means

$$v(t+T_\alpha, i) \geq \underline{v}_2(t,i), \ \forall t \geq 0, i \in [-R+I(t), R+I(t)] \cap \mathbb{Z}.$$

Note that $0 \leq I(t) - [I(t)] < 1$. Letting $R \geq 2$ be large enough, there exists $\varepsilon_1 > 0$ such that

$$\begin{aligned} \liminf_{t\to\infty} v\left(t+T_\alpha, [I(t)]\right) &\geq \liminf_{t\to\infty} \underline{v}_2(t, [I(t)]) \\ &= \liminf_{t\to\infty} \eta e^{a(t)} e^{-\gamma([I(t)]-I(t))} \cos \frac{\pi\left([I(t)] - I(t)\right)}{2R} \\ &> 2\varepsilon_1. \end{aligned}$$

Since $t \to \infty$, then one can get rid of T_α in the above limit and obtain that

$$\liminf_{t\to\infty} v\left(t, [I(t)]\right) > \varepsilon_1.$$

The proof of **Step 2** is completed.

Step 3: Show that

$$\liminf_{t\to\infty} \inf_{i \in [0, [kI(t)]] \cap \mathbb{Z}} v(t,i) > 0, \ \forall k \in (0,1).$$

In this step, we apply a similar idea in the proof of Lemma 2.6 in [59], as well Lemma 5.3.18 in this paper.

By contradiction we assume that there exist $k \in (0,1)$, $k_n \in [0,k]$ and sequence $(t_n)_n$ satisfying $t_n \to \infty$ as $n \to \infty$ such that

$$v(t_n, [k_n I(t_n)]) \to 0 \text{ as } n \to \infty. \tag{5.5.44}$$

Firstly, let us observe that $[k_n I(t_n)] \to \infty$ as $n \to \infty$. If not, then $[k_n I(t_n)] \to l \in \mathbb{Z}$ which may happen when $k_n \to 0$. As discussed in Section 5.3.3, one can extract sub-sequence such that

$$v(t+t_n, i) \to v_\infty(t,i), \text{ locally uniformly for } (t,i) \in \mathbb{R} \times \mathbb{Z} \text{ as } n \to \infty.$$

Recalling (5.5.35), one can observe that $v(t+t_n,i)$ satisfies, for all $t \geq -t_n + T_\alpha$ and $|i| \leq c'(t+t_n)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t+t_n,i) \geq \sum_{j\in\mathbb{Z}} J_2(t+t_n,j)\left[v(t+t_n,i-j)-v(t+t_n,i)\right] + r(t+t_n)v(t+t_n,i)\big(1-L\alpha-L(1+M_\alpha)v(t+t_n,i)\big).$$

Similar to Section 5.3.3, one can derive that v_∞ satisfies for all $t\in\mathbb{R}$ and $i\in\mathbb{Z}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}v_\infty(t,i) \geq \sum_{j\in\mathbb{Z}} \tilde{J}_2(t,j)\left[v_\infty(t,i-j)-v_\infty(t,i)\right]+\tilde{r}(t)v_\infty(t,i)\big(1-L\alpha-L(1+M_\alpha)v_\infty(t,i)\big),$$

where

$$J_2(t+t_n,j) \to \tilde{J}_2(t,j) \text{ as } n\to\infty \text{ in weak-}\star \text{ topology of } L^\infty_{\mathrm{loc}}(\mathbb{R}) \text{ for all } j\in\mathbb{Z},$$
$$r(t+t_n)\to\tilde{r}(t) \text{ as } n\to\infty \text{ in local uniform topology of } C(\mathbb{R};\mathbb{R}).$$

Note that (5.5.44) implies $v_\infty(0,l)=0$. Hence, the strong maximum principle applies and ensures that $v_\infty \equiv 0$. This contradicts the property $\liminf_{t\to\infty} v(t,0) > \varepsilon_0$ in **Step 1**. Thus, we obtain that $[k_n I(t_n)] \to \infty$ as $n\to\infty$. This ensures that one can choose a sub-sequence such that $[k_n I(t_n)] \geq 2$ for all $n\geq 1$.

Possibly up to a sub-sequence, (5.5.44) can also be rewritten as

$$v(t_n,[k_n I(t_n)]-1) \to 0 \text{ as } n\to\infty. \tag{5.5.45}$$

Again, up to a sub-sequence, one may assume that $I(0)=0<[k_nI(t_n)]-1<[I(t_n)]$ for all $n\geq 1$. There exists $t_n' < t_n$ such that

$$[I(t_n')] = [k_nI(t_n)]-1.$$

One can also observe that $t_n'\to\infty$ as $n\to\infty$. From the chosen of t_n', one has

$$v(t_n',[I(t_n')]) = v(t_n',[k_nI(t_n)]-1).$$

Recalling **Step 2**, there exists $\varepsilon_1>0$ such that

$$\liminf_{n\to\infty} v(t_n',[I(t_n')]) \geq \varepsilon_1 > 0.$$

Let $\tilde{\varepsilon}_0>0$ be chosen later. We define

$$t_n'' := \sup\left\{t\in(t_n',t_n):\ v(t,[I(t_n')]) \geq \frac{\min\{\varepsilon_1,\tilde{\varepsilon}_0\}}{2}\right\}.$$

The chosen of t_n' and (5.5.45) implies that

$$v(t_n,[I(t_n')]) = v(t_n,[k_nI(t_n)]-1) \to 0 \text{ as } n\to\infty.$$

Then one may assume that for all $n\geq1$ large enough one has

$$\begin{cases} v(t_n'',[I(t_n')]) = \frac{\min\{\varepsilon_1,\tilde{\varepsilon}_0\}}{2}, \\ v(t,[I(t_n')]) \leq \frac{\min\{\varepsilon_1,\tilde{\varepsilon}_0\}}{2}, \quad \forall t\in[t_n'',t_n], \\ v(t_n,[I(t_n')]) \leq \frac{1}{n}. \end{cases}$$

Next we claim that $t_n - t''_n \to \infty$ as $n \to \infty$. If not, we assume that $t_n - t''_n \to \tau \in \mathbb{R}$. Set

$$v^n(t,i) := v(t + t''_n, i + [I(t'_n)]), \text{ for all } t \geq -t''_n + T_\alpha \text{ and } |i + [I(t'_n)]| \leq c'(t + t''_n).$$

The regularity of v ensures that one can extract sub-sequence such that

$$v^n(t,i) \to v^\infty(t,i) \text{ locally uniformly for } (t,i) \in \mathbb{R} \times \mathbb{Z} \text{ as } n \to \infty.$$

Let us observe that $v^\infty(t,i)$ is defined in $\mathbb{R} \times \mathbb{Z}$. Indeed, since $t'_n < t''_n < t_n$ and sequence $t'_n \to \infty$ as $n \to \infty$, then one has $t''_n \to \infty$ as $n \to \infty$. Note that $\langle c_{R,B}(\hat{\gamma})\rangle < c'$. From the definition of mean value and $I(t)$, one has

$$c't''_n - [I(t'_n)] \geq c't'_n - [I(t'_n)] = \left(c' - \frac{[I(t'_n)]}{t'_n}\right) t'_n \to \infty \text{ as } n \to \infty.$$

Also, one can observe that

$$v^n(t,i) = v(t + t''_n, i + [I(t'_n)]), \text{ for } |i| \leq c't + c't''_n - [I(t'_n)] \text{ and } t \geq -t''_n + T_\alpha.$$

Letting $n \to \infty$, we obtain that $v^\infty(t,i)$ is defined in $\mathbb{R} \times \mathbb{Z}$.

As discussed previously, we obtain that $v^\infty(t,i)$ satisfies for all $(t,i) \in \mathbb{R} \times \mathbb{Z}$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} v^\infty(t,i) \geq & \sum_{j \in \mathbb{Z}} J_2^\infty(t,j) \left[v^\infty(t,i-j) - v^\infty(t,i)\right] \\ & + r^\infty(t) v^\infty(t,i) \big(1 - L\alpha - L(1+M_\alpha) v^\infty(t,i)\big), \end{aligned} \tag{5.5.46}$$

where

$$\begin{aligned} & J_2(t + t''_n, j) \to J_2^\infty(t,j) \text{ as } n \to \infty \text{ in weak-}\star \text{ topology of } L^\infty_{\text{loc}}(\mathbb{R}), \text{ for all } j \in \mathbb{Z}, \\ & r(t + t''_n) \to r^\infty(t) \text{ as } n \to \infty \text{ in local uniform topology of } C(\mathbb{R};\mathbb{R}). \end{aligned}$$

One can note that

$$v^\infty(0,0) = \frac{\min\{\varepsilon_1, \tilde{\varepsilon}_0\}}{2} > 0. \tag{5.5.47}$$

Recall that the proof of Lemma 5.3.10 and Remark 5.3.12. Similarly, one can show that there exists a constant $\tilde{\varepsilon}_0 > 0$ (which is independent of initial condition $v^\infty(0,i) \not\equiv 0$ and independent of the time shift limit functions J_2^∞ and r^∞) such that

$$\liminf_{t \to \infty} v^\infty(t,0) > \tilde{\varepsilon}_0. \tag{5.5.48}$$

Due to the assumption $t_n - t''_n \to \tau$, one has

$$v^\infty(\tau,0) = \lim_{n \to \infty} v(t_n - t''_n + t''_n, [I(t'_n)]) = 0.$$

Then the strong maximum principle implies that $v^\infty \equiv 0$. This contradicts (5.5.47). Hence, we obtain that $t_n - t''_n \to \infty$ as $n \to \infty$. It implies that

$$v^\infty(t,0) \leq \frac{\min\{\varepsilon_1, \tilde{\varepsilon}_0\}}{2}, \ \forall t \geq 0.$$

This contradicts with (5.5.48). So we complete the proof of **Step 3**.

Step 4: Prove that

$$\liminf_{t \to \infty} \inf_{|i| \leq ct} v(t,i) > 0, \ \forall c \in [0, c_*).$$

As we discussed in above, for each given $c'' \in (c, c')$, one can choose $\hat{\gamma} \in (0, \gamma^*)$ (for notation simplicity, still denote by $\hat{\gamma}$) such that

$$c'' < \langle c_{R,B}(\hat{\gamma}) \rangle < c'.$$

The property of mean value implies that there exists some $\tilde{b} \in W^{1,\infty}(0, \infty)$ such that

$$c_{R,B}(\hat{\gamma})(t) + \tilde{b}'(t) > c'', \ \forall t > 0.$$

Recalling the definition of $I(t)$ in (5.5.40), let us choose $C_{\hat{\gamma}}$ large enough such that $C_{\hat{\gamma}} \geq 2\|\tilde{b}\|_\infty \geq \tilde{b}(t + T_\alpha) - \tilde{b}(T_\alpha)$ for all $t > -T_\alpha$. One can observe that

$$I(t) = \int_0^t c_{R,B}(\hat{\gamma})(s + T_\alpha)\mathrm{d}s + C_{\hat{\gamma}} \geq \int_0^t c_{R,B}(\hat{\gamma})(s + T_\alpha) + \tilde{b}'(s + T_\alpha)\mathrm{d}s \geq c''t, \ \forall t > 0.$$

Set $k_0 = \frac{c}{c''} \in (0, 1)$. Note that

$$k_0 I(t) \geq k_0 c'' t = ct, \ \forall t > 0.$$

Hence, one has

$$kI(t) \geq ct, \ \forall k \in (k_0, 1), \ \forall t > 0.$$

Recall in **Step 3** that

$$\liminf_{t\to\infty} \inf_{i\in[0,[kI(t)]]\cap\mathbb{Z}} v\,(t, i) > 0.$$

We obtain that

$$\liminf_{t\to\infty} \inf_{i\in[0,ct]\cap\mathbb{Z}} v\,(t, i) > 0, \ \forall c \in [0, c_*).$$

By a symmetric argument, one has

$$\liminf_{t\to\infty} \inf_{|i|\leq ct} v\,(t, i) > 0, \ \forall c \in [0, c_*).$$

The proof of **Step 4** is completed.

