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About Berge–Füredi’s conjecture on the chromatic index of hypergraphs

Sur la conjecture de Berge–Füredi sur l’indice chromatique des hypergraphes

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Abstract. We prove that the chromatic index of a hypergraph \mathcal{H} satisfies the Berge–Füredi conjectured bound $q(\mathcal{H}) \leq \Delta([\mathcal{H}]_2) + 1$ under certain hypotheses on the antirank $\text{ar}(\mathcal{H})$ or on the maximum degree $\Delta(\mathcal{H})$. This provides sharp information in connection with the Erdős–Faber–Lovász conjecture which deals with the coloring of a family of cliques that intersect pairwise in at most one vertex.

Résumé. Nous montrons que l’indice chromatique d’un hypergraphe satisfait la borne conjecturée indépendamment par Berge et Füredi, $q(\mathcal{H}) \leq \Delta([\mathcal{H}]_2) + 1$ sous certaines conditions portant sur son antirang $\text{ar}(\mathcal{H})$ ou son degré maximum $\Delta(\mathcal{H})$. Ces résultats fournissent des informations sur la conjecture de Erdős, Faber and Lovász qui traite de la coloration d’une famille de cliques se coupant deux à deux en au plus un sommet.

Keywords. Hypergraphs, hyperedge coloring, chromatic index.

Mots-clés. Hypergraphes, coloration des hyperarêtes, indice chromatique.

2020 Mathematics Subject Classification. 05C15.

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1. Introduction

Berge–Füredi’s conjecture (cf. [1,2,6] and Meyniel, unpublished communication) extends to linear hypergraphs Vizing’s theorem on the chromatic index of graphs (see [3, Theorem 7.2.2]). It asserts that the chromatic index of any linear hypergraph $\mathcal{H} = (V, E)$ without loop satisfies

$$q(\mathcal{H}) \leq \Delta([\mathcal{H}]_2) + 1. \quad (1)$$

This bound was first conjectured around 1986. Since then, to our knowledge, only few advances have been made. It has been recently proved [8] that (1) holds whenever $\Delta([\mathcal{H}]_2) \leq 5$. In the present paper we obtain new conditional results where the antirank $\text{ar}(\mathcal{H})$ and the maximum degree $\Delta(\mathcal{H})$ are involved together with the maximum degree $\Delta([\mathcal{H}]_2)$ of the 2-section of \mathcal{H} .

If \mathcal{H} is linear then $\Delta([\mathcal{H}]_2) \leq |V| - 1$. Hence bound (1) implies Erdős–Faber–Lovász conjecture asserting that $q(\mathcal{H}) \leq |V|$ for any linear hypergraph \mathcal{H} .

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2. Notation and statements

Let $\mathcal{H} = (V, E)$ be a hypergraph. For $x \in V$ and $e \in E$, we let

$$\deg_{\mathcal{H}}(x) = \left| \{a \in E : x \in a\} \right|, \quad d_{\mathcal{H}}(e) = \left| \{a \in E \setminus \{e\} : a \cap e \neq \emptyset\} \right|,$$

be the *degree of vertex* x and the *degree of hyperedge* e in \mathcal{H} , respectively. Then, we define respectively the *rank*, the *antirank*, the *maximum degree* and the *minimum degree* of \mathcal{H} by

$$r(\mathcal{H}) = \max_{e \in E} |e|, \quad ar(\mathcal{H}) = \min_{e \in E} |e|, \quad \Delta(\mathcal{H}) = \max_{x \in V} \deg_{\mathcal{H}}(x), \quad \delta(\mathcal{H}) = \min_{x \in V} \deg_{\mathcal{H}}(x).$$

A *loop* in a hypergraph \mathcal{H} is a hyperedge e such that $|e| = 1$.

