



**HAL**  
open science

# Fading regularization method and DKQ plate elements for the Cauchy problem associated with the biharmonic equation

Mohamed Aziz Boukraa, Franck Delvare

► **To cite this version:**

Mohamed Aziz Boukraa, Franck Delvare. Fading regularization method and DKQ plate elements for the Cauchy problem associated with the biharmonic equation. 45eme Congrès National d'Analyse Numérique (CANUM20), Jun 2022, Evian-Les-Bains, France. hal-04310355

**HAL Id: hal-04310355**

**<https://normandie-univ.hal.science/hal-04310355>**

Submitted on 27 Nov 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



# Fading regularization method and DKQ plate elements for the Cauchy problem associated with the biharmonic equation

Mohamed Aziz Boukraa<sup>1,2</sup>, Franck Delvare<sup>2</sup>

<sup>1</sup> Université Clermont Auvergne, CNRS, LMBP, F-63000 Clermont-Ferrand, France

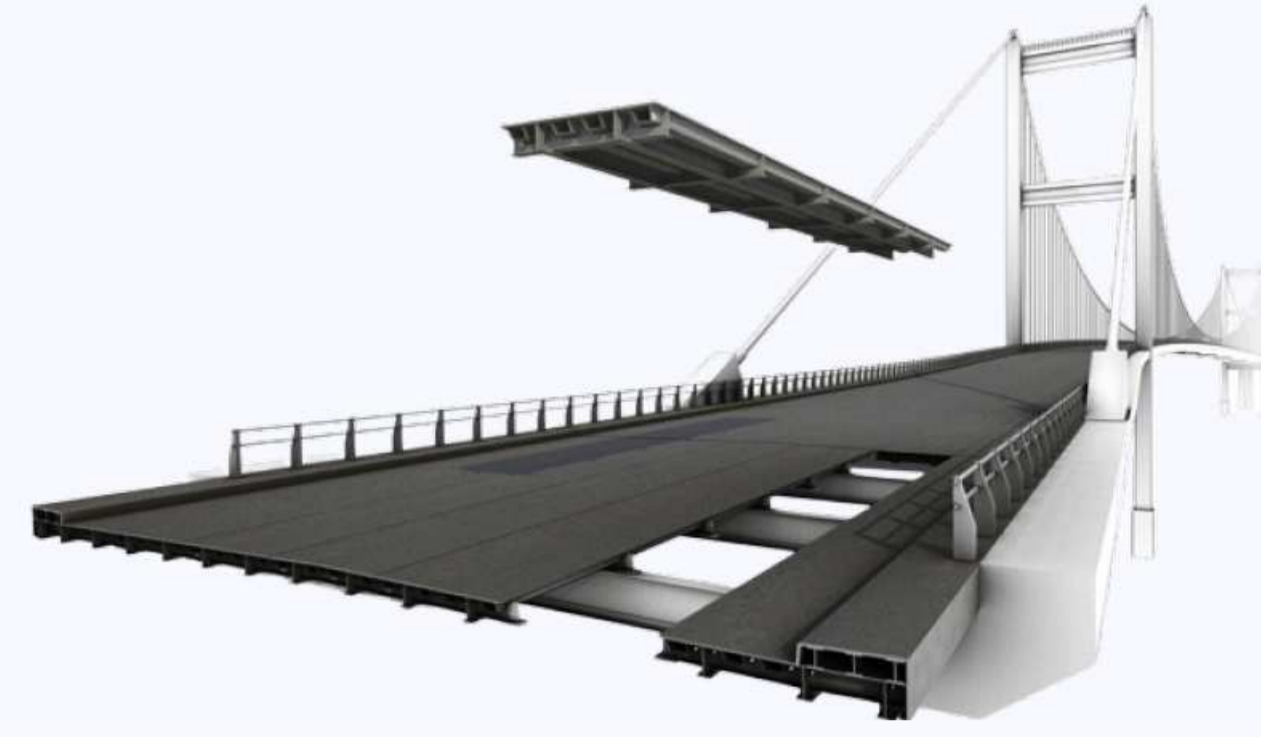
<sup>2</sup> Normandie Univ, UNICAEN, CNRS, LMNO, 14000 Caen, France

mohamed.boukraa@uca.fr

## Motivation: Inverse problems in thin plate bending

Thin plates find applications in several fields of science and engineering. We recall here the case of plate elements for a bridge deck. In some situations, it may be mandatory to ensure the embedding and support of these plates for the correct installation of this structure. However, direct measurements may be impossible or difficult to obtain during construction or during maintenance. An inverse problem then arises!