Step 5: Prove that

$$\limsup_{t\to\infty} \sup_{|i|\leq ct} u\,(t, i) < 1, \ \forall c \in [0, c_*).$$

To do this, we proceed by contradiction argument. Assume that there exist $\tilde{c} \in [0, c_*)$ and sequences $(t_n)_n$ and $(i_n)_n$ such that

$$\begin{aligned} &|i_n| \leq \tilde{c} t_n, \\ &t_n \to \infty \text{ and } u(t_n, i_n) \to 1, \quad \text{as } n \to \infty. \end{aligned}$$

As analysis in Section 5.3.3, one can extract sub-sequence such that $(u(t+t_n, i+i_n), v(t+t_n, i+i_n))$ converges to $(u_\infty(t, i), v_\infty(t, i))$ which satisfies $(\mathbf{P}_{\tilde{\sigma}})$ (see (5.3.18)) with suitable $\tilde{\sigma} \in \Sigma$. Note that $u_\infty(0, 0) = 1$ and $0 \leq u_\infty \leq 1$. We can apply the strong maximum principle to obtain that $u_\infty \equiv 1$. One can observe that the first equation in $(\mathbf{P}_{\tilde{\sigma}})$ implies that

$$\tilde{f}(t, 1, v) = 0, \ \forall t \in \mathbb{R}.$$

Also, we have proved that

$$\liminf_{t\to\infty} \inf_{|i|\leq \hat{c}t} v(t, i) > 0, \ \forall \hat{c} \in [0, c_*).$$

Similar as proof of Claim 5.5.4, one can show that

$$\inf_{(t,i)\in\mathbb{R}\times\mathbb{Z}} v_\infty(t,i) > 0.$$

From Assumption 5.2.3 $(f5)$, one has $\sup_{t\in\mathbb{R}} \tilde{f}(t,1,v_\infty) < 0$ for all $v_\infty > 0$. This is a contradiction. We complete the proof of **Step 5**.

Thus, the proof of Theorem 5.2.10 (iii) and 5.2.11 (ii) is completed. □

Now we show that the proof of Lemma 5.5.5 and 5.5.7.

Proof of Lemma 5.5.5. Let $c' \in (0, c_*)$ be given. For each fixed $R > 0$, one can choose $T_\alpha > 0$ large enough such that $c'T_\alpha > R$. Hence, one has

$$[-R, R] \subset [-c'(t+T_\alpha), c'(t+T_\alpha)].$$

We define $x_r := x - [x]$, where $[x]$ means taking the maximal integer not larger than x. Let us observe that for $x \in [-R, R]$,

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} J_2(t+T_\alpha, j)\underline{v}_1(t, x-j) &= \sum_{j\in\mathbb{Z}} J_2(t+T_\alpha, j)\underline{v}_1(t, x_r + [x] - j) \\
&= \sum_{j\in\mathbb{Z}} J_2(t+T_\alpha, [x]-j)\underline{v}_1(t, x_r + j) \\
&= \sum_{j\in[-R-x_r, R-x_r]\cap\mathbb{Z}} J_2(t+T_\alpha, [x]-j)\eta\cos\frac{\pi(x_r+j)}{2R} \\
&\geq \sum_{j\in[-R-x_r, R-x_r]\cap\mathbb{Z}} J_{2,B}(t+T_\alpha, [x]-j)\eta\cos\frac{\pi(x_r+j)}{2R} \\
&\geq \sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, [x]-j)\eta\cos\frac{\pi(x_r+j)}{2R}.
\end{aligned} \tag{5.5.49}$$

Herein $J_{2,B}$ is defined in (5.5.38). In (5.5.49), the first inequality is ensured by $J_2 \geq J_{2,B}$ and $\cos\frac{\pi(x_r+j)}{2R} \geq 0$ for all $x_r + j \in [-R, R]$. Note that $\cos\frac{\pi(x_r+j)}{2R} \leq 0$ for all $R \leq |x_r + j| \leq 2R$. And one can observe that if the integer $[x] \in [-R, R] \cap \mathbb{Z}$ and $|x_r + j| \geq 2R$, then $|[x] - j| \geq R - 1 > B$ and $J_{2,B}(t+T_\alpha, [x]-j) = 0$. So the second inequality holds. The last term in (5.5.49) can rewrite as

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, [x]-j)\eta\cos\frac{\pi(x_r+j)}{2R} \\
&= \sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, j)\eta\cos\frac{\pi(x_r+[x]-j)}{2R} \\
&= \sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, j)\eta\left(\cos\frac{\pi x}{2R}\cos\frac{\pi j}{2R} + \sin\frac{\pi x}{2R}\sin\frac{\pi j}{2R}\right).
\end{aligned} \tag{5.5.50}$$

For easy of writing, we set $\overline{J}_2(t+T_\alpha) := \sum_{j\in\mathbb{Z}} J_2(t+T_\alpha, j)$ and define the operator $\mathscr{L}$ given by

$$\begin{aligned}
\mathscr{L}\phi(t,x) :=& \frac{\mathrm{d}}{\mathrm{d}t}\phi(t,x) - \sum_{j\in\mathbb{Z}} J_2(t+T_\alpha, j)\phi(t, x-j) + \overline{J}_2(t+T_\alpha)\phi(t,x) \\
&- r(t+T_\alpha)(1-L\alpha)\phi(t,x).
\end{aligned} \tag{5.5.51}$$

From (5.5.49) and (5.5.50), one can observe that for all $t \geq 0$ and $x \in [-R, R]$,

$$\begin{aligned}\mathscr{L}\underline{v}_1(t,x) \leq & -\sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, j)\eta\left(\cos\frac{\pi x}{2R}\cos\frac{\pi j}{2R} + \sin\frac{\pi x}{2R}\sin\frac{\pi j}{2R}\right)\\ & + \overline{J}_2(t+T_\alpha)\eta\cos\frac{\pi x}{2R} - r(t+T_\alpha)(1-L\alpha)\eta\cos\frac{\pi x}{2R}\\ = & \, \eta\cos\frac{\pi x}{2R}\left(-\sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, j)\cos\frac{\pi j}{2R} + \overline{J}_2(t+T_\alpha) - r(t+T_\alpha)(1-L\alpha)\right)\\ & - \eta\sin\frac{\pi x}{2R}\sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, j)\sin\frac{\pi j}{2R}.\end{aligned}$$

Since J_2 is symmetric, then

$$\sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, j)\sin\frac{\pi j}{2R} = 0.$$

Due to $J_2 \in l^1(\mathbb{Z}, L^\infty(\mathbb{R}^+))$, one has

$$\left|\sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, j)\cos\frac{\pi j}{2R}\right| \leq \sum_{j\in\mathbb{Z}} \|J_2(\cdot, j)\|_\infty < \infty.$$

Applying the Lebesgue dominated convergence theorem, one has

$$\lim_{\substack{B\to\infty\\ R\to\infty}} \sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, j)\cos\frac{\pi j}{2R} = \sum_{j\in\mathbb{Z}} J_2(t+T_\alpha, j) = \overline{J}_2(t+T_\alpha), \text{ uniformly for } t \geq 0.$$

Note that Assumption 5.2.4 ($g3$) yields $\inf_{t\geq 0} r(t+T_\alpha) > 0$. Then one can choose $R > 0$ and $B > 0$ large enough such that for some $\theta_0 > 0$, the following inequality holds true

$$\sup_{t\geq 0}\left\{-\sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, j)\cos\frac{\pi j}{2R} + \overline{J}_2(t+T_\alpha) - r(t+T_\alpha)(1-L\alpha)\right\} \leq -\theta_0.$$

Hence for all $t \geq 0$ and $x \in [-R, R]$ one has

$$\mathscr{L}\underline{v}_1(t,x) \leq -\theta_0\eta\cos\frac{\pi x}{2R} = -\theta_0\underline{v}_1(t,x).$$

Let us choose $\eta > 0$ small enough such that

$$r(t+T_\alpha)L(1+M_\alpha)\underline{v}_1(t,x) \leq \eta\|r\|_\infty L(1+M_\alpha) < \theta_0, \ \forall t \geq 0, x \in [-R, R].$$

So that for all $t \geq 0$ and $x \in [-R, R]$, one has

$$-\theta_0\underline{v}_1(t,x) \leq -r(t+T_\alpha)L(1+M_\alpha)\underline{v}_1^2(t,x).$$

Hence we obtain that $\mathscr{L}\underline{v}_1(t,x) \leq -r(t+T_\alpha)L(1+M_\alpha)\underline{v}_1^2(t,x)$, namely, $\underline{v}_1(t,x)$ is a sub-solution of (5.5.36). This completes the proof of Lemma 5.5.5. □

Next we prove that Lemma 5.5.7.

Proof of Lemma 5.5.7. By the same analysis in (5.5.49), one can obtain that for all $x \in [-R+I(t), R+I(t)]$ and $t \geq 0$,

$$\sum_{j\in\mathbb{Z}} J_2(t+T_\alpha, j)\underline{v}_2(t, x-j) \geq \sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, j)\eta e^{a(t)} e^{-\hat{\gamma}(x-j-I(t))} \cos\frac{\pi(x-j-I(t))}{2R}.$$

Recalling operator $\mathscr{L}$ defined in (5.5.51), through direct computation, one can observe that for all $t \geq 0$ and $x \in [-R+I(t), R+I(t)]$,

$$\begin{aligned}
\mathscr{L}\underline{v}_2(t,x) \leq{}& \eta e^{a(t)} e^{-\hat{\gamma}(x-I(t))} \cos\frac{\pi\,(x-I(t))}{2R}\,(a'(t)+\hat{\gamma}I'(t)) \\
&+ \frac{\pi}{2R} I'(t)\eta e^{a(t)} e^{-\hat{\gamma}(x-I(t))} \sin\frac{\pi\,(x-I(t))}{2R} \\
&- \eta e^{a(t)} e^{-\hat{\gamma}(x-I(t))} \sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, j) e^{\hat{\gamma}j} \cos\frac{\pi\,(x-I(t))}{2R}\cos\frac{\pi j}{2R} \\
&- \eta e^{a(t)} e^{-\hat{\gamma}(x-I(t))} \sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, j) e^{\hat{\gamma}j} \sin\frac{\pi\,(x-I(t))}{2R}\sin\frac{\pi j}{2R} \\
&+ \overline{J}_2(t+T_\alpha)\eta e^{a(t)} e^{-\hat{\gamma}(x-I(t))} \cos\frac{\pi(x-I(t))}{2R} \\
&- r(t+T_\alpha)(1-L\alpha)\eta e^{a(t)} e^{-\hat{\gamma}(x-I(t))} \cos\frac{\pi(x-I(t))}{2R} \\
={}& \eta e^{a(t)} e^{-\hat{\gamma}(x-I(t))} \cos\frac{\pi(x-I(t))}{2R}\Bigg\{ a'(t) + \hat{\gamma} c_{R,B}(\hat{\gamma})(t+T_\alpha) \\
&\quad - \sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, j) e^{\hat{\gamma}j}\cos\frac{\pi j}{2R} + \overline{J}_2(t+T_\alpha) - r(t+T_\alpha)(1-L\alpha)\Bigg\} \\
&+ \eta e^{a(t)} e^{-\hat{\gamma}(x-I(t))} \sin\frac{\pi(x-I(t))}{2R}\Bigg\{ \frac{\pi}{2R} c_{R,B}(\hat{\gamma})(t+T_\alpha) \\
&\quad - \sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, j) e^{\hat{\gamma}j}\sin\frac{\pi j}{2R}\Bigg\}.
\end{aligned}$$