The *2-section* of $\mathcal{H} = (V, E)$ is the loopless multigraph $[\mathcal{H}]_2 = (V, A)$ with k many edges between $x, y \in V, x \neq y$, if there are k hyperedges in \mathcal{H} containing both x and y . We have

$$\Delta([\mathcal{H}]_2) \geq (ar(\mathcal{H}) - 1)\Delta(\mathcal{H}). \tag{2}$$

The *chromatic index* of \mathcal{H} , denoted by $q(\mathcal{H})$, is the least integer q such that there exists a map $c: E \rightarrow \{1, 2, \dots, q\}$ satisfying

$$\forall e, e' \in E, (c(e) = c(e') \implies e \cap e' = \emptyset \text{ or } e = e').$$

The map c is called a *hyperedge-coloring* of \mathcal{H} and the image of E through c forms the set of *colors*.

A hypergraph $\mathcal{H} = (V, E)$ is said *linear* if for any pair of distinct hyperedges $e, e' \in E$, we have $|e \cap e'| \leq 1$. In this case $[\mathcal{H}]_2$ reduces to a simple graph.

Recently it has been established that if \mathcal{H} is a linear hypergraph satisfying

$$ar(\mathcal{H}) \geq \sqrt{|V|}, \tag{3}$$

then $q(\mathcal{H}) \leq |V|$ (cf. [7]). This upper bound has been improved in [4] where it is shown that under the same hypotheses on \mathcal{H} , then (1) actually holds. In this note we show that condition (3) can be weakened by proving the following sharper result.

Theorem 1. *Let $\mathcal{H} = (V, E)$ be a loopless hypergraph such that $ar(\mathcal{H}) \geq \sqrt{\Delta([\mathcal{H}]_2) + 1}$. Then $q(\mathcal{H}) \leq \Delta([\mathcal{H}]_2) + 1$.*

Note that we do not assume in the above statement that \mathcal{H} is linear. We will propose an approach supported by the notion of critical hypergraph (see also Section 6.1 for additional comments).

We will prove the following key result which has its own interest. \mathcal{H} is said *k-uniform* (resp. *d-regular*) if $\forall e \in E, |e| = k$ (resp. $\forall x \in V, \deg_{\mathcal{H}}(x) = d$).

Theorem 2. *Let $k \geq 2$ and $\mathcal{H} = (V, E)$ be a k -uniform and $(k + 1)$ -regular linear hypergraph. Then $q(\mathcal{H}) \leq \Delta([\mathcal{H}]_2) + 1$.*

The following corollary will be derived from Theorem 1.

Corollary 3. *Let $\mathcal{H} = (V, E)$ be a loopless hypergraph such that*

$$\Delta(\mathcal{H}) \leq \sqrt{\Delta([\mathcal{H}]_2) + 1} + 1. \tag{4}$$

Then $q(\mathcal{H}) \leq \Delta([\mathcal{H}]_2) + 1$.

3. Proof of Theorem 1

In [4] we exploited the notion of critical hyperedge. A hyperedge e is said *critical* if $q(\mathcal{H} \setminus e) = q(\mathcal{H}) - 1$. Hypergraph \mathcal{H} is a *critical hypergraph* if all its hyperedges are critical. Since the addition of e to $\mathcal{H} \setminus e$ needs a new color, e must intersect every color of $\mathcal{H} \setminus e$. So we have the following.

Lemma 4 ([4, Lemma 3.2]). *Let $\mathcal{H} = (V, E)$ be a hypergraph without loop and $e \in E$ be a critical hyperedge. Then $q(\mathcal{H}) - 1 \leq d_{\mathcal{H}}(e)$.*