**Objectives :** The aim of this study is to solve the inverse Cauchy-type problem for the bilaplacian operator, which governs the bending of thin plates, such as boundary conditions are available only on part of the boundary. We propose to apply the fading regularization method and to implement it numerically using Discrete Kirchhoff (DK) plate elements.



canambridges.com

Figure 1: Plate elements for a bridge deck

## Ill-posed Cauchy problem associated with the bilaplacian

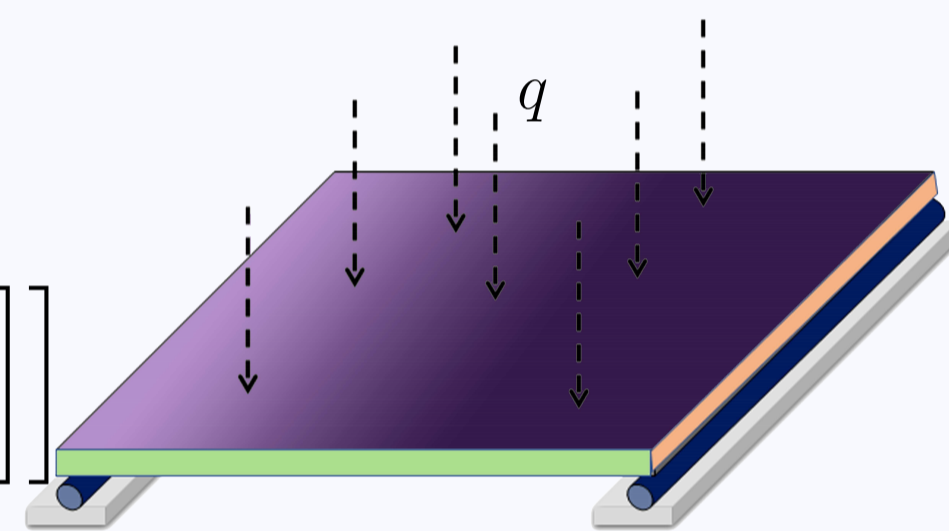
The constitutive equation for the bending plate problem, under Kirchhoff-Love theory, is :

$$\int_{\Omega} \underbrace{(\mathbf{L}\nabla)^t \mathbf{D}(\mathbf{L}\nabla w)}_{D\Delta^2 w} \delta w \, dx dy = \int_{\Omega} q(x,y) \delta w \, dx dy + \int_{\Gamma} \left[ \mathcal{M}_n \frac{\partial \delta w}{\partial n} - \mathcal{V}_n \delta w \right] ds + \sum_i \delta w_i R_i.$$

It is characterized by the biharmonic operator to which boundary conditions, related to the transverse displacement  $w$ , the normal rotation  $\frac{\partial w}{\partial n}$ , the moment  $\mathcal{M}_n$  and the shear force  $\mathcal{V}_n$ , are added at any point of the boundary.

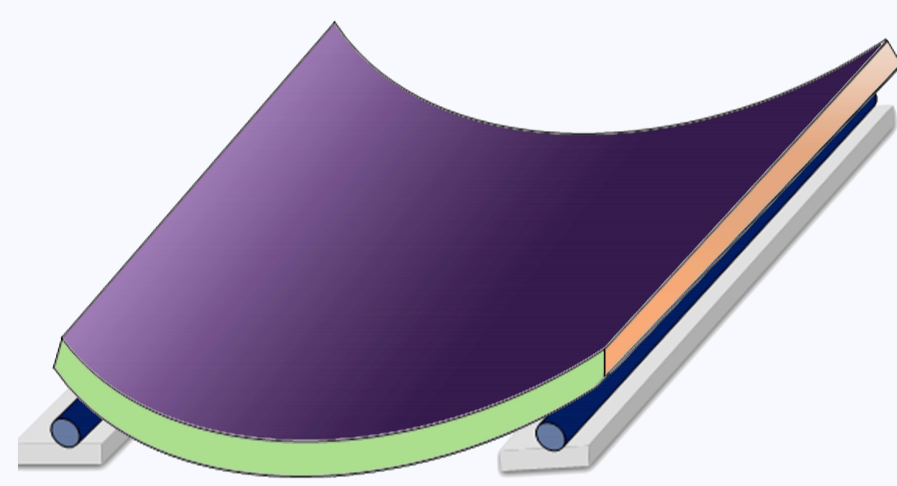
$$\mathcal{M}_n = -D \left[ \Delta w + (1-\nu) \left( 2n_x n_y \frac{\partial^2 w}{\partial x \partial y} - n_y^2 \frac{\partial^2 w}{\partial x^2} - n_x^2 \frac{\partial^2 w}{\partial y^2} \right) \right]$$

$$\mathcal{V}_n = -D \left[ \frac{\partial \Delta w}{\partial n} + (1-\nu) \frac{\partial}{\partial s} \left[ n_x n_y \left( \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2} \right) + (n_x^2 - n_y^2) \frac{\partial^2 w}{\partial x \partial y} \right] \right]$$