Recalling that the definition of $c_{R,B}(\hat{\gamma})$ in (5.5.39), the last term in above equation disappear. Next, let us consider the remain term in above equation. Due to $\hat{\gamma} \in (0, \gamma^*)$, Proposition 5.2.7 (ii) ensures that

$$\left.\frac{\mathrm{d}\langle c_v(\gamma)\rangle}{\mathrm{d}\gamma}\right|_{\gamma=\hat{\gamma}} = \frac{\left\langle \hat{\gamma}\sum_{j\in\mathbb{Z}} J_2(\cdot, j)e^{\hat{\gamma}j}j - \sum_{j\in\mathbb{Z}} J_2(\cdot, j)e^{\hat{\gamma}j} + \overline{J}_2(\cdot) - r(\cdot)\right\rangle}{\hat{\gamma}^2} < 0. \tag{5.5.52}$$

Thus, for some $\theta_0 > 0$, there exist some $\alpha > 0$ small enough such that

$$\left\langle \hat{\gamma}\sum_{j\in\mathbb{Z}} J_2(\cdot, j)e^{\hat{\gamma}j}j - \sum_{j\in\mathbb{Z}} J_2(\cdot, j)e^{\hat{\gamma}j} + \overline{J}_2(\cdot) - r(\cdot)\right\rangle + \|r\|_\infty L\alpha < -2\theta_0.$$

The Lebesgue dominated convergence theorem ensures that

$$\lim_{\substack{B\to\infty\\ R\to\infty}} \sum_{j\in\mathbb{Z}} J_{2,B}(t+T_\alpha, j)e^{\hat{\gamma}j}\hat{\gamma}\frac{2R}{\pi}\sin\frac{\pi j}{2R} = \hat{\gamma}\sum_{j\in\mathbb{Z}} J_2(t+T_\alpha, j)e^{\hat{\gamma}j}j, \text{ uniformly for } t \geq 0,$$

and

$$\lim_{\substack{B\to\infty\\R\to\infty}}\sum_{j\in\mathbb{Z}}J_{2,B}(t+T_\alpha,j)e^{\hat{\gamma}j}\cos\frac{\pi j}{2R}=\sum_{j\in\mathbb{Z}}J_2(t+T_\alpha,j)e^{\hat{\gamma}j},\ \text{uniformly for } t\geq 0.$$

Hence, for $R>0$ and $B>0$ large enough, the property of mean value ensures that there exists $a\in W^{1,\infty}(0,\infty)$ such that for all $t\geq 0$,

$$\begin{aligned}a'(t)+\hat{\gamma}\frac{2R}{\pi}\sum_{j\in\mathbb{Z}}J_{2,B}(t+T_\alpha,j)e^{\hat{\gamma}j}\sin\frac{\pi j}{2R}-\sum_{j\in\mathbb{Z}}J_{2,B}(t+T_\alpha,j)e^{\hat{\gamma}j}\cos\frac{\pi j}{2R}+\overline{J}_2(t+T_\alpha)\\-r(t+T_\alpha)(1-L\alpha)\leq-\theta_0.\end{aligned}$$

So one has $\mathscr{L}\underline{v}_2(t,x)\leq-\theta_0\underline{v}_2(t,x)$ for $t\geq 0$ and $x\in[-R+I(t),R+I(t)]$. Let $\eta>0$ be small enough such that

$$r(t+T_\alpha)L(1+M_\alpha)\underline{v}_2(t,x)\leq\eta\|r\|_\infty L(1+M_\alpha)e^{\|a\|_\infty}\sup_{x\in[-R,R]}e^{-\gamma x}<\theta_0.$$

This means that

$$-\theta_0\underline{v}_2(t,x)\leq-r(t+T_\alpha)L(1+M_\alpha)\underline{v}_2^2(t,x),\ \forall t\geq 0,x\in[-R+I(t),R+I(t)].$$

Hence, we obtain that $\underline{v}_2$ is the sub-solution of (5.5.36). The proof is completed. □

5.6 Proof of Proposition 5.2.15

In this section, we show that the solution of (5.1.1) is bounded with additional Assumption 5.2.13.

Proof of Proposition 5.2.15. As discussed in Remark 5.2.16, one has $0\leq u(t,i)\leq 1$ and $v(t,i)\geq 0$ for all $t\geq 0$ and $i\in\mathbb{Z}$. Now we show that v is bounded. Set $W:=u+\varepsilon v$. Herein $\varepsilon>0$ is given in Assumption 5.2.13. Note that W satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}W(t,i)=&\sum_{j\in\mathbb{Z}}J_2(t,j)[W(t,i-j)-W(t,i)]+\sum_{j\in\mathbb{Z}}[J_1(t,j)-J_2(t,j)]\,[u(t,i-j)-u(t,i)]\\&+u(t,i)f(t,u,v)+\varepsilon v(t,i)g(t,u,v).\end{aligned}$$

From Assumption 5.2.13 and $0\leq u\leq 1$, one can observe that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}W(t,i)&\leq\sum_{j\in\mathbb{Z}}J_2(t,j)[W(t,i-j)-W(t,i)]+\|\overline{J}_1\|_\infty+\|\overline{J}_2\|_\infty+\mathcal{M}-\eta\frac{W(t,i)-u(t,i)}{\varepsilon},\\&\leq\sum_{j\in\mathbb{Z}}J_2(t,j)[W(t,i-j)-W(t,i)]+\|\overline{J}_1\|_\infty+\|\overline{J}_2\|_\infty+\mathcal{M}+\frac{\eta}{\varepsilon}-\frac{\eta}{\varepsilon}W(t,i),\end{aligned}$$

where η and $\mathcal{M}$ are given in Assumption 5.2.13. Let $K_0>0$ be a constant such that $K_0\geq u_0(i)+\varepsilon v_0(i)$ for all $i\in\mathbb{Z}$. This is due to u_0 and v_0 are bounded. One can observe that

$$\overline{W}(t):=\frac{\varepsilon}{\eta}\left(\|\overline{J}_1\|_\infty+\|\overline{J}_2\|_\infty+\mathcal{M}+\frac{\eta}{\varepsilon}\right)(1-e^{-\frac{\eta}{\varepsilon}t})+K_0e^{-\frac{\eta}{\varepsilon}t},\ \forall t\geq 0,$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{W}(t)=\|\overline{J}_1\|_\infty+\|\overline{J}_2\|_\infty+\mathcal{M}+\frac{\eta}{\varepsilon}-\frac{\eta}{\varepsilon}\overline{W}(t,i),\quad\overline{W}(0)=K_0.$$

The comparison principle implies that

$$W(t,i)=u(t,i)+\varepsilon v(t,i)\leq\overline{W}(t),\ \forall t\geq 0,i\in\mathbb{Z}.$$

Since $\overline{W}$ is bounded, then we obtain that v is bounded for $t\geq 0$ and $i\in\mathbb{Z}$. □

5.7 Appendix A: Maximum principles

5.7.1 Proof of Proposition 5.3.3

Proof of Proposition 5.3.3. For notation simplicity, we assume that $t_0 = 0$. Since $a(t,i)$ is bounded and $\overline{J}(t) := \sum_{j\in\mathbb{Z}} J(t,j) \in L^\infty(0,T)$, then one can choose $K > 0$ large enough such that $K - \overline{J}(t) + a(t,i) \geq 1$ for all $t \in [0,T]$ and $i \in \mathbb{Z}$. Set $v(t,i) := e^{Kt}u(t,i)$. Note that $v(t,i)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}v(t,i) \geq \sum_{j\in\mathbb{Z}} J(t,j)v(t,i-j) + \big(K - \overline{J}(t) + a(t,i)\big)v(t,i), & \forall t \in (0,T],\ \forall i \in \mathbb{Z}, \\ v(0,i) \geq 0, & \forall i \in \mathbb{Z}. \end{cases} \tag{5.7.53}$$

Due to $u(t,i)$ is bounded, so $v(t,i)$ is bounded for all $t \in [0,T]$ and $i \in \mathbb{Z}$.

It is sufficiently to show that $v(t,i) \geq 0$ for all $t \in [0,T]$ and $i \in \mathbb{Z}$. Let us denote

$$M := \sup_{(t,i)\in[0,T]\times\mathbb{Z}} \{K + a(t,i)\} \text{ and } \tau := \min\left\{T, \frac{1}{2M}\right\}.$$

We first claim that $v(t,i) \geq 0$ for all $t \in [0,\tau]$ and $i \in \mathbb{Z}$. By contradiction argument, we assume that there exists some point in $(0,\tau]\times\mathbb{Z}$ such that $v < 0$. Set $w(t) := \inf_{i\in\mathbb{Z}} v(t,i)$. One can assume there exists some $t_* \in (0,\tau]$ such that $w(t_*) = \inf_{t\in(0,\tau]} w(t) < 0$. Let us observe that for all $(t,i) \in [0,T]\times\mathbb{Z}$,

$$\big(K - \overline{J}(t) + a(t,i)\big)v(t,i) \geq \big(K - \overline{J}(t) + a(t,i)\big)w(t) \geq G(t)w(t),$$

where $G(t)$ is defined by

$$G(t) := \begin{cases} K - \overline{J}(t) + \inf_{i\in\mathbb{Z}} a(t,i), & w(t) \geq 0, \\ K - \overline{J}(t) + \sup_{i\in\mathbb{Z}} a(t,i), & w(t) < 0. \end{cases}$$

Since $K - \overline{J}(t) + a(t,i) \geq 1$ for all $(t,i) \in [0,T]\times\mathbb{Z}$, then $\inf_{t\in[0,T]} G(t) > 0$.