We may assume that \mathcal{H} is connected and critical since $\text{ar}(\mathcal{H}^l) \geq \text{ar}(\mathcal{H})$ and $\Delta([\mathcal{H}^l]_2) \leq \Delta([\mathcal{H}]_2)$ for any partial hypergraph \mathcal{H}^l of \mathcal{H} . Let $e \in E$ be of minimum cardinality $k := \text{ar}(\mathcal{H})$. We have

$$d_{\mathcal{H}}(e) = \sum_{\substack{a \in E \setminus \{e\} \\ a \cap e \neq \emptyset}} 1 \leq \sum_{a \in E \setminus \{e\}} |a \cap e| = \sum_{x \in V} \sum_{\substack{a \in E \setminus \{e\} \\ a \ni x}} 1 = \sum_{x \in e} (\text{deg}_{\mathcal{H}}(x) - 1), \quad (5)$$

and equality occurs when \mathcal{H} is linear. Then by Lemma 4 we get

$$q(\mathcal{H}) - 1 = q(\mathcal{H} \setminus e) \leq d_{\mathcal{H}}(e) \leq k(\Delta(\mathcal{H}) - 1) = (k - 1)\Delta(\mathcal{H}) - (k - \Delta(\mathcal{H})),$$

so

$$q(\mathcal{H}) \leq \Delta([\mathcal{H}]_2) + 1 - (k - \Delta(\mathcal{H})). \quad (6)$$

Our hypothesis implies $\Delta([\mathcal{H}]_2) \leq k^2 - 1$. We distinguish two cases.

- If $\Delta([\mathcal{H}]_2) < k^2 - 1$ we get by (2) that $\Delta(\mathcal{H}) < k + 1$ thus $\Delta(\mathcal{H}) \leq k$, yielding the required upper bound for $q(\mathcal{H})$ by (6).
- Assume that $\Delta([\mathcal{H}]_2) = k^2 - 1$. We thus have to prove that $q(\mathcal{H}) \leq k^2$. If there exists $e \in E$ such that $|e| = k$ and $d_{\mathcal{H}}(e) \leq k^2 - 1$ then $q(\mathcal{H}) \leq d_{\mathcal{H}}(e) + 1 \leq k^2$ by Lemma 4 and we are done. Otherwise

$$d_{\mathcal{H}}(e) \geq k^2 \quad \text{for any hyperedge } e \text{ of cardinality } k. \quad (7)$$

We shall see that \mathcal{H} is a very special hypergraph.

- Let $e \in E$ be of cardinality k . By (2) we have $\Delta(\mathcal{H}) \leq k + 1$, thus by (7) and (5) we get

$$k^2 \leq d_{\mathcal{H}}(e) \leq \sum_{x \in e} (\text{deg}_{\mathcal{H}}(x) - 1) \leq k(\Delta(\mathcal{H}) - 1) \leq k(k + 1 - 1) = k^2.$$

We infer

$$d_{\mathcal{H}}(e) = k^2 \quad \text{and} \quad \text{deg}_{\mathcal{H}}(x) - 1 = k \quad \text{for any } x \in e. \quad (8)$$

- Let $a \in E$ such that $a \cap e \neq \emptyset$. Let $x \in a \cap e$. We have

$$\begin{aligned} k^2 - 1 = \Delta([\mathcal{H}]_2) &\geq \text{deg}_{[\mathcal{H}]_2}(x) = \sum_{\substack{b \in E \\ b \ni x}} (|b| - 1) \\ &= |a| - 1 + \sum_{\substack{b \in E \setminus \{a\} \\ b \ni x}} (|b| - 1) \\ &\geq |a| - 1 + (\text{deg}_{\mathcal{H}}(x) - 1)(k - 1) \\ &= |a| - 1 + k(k - 1) \\ &\geq k^2 - 1, \end{aligned}$$

giving $|a| = k$.

- Since \mathcal{H} is connected we infer that $|a| = k$ for any $a \in E$. Hence \mathcal{H} is k -uniform and $(k + 1)$ -regular. By (5) and (8), we have $|a \cap e| = 1$ whenever $a \cap e \neq \emptyset$. Thus \mathcal{H} is linear. The proof will be completed when Theorem 2 will be established.

4. Proof of Theorem 2

By assumption we have

$$k|e| = (k+1)|V|, \quad \Delta([\mathcal{H}]_2) = k^2 - 1 \quad \text{and} \quad \forall e \in E, \quad d_{\mathcal{H}}(e) = k^2. \quad (9)$$

Since \mathcal{H} is linear, its *line-graph* $L(\mathcal{H})$ is the simple graph whose vertices are hyperedges of \mathcal{H} and such that there is an edge between $e, e' \in E, e \neq e'$, if $e \cap e' \neq \emptyset$. We have $\deg_{L(\mathcal{H})}(e) = d_{\mathcal{H}}(e)$ for any $e \in E$.