Integrating (5.7.53) from $t = 0$ to $t = t_*$, one has

$$\begin{aligned} v(t_*,i) &\geq v(0,i) + \int_0^{t_*} \sum_{j\in\mathbb{Z}} J(t,j)v(t,i-j)\mathrm{d}t + \int_0^{t_*} \big(K - \overline{J}(t) + a(t,i)\big)\, v(t,i)\mathrm{d}t \\ &\geq v(0,i) + \int_0^{t_*} \sum_{j\in\mathbb{Z}} J(t,j)v(t,i-j)\mathrm{d}t + \int_0^{t_*} G(t)w(t)\mathrm{d}t. \end{aligned}$$

Recalling that $w(t_*) = \inf_{(t,i)\in[0,\tau]\times\mathbb{Z}} v(t,i)$, and taking infimum with respect to $i \in \mathbb{Z}$ in above inequality, one has

$$w(t_*) \geq w(0) + w(t_*) \int_0^\tau \left[\overline{J}(t) + G(t)\right] \mathrm{d}t$$

Recalling the definition of M and $G(t)$, note that $\sup_{t\in[0,T]} \left\{\overline{J}(t) + G(t)\right\} \leq M$. Since $w(t_*) < 0$ and $w(0) \geq 0$, then

$$w(t_*) \geq M\tau w(t_*) \geq \frac{1}{2}w(t_*).$$

This contradicts $w(t_*) < 0$. Hence $v(t,i) \geq 0$ for all $t \in [0,\tau]$ and $i \in \mathbb{Z}$. The same argument can be repeated for $t \in [\tau,T]$, we obtain the result. □

5.7.2 Proof of Proposition 5.3.5

Proof of Proposition 5.3.5. For notation simplicity, we assume that $t_0 = 0$. Since $a(t,i)$ is bounded in Ω_T and $\overline{J}(t) := \sum_{j\in\mathbb{Z}} J(t,j) \in L^\infty(0,T)$, then one can choose $K > 0$ large enough such that $K - \overline{J}(t) + a(t,i) \geq 1$ for all $(t,i) \in \Omega_T$. Let $v(t,i) = e^{Kt}u(t,i)$. Note that $v(t,i)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}v(t,i) \geq \displaystyle\sum_{j\in\mathbb{Z}} J(t,j)v(t,i-j) + \big(K - \overline{J}(t) + a(t,i)\big)\, v(t,i), & \forall (t,i) \in \Omega_T, \\ v(t,i) \geq 0, & \forall (t,i) \in \{(0,T]\times\mathbb{Z}\} \setminus \Omega_T, \\ v(0,i) \geq 0, & \forall i \in \big[I_1(0), I_2(0)\big] \cap \mathbb{Z}. \end{cases}$$

As well as one can observe that $v(t,i)$ is bounded for all $t \in [0,T]$ and $i \in \mathbb{Z}$. This is due to u is bounded.

It is sufficiently to show that $v(t,i) \geq 0$ for all $(t,i) \in \Omega_T$. Let us denote

$$M := \sup_{(t,i)\in\Omega_T} \{K + a(t,i)\} \text{ and } \eta := \min\left\{T, \frac{1}{2M}\right\}.$$

We firstly claim that $v(t,i) \geq 0$ for all $(t,i) \in \Omega_\eta$, namely $v(t,i) \geq 0$ for all $t \in (0,\eta]$ and $i \in (I_1(t), I_2(t)) \cap \mathbb{Z}$. By contradiction argument, assume that there exists some point in Ω_η such that $v < 0$. Thus one can find $(t_*, i_*) \in \Omega_\eta$ such that

$$v(t_*, i_*) = \min_{(t,i)\in\Omega_\eta} v(t,i) < 0.$$

Recalling that

$$A_\eta(i_*) = \left\{t \in [0,\eta] : (t, i_*) \in \overline{\Omega}_\eta\right\},$$

one can find some $\hat{t}_1, \hat{t}_2 \in [0,\eta]$ such that $[\hat{t}_1, \hat{t}_2] = A_\eta(i_*)$. Since $(t_*, i_*) \in \Omega_\eta$, then $\hat{t}_1 < t_* \leq \hat{t}_2$. Integrating the differential inequality satisfied by v from $t = \hat{t}_1$ to $t = t_*$ for $i = i_*$, one has

$$v(t_*, i_*) \geq v(\hat{t}_1, i_*) + \int_{\hat{t}_1}^{t_*} \sum_{j\in\mathbb{Z}} J(t, i_* - j)v(t,j)\mathrm{d}t + \int_{\hat{t}_1}^{t_*} \big(K - \overline{J}(t) + a(t,i_*)\big)\, v(t,i_*)\mathrm{d}t.$$

Due to v is non-negative in the outside of Ω_T, one can note that

$$\begin{aligned} \int_{\hat{t}_1}^{t_*} \sum_{j\in\mathbb{Z}} J(t, i_* - j)v(t,j)\mathrm{d}t &\geq \int_{\hat{t}_1}^{t_*} \sum_{j\in(I_1(t),I_2(t))\cap\mathbb{Z}} J(t, i_* - j)v(t,j)\mathrm{d}t \\ &\geq v(t_*, i_*) \int_{\hat{t}_1}^{t_*} \sum_{j\in(I_1(t),I_2(t))\cap\mathbb{Z}} J(t, i_* - j)\mathrm{d}t. \end{aligned}$$

Since $(\hat{t}_1, i_*)$ is the boundary point of Ω_η, then $v(\hat{t}_1, i_*) \geq 0$. Due to $v(t_*, i_*) < 0$ and $\sum_{j\in(I_1(t),I_2(t))\cap\mathbb{Z}} J(t, i_* - j) \leq \overline{J}(t)$, one has

$$v(t_*, i_*) \geq v(t_*, i_*) \int_{\hat{t}_1}^{t_*} \left\{\overline{J}(t) + \big(K - \overline{J}(t) + a(t,i_*)\big)\right\}\mathrm{d}t.$$

From the definition of M and $t_* - \hat{t}_1 \leq \eta$, one has

$$v(t_*, i_*) \geq M\eta v(t_*, i_*) \geq \frac{1}{2}v(t_*, i_*).$$

This contradicts $v(t_*, i_*) < 0$. Hence one has $v(t,i) \geq 0$ for $(t,i) \in \Omega_\eta$. Repeating the same argument, the result follows. ☐

5.7.3 Proof of Proposition 5.3.7

Proof of Proposition 5.3.7. For notation simplicity, we assume that $t_0 = 0$. From Proposition 5.3.3, one has $u(t,i) \geq 0$ for all $t \geq 0$ and $i \in \mathbb{Z}$. Similarly to prove Proposition 5.3.3, one can find $K > 0$ large enough such that $K - \overline{J}(t) + a(t,i) \geq 1$ for all $t \geq 0$ and $i \in \mathbb{Z}$. Let us define $v(t,i) := e^{Kt}u(t,i)$. Note that $v(t,i)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}v(t,i) \geq \displaystyle\sum_{j\in\mathbb{Z}} J(t,j)v(t,i-j) + \big(K - \overline{J}(t) + a(t,i)\big)v(t,i), & \forall t > 0,\ \forall i \in \mathbb{Z}, \\ v(0,i) \geq 0, & \forall i \in \mathbb{Z}. \end{cases}$$

Set $\tilde{J}(j) = \inf_{t\geq 0} J(t,j)$ and let $\tilde{v}$ be the solution of

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{v}(t,i) = \displaystyle\sum_{j\in\mathbb{Z}} \tilde{J}(j)\tilde{v}(t,i-j), & \forall t > 0, \forall i \in \mathbb{Z}, \\ \tilde{v}(0,i) = v(0,i), & \forall i \in \mathbb{Z}. \end{cases}$$

Since $\big(K - \overline{J}(t) + a(t,i)\big)v(t,i) \geq 0$ for all $t \geq 0$ and $i \in \mathbb{Z}$. Then applying the comparison principle, one has

$$v(t,i) \geq \tilde{v}(t,i),\ \forall t \geq 0, \forall i \in \mathbb{Z}.$$

It is sufficiently to show that $\tilde{v}(t,i) > 0$ for all $t > 0$ and $i \in \mathbb{Z}$. Through iteration one has

$$\tilde{v}(t,i) = v(0,i) + \sum_{n=1}^{+\infty} \frac{t^n}{n!}\tilde{J}^{*n} * v(0,i),$$

where $\tilde{J}^{*n} * v(0,i)$ is given by

$$\tilde{J}^{*n} * v(0,i) = \sum_{(j_1,\cdots,j_n)\in\mathbb{Z}^n} \tilde{J}(j_n)\cdots\tilde{J}(j_1)v(0,i-j_1\cdots - j_n).$$

Note that there exists $i_0 \in \mathbb{Z}$ such that $v(0,i_0) > 0$ due to $v(0,i) \not\equiv 0$. Recalling the assumption $\inf_{t\geq 0} J(t,\pm 1) > 0$, this can rewrite as $\tilde{J}(\pm 1) > 0$. Then one has

$$\tilde{J} * v(0,i) > 0 \text{ for all } i \in \{i_0 - 1, i_0 + 1\}.$$

By induction, one also has for all $n \geq 1$

$$\tilde{J}^{*n} * v(0,i) > 0 \text{ for all } i \in [i_0 - n, i_0 + n] \cap \mathbb{Z}.$$

Since $\mathbb{Z} = \cup_{n=1}^{\infty} [i_0 - n, i_0 + n] \cap \mathbb{Z}$, we obtain that $\tilde{v}(t,i) > 0$ for all $t > 0$ and $i \in \mathbb{Z}$. Hence we end-up with $u(t,i) > 0$ for all $t > 0$ and $i \in \mathbb{Z}$, which completes the proof. □

5.8 Appendix B: Spreading speed for Fisher-KPP equations

5.8.1 Proof of Proposition 5.3.14

We will apply the key persistence Lemma 5.3.18 to prove Proposition 5.3.14. To do this, we first introduce some notations. For each $B > 0$, we define $J_B := J_B(t,i)$ given by

$$J_B(t,i) := \begin{cases} J(t,i), & \forall t \geq 0, i \in [-B,B] \cap \mathbb{Z}, \\ 0, & \textit{else}. \end{cases} \tag{5.8.54}$$

For all $B > 0$, $R > 0$ and $\mu \geq 0$ given, we define the following function $t \mapsto \hat{c}_{R,B}(\mu)(t) \in L^\infty(0,\infty)$ given by

$$\hat{c}_{R,B}(\mu)(t) := \frac{2R}{\pi} \sum_{j\in\mathbb{Z}} \sin\frac{\pi j}{2R} J_B(t,j) e^{\mu j}. \tag{5.8.55}$$

For some constant $C_\mu > 0$, set

$$X(t) := \int_0^t \hat{c}_{R,B}(\mu)(s)\mathrm{d}s + C_\mu. \tag{5.8.56}$$

Recall that Definition 5.3.16 and $\tilde{w}$ satisfies (5.3.22) for $\tilde{w} \in \mathcal{H}(w) \setminus \{0\}$.

We state the following lemma about the sub-solution defined in $[0,\infty) \times \mathbb{R}$.

Lemma 5.8.1. *Let assumptions in Proposition 5.3.14 be satisfied. For some $B_0 > 0$ large enough, for all $B > B_0$, there exist $R_0(B) > 0$ large enough enjoying following properties:*

For some given $\mu > 0$, for all $B > B_0$ and $R > \max(R_0(B), B+1)$, for some $\eta > 0$ small enough, for some $b \in W^{1,\infty}(0,\infty)$, we define

$$\underline{w}(t,x) := \begin{cases} \eta e^{b(t)} e^{-\mu(x-X(t))} \cos\frac{\pi(x-X(t))}{2R}, & x \in [-R+X(t), R+X(t)], t \geq 0, \\ 0, & \textit{else}. \end{cases} \tag{5.8.57}$$

Then $\underline{w}$ is the sub-solution of (5.3.19), that is $\underline{w}(t,x)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{w}(t,x) \leq \sum_{j\in\mathbb{Z}} J(t,j)[\underline{w}(t,x-j) - \underline{w}(t,x)] + m(t)\underline{w}(t,x)\left(1 - l\underline{w}(t,x)\right),$$

for all $x \in [-R+X(t), R+X(t)]$ and $t \geq 0$.

Indeed, the proof is similar as Lemma 5.5.7. For the reader convenience, we prove it below.

Proof. Since $\langle c_w(\mu)\rangle$ takes the minimal value c_w^* at $\mu = \mu^*$ where $\mu^* \in (0, \mathrm{abs}(J))$, then

$$\left.\frac{\mathrm{d}\langle c_w(\mu)\rangle}{\mathrm{d}\mu}\right|_{\mu=\mu^*} = \frac{1}{\mu^*}\left(\sum_{j\in\mathbb{Z}} \langle J(\cdot,j)\rangle e^{\mu^* j} j - \langle c_w(\mu^*)\rangle\right) = 0,$$

and

$$\frac{\mathrm{d}\langle c_w(\mu)\rangle}{\mathrm{d}\mu} = \frac{\left\langle \mu \sum\limits_{j\in\mathbb{Z}} J(\cdot,j)e^{\mu j}j - \sum\limits_{j\in\mathbb{Z}} J(\cdot,j)e^{\mu j} + \overline{J}(\cdot) - m(\cdot)\right\rangle}{\mu^2} < 0,\ \forall \mu \in (0,\mu^*),$$

where $\overline{J}(\cdot) = \sum_{j\in\mathbb{Z}} J(\cdot,j)$.

From Lebesgue dominated convergence theorem, one can observe that

$$\lim_{\substack{B\to\infty \\ R\to\infty}} \sum_{j\in\mathbb{Z}} J_B(t,j)e^{\mu j}\left(\mu\frac{2R}{\pi}\sin\frac{\pi j}{2R} - \cos\frac{\pi j}{2R}\right) = \mu\sum_{j\in\mathbb{Z}} J(t,j)e^{\mu j}j - \sum_{j\in\mathbb{Z}} J(t,j)e^{\mu j},$$

uniformly for $t \geq 0$. Hence for some $R_0 > 0$ and $B_0 > 0$ large enough, for all $R > R_0$ and $B > B_0$, for the given $\mu \in (0,\mu^*)$, one can choose some $\hat{\theta}_0 > 0$ and function $b \in W^{1,\infty}(0,\infty)$ such that for all $t \geq 0$,

$$b'(t) + \mu\frac{2R}{\pi}\sum_{j\in\mathbb{Z}} J_B(t,j)e^{\mu j}\sin\frac{\pi j}{2R} - \sum_{j\in\mathbb{Z}} J_B(t,j)e^{\mu j}\cos\frac{\pi j}{2R} + \overline{J}(t) - m(t) \leq -\hat{\theta}_0. \tag{5.8.58}$$

By the same analysis in (5.5.49), one can observe that for all $x \in [-R+X(t), R+X(t)]$ and $t \geq 0$,

$$\sum_{j\in\mathbb{Z}} J(t,j)\underline{w}(t,x-j) \geq \sum_{j\in\mathbb{Z}} J_B(t,j)\eta e^{b(t)} e^{-\mu(x-j-X(t))} \cos\frac{\pi(x-j-X(t))}{2R}.$$

Let $\mathcal{L}$ be the operator defined as

$$\mathcal{L}\phi(t,x) := \frac{\mathrm{d}}{\mathrm{d}t}\phi(t,x) - \sum_{j\in\mathbb{Z}} J(t,j)\phi(t,x-j) + \overline{J}(t)\phi(t,x) - m(t)\phi(t,x).$$

Through computation, for $t \geq 0$ and $x \in [-R+X(t), R+X(t)]$, one has

$$\begin{aligned}\mathcal{L}\underline{w}(t,x) \leq{}& \eta e^{b(t)} e^{-\mu(x-X(t))} \cos\frac{\pi(x-X(t))}{2R}\Big\{b'(t) + \mu\hat{c}_{R,B}(\mu)(t) - \sum_{j\in\mathbb{Z}} J_B(t,j)e^{\mu j}\cos\frac{\pi j}{2R} \\ &+ \overline{J}(t) - m(t)\Big\} \\ &+ \eta e^{b(t)} e^{-\mu(x-X(t))} \sin\frac{\pi(x-X(t))}{2R}\Big\{\frac{\pi}{2R}\hat{c}_{R,B}(\mu)(t) - \sum_{j\in\mathbb{Z}} J_B(t,j)e^{\mu j}\sin\frac{\pi j}{2R}\Big\}\end{aligned}$$

Due to (5.8.55), the last term in the above equation vanished. The inequality (5.8.58) ensures that

$$\mathcal{L}\underline{w}(t,x) \leq -\hat{\theta}_0\underline{w}(t,i), \ \forall t \geq 0 \text{ and } i \in [-R+I(t), R+I(t)] \cap \mathbb{Z}.$$

One can choose $\eta > 0$ small enough such that

$$\eta\|m\|_\infty l e^{\|b\|_\infty} \sup_{x\in[-R,R]} e^{-\mu x} < \hat{\theta}_0.$$

Hence, one has

$$m(t)l\underline{w}^2(t,x) \leq \hat{\theta}_0\underline{w}(t,x), \ \forall t \geq 0 \text{ and } i \in [-R+X(t), R+X(t)] \cap \mathbb{Z}.$$

So that one obtains $\mathcal{L}\underline{w}(t,x) \leq -m(t)l\underline{w}^2(t,x)$ for all $x \in [-R+X(t), R+X(t)] \cap \mathbb{Z}$ and $t \geq 0$. This means that $\underline{w}(t,x)$ is the sub-solution of (5.3.19). The proof is completed. □

Now we complete the proof of Proposition 5.3.14.

Proof of Proposition 5.3.14. Firstly, similar to Section 5.4, for all $c > c' > c_w^*$, one can construct a super-solution $\overline{w}$ by

$$\overline{w}(t,i) := Ae^{-\mu^* a(t)} e^{-\mu^*(i-c't)},$$

where $A > 0$ is sufficiently large and $a \in W^{1,\infty}(0,\infty)$ satisfies

$$c' \geq \frac{\sum_{j\in\mathbb{Z}} J(t,j)[e^{\mu^* j} - 1] + m(t)}{\mu^*} + a'(t), \forall t \geq 0.$$

One can verify that $\overline{w}$ is a super-solution of (5.3.19). Let $A > 0$ be large enough such that $\overline{w}(0,\cdot) \geq w_0(\cdot)$. Then the comparison principle ensures that for all $c > c' > c_w^*$

$$\limsup_{t\to\infty}\sup_{i\geq ct} w(t,i) \leq \limsup_{t\to\infty}\sup_{i\geq ct} \overline{w}(t,i) \leq \lim_{t\to\infty} Ae^{-\mu^* a(t)} e^{-\mu^*(ct-c't)} = 0.$$

One can prove similarly for $i \leq -ct$. Thus we obtain that

$$\lim_{t\to\infty} \sup_{|i|\geq ct} w(t,i) = 0, \ \forall c > c_w^*.$$

Recalling sub-solutions defined in Lemma 5.8.1, the maximum principle implies that assumptions $(H3)$ in Lemma 5.3.18 is satisfied. From Lemma 5.3.10, one can observe that assumptions $(H1)$ in Lemma 5.3.18 is satisfied. $(H2)$ can be proved similarly. Hence, one can apply Lemma 5.3.18 to obtain that

$$\liminf_{t\to\infty} \inf_{i\in[0,[kX(t)]]\cap\mathbb{Z}} w(t,i) > 0, \ \forall k \in (0,1).$$

Similar to **Step 4** in Section 5.5.3, for the given $c \in [0, c_w^*)$ for some $k_0 \in (0,1)$, for all $k \in (k_0, 1)$, one can choose $C_\mu > 0$ in $X(t)$ (see (5.8.56)) large enough such that $kX(t) \geq ct$ for all $t \geq 0$. Hence one obtains

$$\liminf_{t\to\infty} \inf_{i\in[0,ct]\cap\mathbb{Z}} w\,(t,i) > 0, \ \forall c \in [0, c_w^*).$$

By a symmetric argument, one has

$$\liminf_{t\to\infty} \inf_{|i|\leq ct} w\,(t,i) > 0, \ \forall c \in [0, c_w^*).$$

Similarly to the proof of Lemma 5.5.3, one can complete the proof of Proposition 5.3.14. □

5.8.2 Proof of Lemma 5.3.18

Lastly, let us show the proof of key persistence lemma 5.3.18. The idea is close to **Step 3** in Section 5.5.3.

Proof of Lemma 5.3.18. By contradiction, we assume that there exist $k \in (0,1)$, $k_n \in [0,k]$ and sequence $(t_n)_n$ satisfying $t_n \to \infty$ as $n \to \infty$ such that

$$w(t_n, [k_n X(t_n)]) \to 0 \text{ as } n \to \infty. \tag{5.8.59}$$

Firstly, let us observe that $[k_n X(t_n)] \to \infty$ as $n \to \infty$. If not, then $[k_n X(t_n)] \to l \in \mathbb{Z}$ which may happen when $k_n \to 0$. From the regularity of w, one can extract a sub-sequence such that

$$w(t + t_n, i) \to w_\infty(t,i), \text{ locally uniformly for } (t,i) \in \mathbb{R} \times \mathbb{Z} \text{ as } n \to \infty.$$

As well as $w_\infty \in \mathcal{H}(w)$ and w_∞ satisfies (5.3.22) with suitable coefficients. Note that (5.8.59) implies $w_\infty(0,l) = 0$. Then the strong maximum principle ensures that $w_\infty \equiv 0$. This is contradicted with assumption $(H1)$ in Lemma 5.3.18. So one has $[k_n X(t_n)] \to \infty$ as $n \to \infty$. With this property, one can choose a sub-sequence such that $[k_n X(t_n)] \geq 2$ for all $n \geq 1$.

Possibly up to a subsequence, assumption (5.8.59) can be rewritten as

$$w(t_n, [k_n X(t_n)] - 1) \to 0 \text{ as } n \to \infty. \tag{5.8.60}$$

Again, up to a sub-sequence, one may assume that $X(0) < [k_n X(t_n)] - 1 < [X(t_n)]$ for all $n \geq 1$. From the continuity of $X(t)$, there exists $t_n' < t_n$ such that

$$[X(t_n')] = [k_n X(t_n)] - 1.$$

One can also observe that $t'_n \to \infty$ as $n \to \infty$. From the chosen of t'_n, one has

$$w(t'_n, [X(t'_n)]) = w(t'_n, [k_n X(t_n)] - 1).$$

Due to assumption $(H3)$ (see (5.3.25)), there exists $\varepsilon_3 > 0$ such that

$$\liminf_{n\to\infty} w(t'_n, [X(t'_n)]) \geq \varepsilon_3 > 0.$$

Let $\varepsilon_2 > 0$ be given in assumption $(H2)$ (see (5.3.24)). We define

$$t''_n := \sup\left\{ t \in (t'_n, t_n) : \ w(t, [X(t'_n)]) \geq \frac{\min\{\varepsilon_2, \varepsilon_3\}}{2} \right\}.$$

The definition of t'_n and assumption (5.8.60) implies that

$$w(t_n, [X(t'_n)]) = w(t_n, [k_n X(t_n)] - 1) \to 0 \text{ as } n \to \infty.$$

Then we assume that for all n large enough

$$\begin{cases} w(t''_n, [X(t'_n)]) = \frac{\min\{\varepsilon_2,\varepsilon_3\}}{2}, \\ w(t, [X(t'_n)]) \leq \frac{\min\{\varepsilon_2,\varepsilon_3\}}{2}, \quad \forall t \in [t''_n, t_n], \\ w(t_n, [X(t'_n)]) \leq \frac{1}{n}. \end{cases}$$

Next we claim that $t_n - t''_n \to \infty$ as $n \to \infty$. If not, we assume that $t_n - t''_n \to \tau \in \mathbb{R}$. Set

$$w^n(t, i) := w(t + t''_n, i + [X(t'_n)]), \ \forall t \geq -t''_n, i \in \mathbb{Z}.$$

The regularity of w ensures that one can extract sub-sequence such that

$$w^n(t, i) \to w^\infty(t, i) \text{ locally uniformly for } (t, i) \in \mathbb{R} \times \mathbb{Z} \text{ as } n \to \infty.$$

As well as $w^\infty \in \mathcal{H}(w)$ satisfies (5.3.22) with suitable coefficients.