By (9), each vertex e of $L(\mathcal{H})$ has degree $\deg_{L(\mathcal{H})}(e) = k^2$. Hence $L(\mathcal{H})$ is k^2 -regular. Since $q(\mathcal{H})$ coincides with $\chi(L(\mathcal{H}))$, namely the chromatic number of the simple graph $L(\mathcal{H})$, we infer from Brooks' theorem (see [3, Theorem 7.1.9, p. 238]) that either

$$q(\mathcal{H}) \leq \Delta(L(\mathcal{H})) = k^2 = \Delta([\mathcal{H}]_2) + 1$$

or $L(\mathcal{H})$ is complete or an odd-length cycle. If $L(\mathcal{H})$ were a complete graph, we would have $|e| = k^2 + 1$ implying by (9) that $(k+1)$ divides $k(k^2 + 1)$, which gives a contradiction since $k \geq 2$. If $L(\mathcal{H})$ were a cycle, we would have $d_{\mathcal{H}}(e) = \deg_{L(\mathcal{H})}(e) = 2$ for any $e \in E$, a contradiction. This ends the proof of Theorem 2.

5. Proof of Corollary 3

Let $\mathcal{H}' = (V, E')$ be a critical partial hypergraph of \mathcal{H} such that $q(\mathcal{H}') = q(\mathcal{H})$.

- If $\text{ar}(\mathcal{H}') \geq \sqrt{\Delta([\mathcal{H}']_2) + 1}$ we infer from Theorem 1 that

$$q(\mathcal{H}) = q(\mathcal{H}') \leq \Delta([\mathcal{H}']_2) + 1 \leq \Delta([\mathcal{H}]_2) + 1.$$

- If $\text{ar}(\mathcal{H}') < \sqrt{\Delta([\mathcal{H}']_2) + 1}$ then we take $e \in E'$ such that $|e| = \text{ar}(\mathcal{H}')$. By Lemma 4 and using (5) with \mathcal{H}' in place of \mathcal{H} , we get

$$\begin{aligned} q(\mathcal{H}) - 1 &= q(\mathcal{H}') - 1 \leq d_{\mathcal{H}'}(e) \\ &\leq \sum_{x \in e} (\deg_{\mathcal{H}'}(x) - 1) \\ &\leq |e|(\Delta(\mathcal{H}') - 1) \\ &= \text{ar}(\mathcal{H}')(\Delta(\mathcal{H}') - 1) \\ &< \sqrt{\Delta([\mathcal{H}']_2) + 1}(\Delta(\mathcal{H}') - 1) \\ &\leq \sqrt{\Delta([\mathcal{H}]_2) + 1}(\Delta(\mathcal{H}) - 1) \\ &\leq \Delta([\mathcal{H}]_2) + 1 \end{aligned}$$

by (4). Hence $q(\mathcal{H}) \leq \Delta([\mathcal{H}]_2) + 1$ as wanted.

6. Remarks and questions

6.1.

We should notice that when $\text{ar}(\mathcal{H}) > \sqrt{\Delta([\mathcal{H}]_2) + 1}$ then the desired bound $q(\mathcal{H}) \leq \Delta([\mathcal{H}]_2) + 1$ can be equally deduced from the greedy algorithm for coloring hyperedges (see for instance [5, p. 586]) which yields

$$q(\mathcal{H}) \leq \max_{r(\mathcal{H}) \geq k \geq \text{ar}(\mathcal{H})} k \left\lfloor \frac{\Delta([\mathcal{H}]_2)}{k-1} - 1 \right\rfloor + 1.$$

Here we use a different approach which allows us to solve in addition the equality case $\text{ar}(\mathcal{H}) = \sqrt{\Delta([\mathcal{H}]_2) + 1}$.