If $t_n - t''_n \to \tau$, then

$$w^\infty(\tau, 0) = \lim_{n\to\infty} w(t_n - t''_n + t''_n, [X(t'_n)]) = 0.$$

Hence the strong maximum principle implies that $w^\infty \equiv 0$. This is contradicted with

$$w^\infty(0, 0) = \frac{\min\{\varepsilon_2, \varepsilon_3\}}{2} > 0.$$

So we obtain $t_n - t''_n \to \infty$. From the construction, one has

$$w^\infty(t, 0) \leq \frac{\min\{\varepsilon_2, \varepsilon_3\}}{2}, \ \forall t \geq 0.$$

Due to $w^\infty(0, 0) > 0$ and $w^\infty \in \mathcal{H}(w)$, assumption $(H2)$ yields that

$$\liminf_{t\to\infty} w^\infty(t, 0) \geq \varepsilon_2 > 0.$$

This is a contradiction. The proof of Lemma 5.3.18 is completed. □

Bibliography

[1] L. Abi Rizk, J.-B. Burie, and A. Ducrot. Asymptotic speed of spread for a nonlocal evolutionary-epidemic system. *Discrete & Continuous Dynamical Systems*, 41(10):4959–4985, 2021.

[2] M. Alfaro. Fujita blow up phenomena and hair trigger effect: the role of dispersal tails. *Annales de l'Institut Henri Poincaré C*, 34(5):1309–1327, 2017.

[3] N. Alikakos, P. Bates, and X. Chen. Periodic traveling waves and locating oscillating patterns in multidimensional domains. *Transactions of the American Mathematical Society*, 351(7):2777–2805, 1999.

[4] B. Ambrosio, A. Ducrot, and S. Ruan. Generalized traveling waves for time-dependent reaction–diffusion systems. *Mathematische Annalen*, 381(1):1–27, 2021.

[5] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, and J. J. Toledo-Melero. *Nonlocal diffusion problems*, volume 165. American Mathematical Soc., 2010.

[6] W. Arendt, C. J. Batty, and F. Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems*, volume 96. Springer Science & Business Media, 2011.

[7] D. G. Aronson and H. F. Weinberger. Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In *Partial differential equations and related topics*, pages 5–49. Springer, 1975.

[8] D. G. Aronson and H. F. Weinberger. Multidimensional nonlinear diffusion arising in population genetics. *Advances in Mathematics*, 30(1):33–76, 1978.

[9] X. Bao. Transition waves for two species competition system in time heterogeneous media. *Nonlinear Analysis: Real World Applications*, 44:128–148, 2018.

[10] X. Bao and W.-T. Li. Existence and stability of generalized transition waves for time-dependent reaction-diffusion systems. *Discrete & Continuous Dynamical Systems-B*, 26(7):3621–3641, 2021.

[11] X. Bao, W.-T. Li, and W. Shen. Traveling wave solutions of Lotka–Volterra competition systems with nonlocal dispersal in periodic habitats. *Journal of Differential Equations*, 260(12):8590–8637, 2016.

[12] X. Bao, W.-T. Li, W. Shen, and Z.-C. Wang. Spreading speeds and linear determinacy of time dependent diffusive cooperative/competitive systems. *Journal of Differential Equations*, 265(7):3048–3091, 2018.

[13] X. Bao, W. Shen, and Z. Shen. Spreading speeds and traveling waves for space-time periodic nonlocal dispersal cooperative systems. *Communications on Pure & Applied Analysis*, 18(1):361–396, 2019.

[14] P. W. Bates and A. Chmaj. A discrete convolution model for phase transitions. *Archive for Rational Mechanics and Analysis*, 150(4):281–368, 1999.

[15] P. W. Bates, P. C. Fife, X. Ren, and X. Wang. Traveling waves in a convolution model for phase transitions. *Archive for Rational Mechanics and Analysis*, 138(2):105–136, 1997.

[16] J. Bell and C. Cosner. Threshold behavior and propagation for nonlinear differential-difference systems motivated by modeling myelinated axons. *Quarterly of Applied Mathematics*, 42(1):1–14, 1984.

[17] H. Berestycki, J. Coville, and H.-H. Vo. Persistence criteria for populations with non-local dispersion. *Journal of Mathematical Biology*, 72(7):1693–1745, 2016.

[18] H. Berestycki and F. Hamel. Front propagation in periodic excitable media. *Communications on Pure and Applied Mathematics*, 55(8):949–1032, 2002.

[19] H. Berestycki and F. Hamel. Generalized travelling waves for reaction-diffusion equations. *Contemporary Mathematics*, 446:101–124, 2007.

[20] H. Berestycki and F. Hamel. Generalized transition waves and their properties. *Communications on Pure and Applied Mathematics*, 65(5):592–648, 2012.

[21] H. Berestycki, F. Hamel, and G. Nadin. Asymptotic spreading in heterogeneous diffusive excitable media. *Journal of Functional Analysis*, 255(9):2146–2189, 2008.

[22] H. Berestycki, F. Hamel, and N. Nadirashvili. The speed of propagation for KPP type problems. I: periodic framework. *Journal of the European Mathematical Society*, 7(2):173–213, 2005.

[23] H. Berestycki and G. Nadin. Asymptotic spreading for general heterogeneous Fisher-KPP type equations. *Memoirs of the American Mathematical Society*, 2019.

[24] C. Besse, G. Faye, J.-M. Roquejoffre, and M. Zhang. The logarithmic Bramson correction for Fisher-KPP equations on the lattice $\mathbb{Z}$. *arXiv preprint arXiv:2206.04358*, 2022.

[25] W.-J. Bo, G. Lin, and S. Ruan. Traveling wave solutions for time periodic reaction-diffusion systems. *Discrete & Continuous Dynamical Systems*, 38(9):4329–4351, 2018.

[26] M. Bramson. Maximal displacement of branching Brownian motion. *Communications on Pure and Applied Mathematics*, 31(5):531–581, 1978.

[27] M. Bramson. *Convergence of solutions of the Kolmogorov equation to travelling waves*, volume 285. American Mathematical Soc., 1983.

[28] G. J. Butler and H. I. Freedman. Periodic solutions of a predator-prey system with periodic coefficients. *Mathematical Biosciences*, 55(1-2):27–38, 1981.

[29] X. Cabré and J.-M. Roquejoffre. The influence of fractional diffusion in Fisher-KPP equations. *Communications in Mathematical Physics*, 320(3):679–722, 2013.

[30] R. S. Cantrell and C. Cosner. *Spatial ecology via reaction-diffusion equations*. John Wiley & Sons, 2004.

[31] F. Cao and W. Shen. Spreading speeds and transition fronts of lattice KPP equations in time heterogeneous media. *Discrete & Continuous Dynamical Systems*, 37(9):4697–4727, 2017.

[32] J. Carr and A. Chmaj. Uniqueness of travelling waves for nonlocal monostable equations. *Proceedings of the American Mathematical Society*, 132(8):2433–2439, 2004.

[33] C. Carrère. Spreading speeds for a two-species competition-diffusion system. *Journal of Differential Equations*, 264(3):2133–2156, 2018.

[34] E. Chasseigne, M. Chaves, and J. D. Rossi. Asymptotic behavior for nonlocal diffusion equations. *Journal de Mathématiques Pures et Appliquées*, 86(3):271–291, 2006.

[35] X. Chen and J.-S. Guo. Existence and asymptotic stability of traveling waves of discrete quasilinear monostable equations. *Journal of Differential Equations*, 184(2):549–569, 2002.

[36] X. Chen and J.-S. Guo. Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics. *Mathematische Annalen*, 326(1):123–146, 2003.

[37] X. Chen and J.-C. Tsai. Spreading speed in a farmers and hunter-gatherers model arising from Neolithic transition in Europe. *Journal de Mathématiques Pures et Appliquées*, 143:192–207, 2020.

[38] Y.-Y. Chen, J.-S. Guo, and F. Hamel. Traveling waves for a lattice dynamical system arising in a diffusive endemic model. *Nonlinearity*, 30(6):2334, 2017.

[39] K.-S. Cheng, S.-B. Hsu, and S.-S. Lin. Some results on global stability of a predator-prey system. *Journal of Mathematical Biology*, 12(1):115–126, 1982.

[40] W. Choi, T. Giletti, and J.-S. Guo. Persistence of species in a predator-prey system with climate change and either nonlocal or local dispersal. *Journal of Differential Equations*, 302:807–853, 2021.

[41] J. Coville, J. Dávila, and S. Martínez. Nonlocal anisotropic dispersal with monostable nonlinearity. *Journal of Differential Equations*, 244(12):3080–3118, 2008.

[42] J. Coville, J. Dávila, and S. Martínez. Pulsating fronts for nonlocal dispersion and KPP nonlinearity. *Annales de L'Institut Henri Poincare Section (C) Non Linear Analysis*, 30(2):179–223, 2013.

[43] J. Coville and L. Dupaigne. Propagation speed of travelling fronts in non local reaction–diffusion equations. *Nonlinear Analysis: Theory, Methods & Applications*, 60(5):797–819, 2005.

[44] J. Coville and L. Dupaigne. On a non-local equation arising in population dynamics. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 137(4):727–755, 2007.

[45] J. M. Cushing. Periodic time-dependent predator-prey systems. *SIAM Journal on Applied Mathematics*, 32(1):82–95, 1977.

[46] T. de Camino-Beck and M. A. Lewis. Invasion with stage-structured coupled map lattices: Application to the spread of scentless chamomile. *Ecological Modelling*, 220(23):3394–3403, 2009.

[47] D. L. DeAngelis, W. M. Post, and C. C. Travis. *Positive feedback in natural systems*, volume 15. Springer Berlin, Heidelberg, 1986.

[48] O. Diekmann. Run for your life. A note on the asymptotic speed of propagation of an epidemic. *Journal of Differential Equations*, 33(1):58–73, 1979.

[49] A. Ducrot. Convergence to generalized transition waves for some Holling–Tanner prey–predator reaction–diffusion system. *Journal de Mathématiques Pures et Appliquées*, 100(1):1–15, 2013.