6.2.

When the maximum degree $\Delta(\mathcal{H})$ satisfies $\Delta(\mathcal{H}) > \sqrt{\Delta([\mathcal{H}]_2)} + 1 + 1$, Corollary 3 is no longer usable. However at the cost of an additional condition on the rank $r(\mathcal{H}) = \max_{e \in E} |e|$, we can notice that (1) remains valid for linear hypergraphs \mathcal{H} . Let $\delta_2 := \sqrt{\Delta([\mathcal{H}]_2)}$ and assume that

$$\begin{aligned} r(\mathcal{H}) &= \delta_2 - u, \quad u > 0, \\ \Delta(\mathcal{H}) &= \delta_2 + 1 + v, \quad v > 0. \end{aligned}$$

Then

$$q(\mathcal{H}) = \chi(L(\mathcal{H})) \leq \Delta(L(\mathcal{H})) + 1 \leq r(\mathcal{H})(\Delta(\mathcal{H}) - 1) + 1 = (\delta_2 - u)(\delta_2 + v) + 1.$$

Thus (1) holds whenever $(v - u)\delta_2 \leq uv$, that is $\frac{1}{u} \leq \frac{1}{v} + \frac{1}{\delta_2}$.

6.3.

If \mathcal{H} is supposed linear, in the *extremal case* $\Delta([\mathcal{H}]_2) = |V| - 1$, $ar(\mathcal{H}) = |V|^{1/2}$ in Theorem 1 the authors obtained in [4] a much better upper bound for $q(\mathcal{H})$, namely $q(\mathcal{H}) \leq \frac{1}{2}(\Delta([\mathcal{H}]_2) + 1 + \sqrt{\Delta([\mathcal{H}]_2)} + 1)$. How can the upper bound for $q(\mathcal{H})$ be improved under the sole condition $ar(\mathcal{H}) = \sqrt{\Delta([\mathcal{H}]_2)} + 1$?

6.4.

Theorem 2 deals with k -uniform and d -regular linear hypergraphs when $d = k + 1$ and provides the required bound (1). For general k, d satisfying $k \leq d - 2$, the line-graph $L(\mathcal{H})$ is $k(d - 1)$ -regular and Brooks' theorem provides the (non sufficiently sharp) inequality $q(\mathcal{H}) = \chi(L(\mathcal{H})) \leq \Delta(L(\mathcal{H})) = k(d - 1)$ while $\Delta([\mathcal{H}]_2) = d(k - 1)$.

6.5.

More generally let \mathcal{H} be a k -uniform linear hypergraph with $k \geq 2$ and set $\delta_2 := \sqrt{\Delta([\mathcal{H}]_2)}$. Then (1) holds if at least one of the following conditions is satisfied:

- (1) $k = 2$: Vizing's theorem (see [3, Theorem 7.2.2, p. 249]);
- (2) $k \geq \sqrt{\Delta([\mathcal{H}]_2)} + 1$: Theorem 1;
- (3) $\Delta([\mathcal{H}]_2) = k^2$ which implies $\Delta(\mathcal{H}) \leq \sqrt{\Delta([\mathcal{H}]_2)} + 1$: Corollary 3;
- (4) $3 \leq k < \delta_2$ and $\frac{1}{\delta_2 - k} \leq \frac{1}{\Delta(\mathcal{H}) - \delta_2 - 1} + \frac{1}{\delta_2}$: see Section 6.2.

In order to solve Berge-Füredi's conjecture for all linear uniform hypergraphs, it thus remains to consider the case where

$$3 \leq k < \delta_2 < \Delta(\mathcal{H}) - 1 \quad \text{and} \quad \frac{1}{\delta_2 - k} > \frac{1}{\Delta(\mathcal{H}) - \delta_2 - 1} + \frac{1}{\delta_2}.$$

We may likewise ask the following question: is Berge-Füredi's conjecture in all its generality implied by its validity for linear uniform hypergraphs?

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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