[50] A. Ducrot. A multi-dimensional bistable nonlinear diffusion equation in a periodic medium. *Mathematische Annalen*, 366(1):783–818, 2016.

[51] A. Ducrot. Spatial propagation for a two component reaction–diffusion system arising in population dynamics. *Journal of Differential Equations*, 260(12):8316–8357, 2016.

[52] A. Ducrot and T. Giletti. Convergence to a pulsating travelling wave for an epidemic reaction-diffusion system with non-diffusive susceptible population. *Journal of Mathematical Biology*, 69(3):533–552, 2014.

[53] A. Ducrot, T. Giletti, J.-S. Guo, and M. Shimojo. Asymptotic spreading speeds for a predator–prey system with two predators and one prey. *Nonlinearity*, 34(2):669–704, 2021.

[54] A. Ducrot, T. Giletti, and H. Matano. Existence and convergence to a propagating terrace in one-dimensional reaction-diffusion equations. *Transactions of the American Mathematical Society*, 366(10):5541–5566, 2014.

[55] A. Ducrot, T. Giletti, and H. Matano. Spreading speeds for multidimensional reaction–diffusion systems of the prey–predator type. *Calculus of Variations and Partial Differential Equations*, 58(4):1–34, 2019.

[56] A. Ducrot, J.-S. Guo, G. Lin, and S. Pan. The spreading speed and the minimal wave speed of a predator–prey system with nonlocal dispersal. *Zeitschrift für angewandte Mathematik und Physik*, 70(5):1–25, 2019.

[57] A. Ducrot and Z. Jin. Spreading speeds for time heterogeneous prey-predator systems with nonlocal diffusion on lattice. *In preparation.*

[58] A. Ducrot and Z. Jin. Generalized travelling fronts for non-autonomous fisher-kpp equations with nonlocal diffusion. *Annali di Matematica Pura ed Applicata (1923-)*, 201(4):1607–1638, 2022.

[59] A. Ducrot and Z. Jin. Spreading properties for non-autonomous Fisher-KPP equations with nonlocal diffusion. *arXiv preprint arXiv:2201.05794*, 2022.

[60] A. Ducrot and Z. Jin. Spreading speeds for time heterogeneous reaction-diffusion systems of the prey-predator type. *Submitted*, 2022.

[61] A. Ducrot, Z. Liu, and P. Magal. Large speed traveling waves for the Rosenzweig–MacArthur predator–prey model with spatial diffusion. *Physica D: Nonlinear Phenomena*, 415:132730, 2021.

[62] S. R. Dunbar. Travelling wave solutions of diffusive Lotka-Volterra equations. *Journal of Mathematical Biology*, 17(1):11–32, 1983.

[63] S. R. Dunbar. Traveling wave solutions of diffusive Lotka-Volterra equations: a heteroclinic connection in $\mathbb{R}^4$. *Transactions of the American Mathematical Society*, 286(2):557–594, 1984.

[64] L. C. Evans. *Partial differential equations*, volume 19. American Mathematical Soc., 2010.

[65] J. Fang, J. Wei, and X.-Q. Zhao. Spreading speeds and travelling waves for non-monotone time-delayed lattice equations. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 466(2119):1919–1934, 2010.

[66] J. Fang, X. Yu, and X.-Q. Zhao. Traveling waves and spreading speeds for time–space periodic monotone systems. *Journal of Functional Analysis*, 272(10):4222–4262, 2017.

[67] P. C. Fife and J. B. McLeod. The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Archive for Rational Mechanics and Analysis*, 65(4):335–361, 1977.

[68] D. Finkelshtein and P. Tkachov. The hair-trigger effect for a class of nonlocal nonlinear equations. *Nonlinearity*, 31(6):2442–2479, 2018.

[69] D. Finkelshtein and P. Tkachov. Accelerated nonlocal nonsymmetric dispersion for monostable equations on the real line. *Applicable Analysis*, 98(4):756–780, 2019.

[70] R. A. Fisher. The wave of advance of advantageous genes. *Annals of eugenics*, 7(4):355–369, 1937.

[71] R. Gardner and C. K. Jones. Stability of travelling wave solutions of diffusive predator-prey systems. *Transactions of the American Mathematical Society*, 327(2):465–524, 1991.

[72] R. A. Gardner. Existence of travelling wave solutions of predator–prey systems via the connection index. *SIAM Journal on Applied Mathematics*, 44(1):56–79, 1984.

[73] J. Garnier. Accelerating solutions in integro-differential equations. *SIAM Journal on Mathematical Analysis*, 43(4):1955–1974, 2011.

[74] J. Garnier, T. Giletti, F. Hamel, and L. Roques. Inside dynamics of pulled and pushed fronts. *Journal de Mathématiques Pures et Appliquées*, 98(4):428–449, 2012.

[75] J. A. Gatica and J. W.-H. So. Predator-prey models with almost periodic coefficients. *Applicable Analysis*, 27(1-3):143–152, 1988.

[76] T. Giletti and L. Rossi. Pulsating solutions for multidimensional bistable and multistable equations. *Mathematische Annalen*, 378(3):1555–1611, 2020.

[77] L. Girardin and K.-Y. Lam. Invasion of open space by two competitors: spreading properties of monostable two-species competition-diffusion systems. *Proceedings of the London Mathematical Society*, 119(5):1279–1335, 2019.

[78] C. Graham. The bramson correction for integro-differential Fisher-KPP equations. *Communications in Mathematical Sciences*, 20(2):563–596, 2022.

[79] J.-S. Guo and F. Hamel. Front propagation for discrete periodic monostable equations. *Mathematische Annalen*, 335(3):489–525, 2006.

[80] J.-S. Guo and X. Liang. The minimal speed of traveling fronts for the Lotka–Volterra competition system. *Journal of Dynamics and Differential Equations*, 23(2):353–363, 2011.

[81] J.-S. Guo and C.-H. Wu. Traveling wave front for a two-component lattice dynamical system arising in competition models. *Journal of Differential Equations*, 252(8):4357–4391, 2012.

[82] J. K. Hale and P. Waltman. Persistence in infinite-dimensional systems. *SIAM Journal on Mathematical Analysis*, 20(2):388–395, 1989.

[83] F. Hamel and G. Nadin. Spreading properties and complex dynamics for monostable reaction–diffusion equations. *Communications in Partial Differential Equations*, 37(3):511–537, 2012.

[84] F. Hamel and N. Nadirashvili. Travelling fronts and entire solutions of the Fisher-KPP equation in $\mathbb{R}^n$. *Archive for Rational Mechanics and Analysis*, 157(2):91–163, 2001.

[85] F. Hamel, J. Nolen, J.-M. Roquejoffre, and L. Ryzhik. A short proof of the logarithmic Bramson correction in Fisher-KPP equations. *Networks and Heterogeneous Media*, 8(1):275–279, 2013.

[86] F. Hamel and L. Roques. Fast propagation for KPP equations with slowly decaying initial conditions. *Journal of Differential Equations*, 249(7):1726–1745, 2010.

[87] F. Hamel and L. Rossi. Admissible speeds of transition fronts for nonautonomous monostable equations. *SIAM Journal on Mathematical Analysis*, 47(5):3342–3392, 2015.

[88] F. Hamel and L. Rossi. Transition fronts for the Fisher-KPP equation. *Transactions of the American Mathematical Society*, 368(12):8675–8713, 2016.

[89] C. S. Holling. Some characteristics of simple types of predation and parasitism. *The Canadian Entomologist*, 91(7):385–398, 1959.

[90] J. Huang, G. Lu, and S. Ruan. Existence of traveling wave solutions in a diffusive predator-prey model. *Journal of Mathematical Biology*, 46(2):132–152, 2003.

[91] W. Huang. Traveling wave solutions for a class of predator–prey systems. *Journal of Dynamics and Differential Equations*, 24(3):633–644, 2012.

[92] V. Hutson, S. Martinez, K. Mischaikow, and G. T. Vickers. The evolution of dispersal. *Journal of Mathematical Biology*, 47(6):483–517, 2003.

[93] Y. Jin and M. A. Lewis. Seasonal influences on population spread and persistence in streams: spreading speeds. *Journal of Mathematical Biology*, 65(3):403–439, 2012.

[94] Y. Jin and X.-Q. Zhao. Spatial dynamics of a periodic population model with dispersal. *Nonlinearity*, 22(5):1167, 2009.

[95] C.-Y. Kao, Y. Lou, and W. Shen. Random dispersal vs. non-local dispersal. *Discrete & Continuous Dynamical Systems*, 26(2):551, 2010.

[96] J. P. Keener. Propagation and its failure in coupled systems of discrete excitable cells. *SIAM Journal on Applied Mathematics*, 47(3):556–572, 1987.

[97] A. N. Kolmogorov, I. G. Petrovsky, and N. S. Piskunov. Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bulletin Université d'Etat de Moscou*, 1(6):1–26, 1937.

[98] K.-S. Lau. On the nonlinear diffusion equation of Kolmogorov, Petrovsky, and Piscounov. *Journal of Differential Equations*, 59(1):44–70, 1985.

[99] M. A. Lewis, B. Li, and H. F. Weinberger. Spreading speed and linear determinacy for two-species competition models. *Journal of Mathematical Biology*, 45(3):219–233, 2002.

[100] H. Li and H. Xiao. Traveling wave solutions for diffusive predator–prey type systems with nonlinear density dependence. *Computers & Mathematics with Applications*, 74(10):2221–2230, 2017.

[101] W.-T. Li, Y.-J. Sun, and Z.-C. Wang. Entire solutions in the Fisher-KPP equation with nonlocal dispersal. *Nonlinear Analysis: Real World Applications*, 11(4):2302–2313, 2010.

[102] X. Liang, H. Wang, Q. Zhou, and T. Zhou. Traveling fronts for Fisher-KPP lattice equations in almost periodic media. *arXiv preprint arXiv:2104.13805*, 2021.

[103] X. Liang, Y. Yi, and X.-Q. Zhao. Spreading speeds and traveling waves for periodic evolution systems. *Journal of Differential Equations*, 231(1):57–77, 2006.

[104] X. Liang and X.-Q. Zhao. Asymptotic speeds of spread and traveling waves for monotone semiflows with applications. *Communications on Pure and Applied Mathematics*, 60(1):1–40, 2007.

[105] X. Liang and X.-Q. Zhao. Spreading speeds and traveling waves for abstract monostable evolution systems. *Journal of Functional Analysis*, 259(4):857–903, 2010.

[106] X. Liang and T. Zhou. Spreading speeds of KPP-type lattice systems in heterogeneous media. *Communications in Contemporary Mathematics*, 22(01):1850083, 2020.

[107] X. Liang and T. Zhou. Spreading speeds of nonlocal KPP equations in almost periodic media. *Journal of Functional Analysis*, 279(9):108723, 2020.

[108] X. Liang and T. Zhou. Spreading speeds of nonlocal KPP equations in heterogeneous media. *Acta Mathematica Sinica, English Series*, 38(1):161–178, 2022.

[109] T. S. Lim and A. Zlatoš. Transition fronts for inhomogeneous Fisher-KPP reactions and non-local diffusion. *Transactions of the American Mathematical Society*, 368(12):8615–8631, 2016.

[110] G. Lin. Spreading speeds of a Lotka–Volterra predator–prey system: the role of the predator. *Nonlinear Analysis: Theory, Methods & Applications*, 74(7):2448–2461, 2011.

[111] G. Lin and W.-T. Li. Asymptotic spreading of competition diffusion systems: the role of interspecific competitions. *European Journal of Applied Mathematics*, 23(6):669–689, 2012.

[112] Q. Liu, S. Liu, and K.-Y. Lam. Asymptotic spreading of interacting species with multiple fronts I: A geometric optics approach. *Discrete & Continuous Dynamical Systems*, 40(6):3683, 2020.

[113] F. Lutscher. *Integrodifference Equations in Spatial Ecology*, volume 49. Springer Nature, 2019.

[114] F. Lutscher, E. Pachepsky, and M. A. Lewis. The effect of dispersal patterns on stream populations. *SIAM Review*, 47(4):749–772, 2005.

[115] S. Ma, P. Weng, and X. Zou. Asymptotic speed of propagation and traveling wavefronts in a non-local delayed lattice differential equation. *Nonlinear Analysis: Theory, Methods & Applications*, 65(10):1858–1890, 2006.

[116] P. Magal and X.-Q. Zhao. Global attractors and steady states for uniformly persistent dynamical systems. *SIAM Journal on Mathematical Analysis*, 37(1):251–275, 2005.

[117] H. Matano. Traveling waves in spatially random media. *RIMS Kokyuroku*, 1337:1–9, 2003.

[118] H. Matano and M. Nara. Large time behavior of disturbed planar fronts in the Allen–Cahn equation. *Journal of Differential Equations*, 251(12):3522–3557, 2011.

[119] R. M. May. Biological populations obeying difference equations: stable points, stable cycles, and chaos. *Journal of Theoretical Biology*, 51(2):511–524, 1975.

[120] A. Mellet, J. Nolen, J.-M. Roquejoffre, and L. Ryzhik. Stability of generalized transition fronts. *Communications in Partial Differential Equations*, 34(6):521–552, 2009.

[121] J. D. Murray. *Mathematical biology: I. An introduction.* Springer, 1993.

[122] J. D. Murray. *Mathematical biology II: spatial models and biomedical applications*, volume 3. Springer New York, 2003.

[123] G. Nadin. Traveling fronts in space–time periodic media. *Journal de Mathématiques Pures et Appliquées*, 92(3):232–262, 2009.

[124] G. Nadin and L. Rossi. Propagation phenomena for time heterogeneous KPP reaction–diffusion equations. *Journal de Mathématiques Pures et Appliquées*, 98(6):633–653, 2012.

[125] G. Nadin and L. Rossi. Transition waves for Fisher–KPP equations with general time-heterogeneous and space-periodic coefficients. *Analysis & PDE*, 8(6):1351–1377, 2015.

[126] H. Ninomiya and M. Taniguchi. Existence and global stability of traveling curved fronts in the Allen–Cahn equations. *Journal of Differential Equations*, 213(1):204–233, 2005.

[127] J. Nolen, J.-M. Roquejoffre, L. Ryzhik, and A. Zlatoš. Existence and non-existence of Fisher-KPP transition fronts. *Archive for Rational Mechanics and Analysis*, 203(1):217–246, 2012.

[128] J. Nolen, M. Rudd, and J. Xin. Existence of KPP fronts in spatially-temporally periodic advection and variational principle for propagation speeds. *Dynamics of Partial Differential Equations*, 2(1):1–24, 2005.

[129] A. Oaten and W. W. Murdoch. Functional response and stability in predator-prey systems. *The American Naturalist*, 109(967):289–298, 1975.

[130] A. Okubo and S. A. Levin. *Diffusion and ecological problems: modern perspectives*, volume 14. Springer, 2001.

[131] G. Peltier. *Analyse mathématique de modèles non-locaux en écologie évolutive.* PhD thesis, Université Montpellier, 2021.

[132] J.-M. Roquejoffre. Large time behaviour in nonlocal reaction-diffusion equations of the Fisher-KPP type. *arXiv preprint arXiv:2204.12246*, 2022.

[133] M. L. Rosenzweig and R. H. MacArthur. Graphical representation and stability conditions of predator-prey interactions. *The American Naturalist*, 97(895):209–223, 1963.

[134] F. Rothe. Convergence to pushed fronts. *The Rocky Mountain Journal of Mathematics*, 11(4):617–633, 1981.

[135] K. Schumacher. Travelling-front solutions for integro-differential equations. I. *Journal für die reine und angewandte Mathematik*, 1980(316):54–70, 1980.

[136] W. Shen. Traveling waves in time almost periodic structures governed by bistable nonlinearities: I. stability and uniqueness. *Journal of Differential Equations*, 159(1):1–54, 1999.

[137] W. Shen. Traveling waves in time almost periodic structures governed by bistable nonlinearities: II. existence. *Journal of Differential Equations*, 159(1):55–101, 1999.

[138] W. Shen. Traveling waves in diffusive random media. *Journal of Dynamics and Differential Equations*, 16(4):1011–1060, 2004.

[139] W. Shen. Spreading and generalized propagating speeds of discrete KPP models in time varying environments. *Frontiers of Mathematics in China*, 4(3):523–562, 2009.

[140] W. Shen. Variational principle for spreading speeds and generalized propagating speeds in time almost periodic and space periodic KPP models. *Transactions of the American Mathematical Society*, 362(10):5125–5168, 2010.

[141] W. Shen. Existence, uniqueness, and stability of generalized traveling waves in time dependent monostable equations. *Journal of Dynamics and Differential Equations*, 23(1):1–44, 2011.

[142] W. Shen. Stability of transition waves and positive entire solutions of fisher-kpp equations with time and space dependence. *Nonlinearity*, 30(9):3466–3491, 2017.

[143] W. Shen and Z. Shen. Transition fronts in nonlocal Fisher-KPP equations in time heterogeneous media. *Communications on Pure & Applied Analysis*, 15(4):1193–1213, 2016.

[144] W. Shen and Z. Shen. Regularity and stability of transition fronts in nonlocal equations with time heterogeneous ignition nonlinearity. *Journal of Differential Equations*, 262(5):3390–3430, 2017.

[145] W. Shen and Z. Shen. Transition fronts in nonlocal equations with time heterogeneous ignition nonlinearity. *Discrete & Continuous Dynamical Systems*, 37(2):1013–1037, 2017.

[146] W. Shen and A. Zhang. Spreading speeds for monostable equations with nonlocal dispersal in space periodic habitats. *Journal of Differential Equations*, 249(4):747–795, 2010.

[147] N. Shigesada and K. Kawasaki. *Biological invasions: theory and practice*. Oxford University Press, UK, 1997.

[148] N. Shigesada, K. Kawasaki, and E. Teramoto. Traveling periodic waves in heterogeneous environments. *Theoretical Population Biology*, 30(1):143–160, 1986.

[149] J. G. Skellam. Random dispersal in theoretical populations. *Biometrika*, 38(1/2):196–218, 1951.

[150] H. L. Smith and H. R. Thieme. *Dynamical systems and population persistence*, volume 118. American Mathematical Soc., 2011.

[151] A. N. Stokes. On two types of moving front in quasilinear diffusion. *Mathematical Biosciences*, 31(3-4):307–315, 1976.

[152] M. Taniguchi. Traveling fronts of pyramidal shapes in the Allen–Cahn equations. *SIAM Journal on Mathematical Analysis*, 39(1):319–344, 2007.

[153] K. Uchiyama. The behavior of solutions of some non-linear diffusion equations for large time. *Journal of Mathematics of Kyoto University*, 18(3):453–508, 1978.

[154] A. I. Volpert, V. A. Volpert, and V. A. Volpert. *Traveling wave solutions of parabolic systems*, volume 140. American Mathematical Soc., 1994.

[155] N. Wang and Z.-C. Wang. Generalized spreading speeds for lattice differential equations with time and space dependence. *Electron. J. Differential Equations*, 2019(74):1–15, 2019.

[156] N. Wang, Z.-C. Wang, and X. Bao. Transition waves for lattice Fisher-KPP equations with time and space dependence. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 151(2):573–600, 2021.

[157] X. Wang and G. Lin. Asymptotic spreading for a time-periodic predator-prey system. *Communications on Pure & Applied Analysis*, 18(6):2983–2999, 2019.

[158] X. Wang and G. Lin. Traveling waves for a periodic Lotka–Volterra predator–prey system. *Applicable Analysis*, 98(14):2619–2638, 2019.

[159] X. Wang, G. Lin, and S. Ruan. Spreading speeds and traveling wave solutions of diffusive vector-borne disease models without monotonicity. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, pages 1–30, 2021.

[160] X. Wang, G. Lin, and S. Ruan. Spatial propagation in a within-host viral infection model. *Studies in Applied Mathematics*, 2022.

[161] Z.-C. Wang, L. Zhang, and X.-Q. Zhao. Time periodic traveling waves for a periodic and diffusive SIR epidemic model. *Journal of Dynamics and Differential Equations*, 30(1):379–403, 2018.

[162] H. F. Weinberger. Long-time behavior of a class of biological models. *SIAM Journal on Mathematical Analysis*, 13(3):353–396, 1982.

[163] H. F. Weinberger. On spreading speeds and traveling waves for growth and migration models in a periodic habitat. *Journal of Mathematical Biology*, 45(6):511–548, 2002.

[164] H. F. Weinberger, M. A. Lewis, and B. Li. Analysis of linear determinacy for spread in cooperative models. *Journal of Mathematical Biology*, 45(3):183–218, 2002.

[165] C.-C. Wu. The spreading speed for a predator–prey model with one predator and two preys. *Applied Mathematics Letters*, 91:9–14, 2019.

[166] W.-B. Xu, W.-T. Li, and S. Ruan. Spatial propagation in an epidemic model with nonlocal diffusion: The influences of initial data and dispersals. *Science China Mathematics*, 63(11):2177–2206, 2020.

[167] W.-B. Xu, W.-T. Li, and S. Ruan. Spatial propagation in nonlocal dispersal Fisher-KPP equations. *Journal of Functional Analysis*, 280(10):108957, 2021.

[168] H. Yagisita. Existence and nonexistence of traveling waves for a nonlocal monostable equation. *Publications of the Research Institute for Mathematical Sciences*, 45(4):925–953, 2010.

[169] G.-B. Zhang and X.-Q. Zhao. Propagation phenomena for a two-species Lotka–Volterra strong competition system with nonlocal dispersal. *Calculus of Variations and Partial Differential Equations*, 59(1):1–34, 2020.

[170] G. Zhao and S. Ruan. Existence, uniqueness and asymptotic stability of time periodic traveling waves for a periodic Lotka–Volterra competition system with diffusion. *Journal de Mathématiques Pures et Appliquées*, 95(6):627–671, 2011.

[171] G. Zhao and S. Ruan. Time periodic traveling wave solutions for periodic advection–reaction–diffusion systems. *Journal of Differential Equations*, 257(4):1078–1147, 2014.

[172] M. Zhao, R. Yuan, Z. Ma, and X. Zhao. Spreading speeds for the predator-prey system with nonlocal dispersal. *Journal of Differential Equations*, 316:552–598, 2022.

[173] X.-Q. Zhao. *Dynamical systems in population biology*, volume 16. Springer, 2003.

[174] B. Zinner, G. Harris, and W. Hudson. Traveling wavefronts for the discrete Fisher's equation. *Journal of Differential Equations*, 105(1):46–62, 1993.