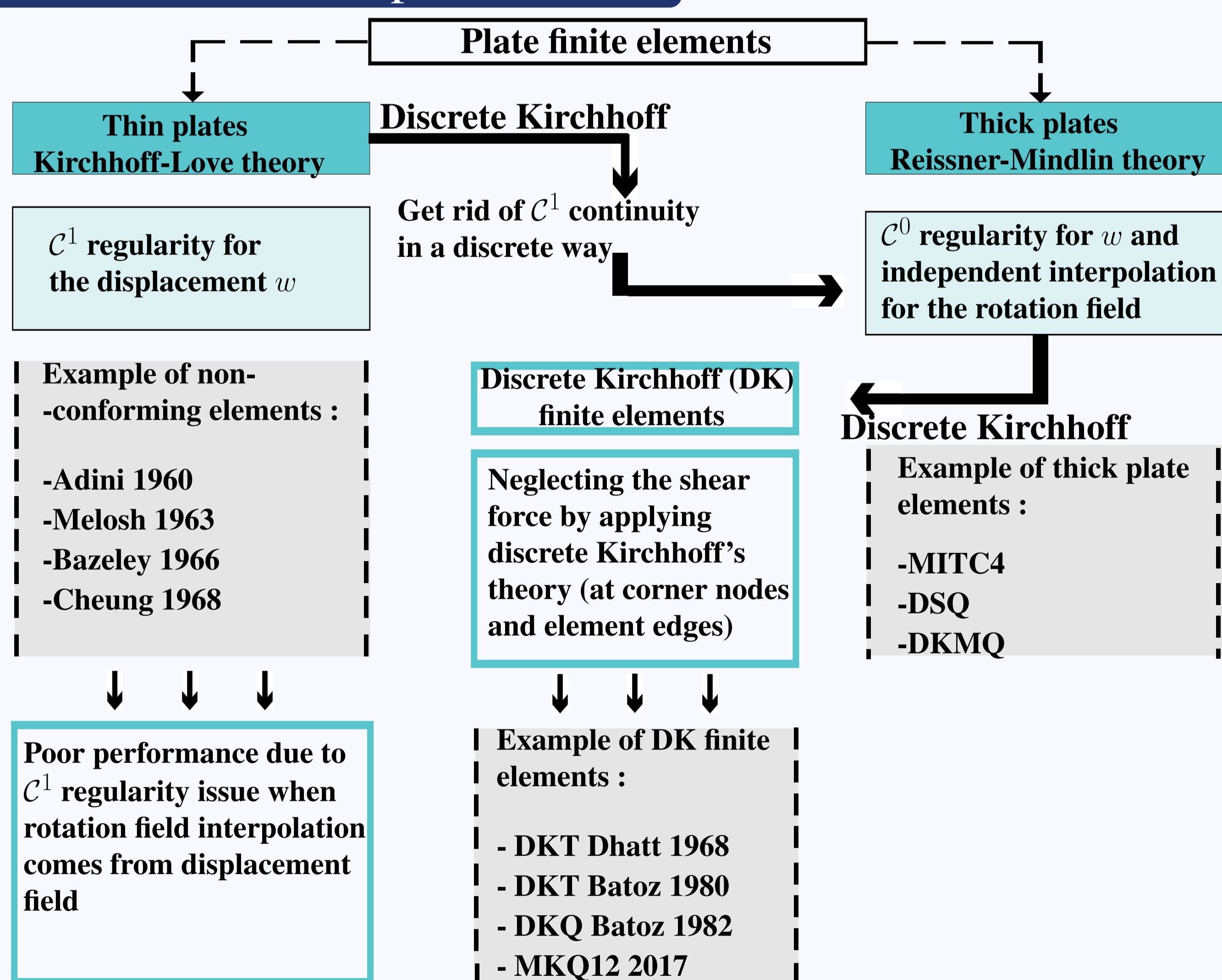
These boundary conditions are not always accessible to measurements. Hence, in the case of an inaccessible part  $\Gamma_i$  of the boundary and an over-specified part of the boundary  $\Gamma_d$ , the resulting Cauchy problem is ill-posed in the sense of Hadamard.

$$\begin{cases} \Delta^2 w = 0 & \forall x \in \Omega \\ w = \varphi_d & \forall x \in \Gamma_d \\ w_n = \psi_d & \forall x \in \Gamma_d \\ \mathcal{M}_n = \mathcal{M}_d & \forall x \in \Gamma_d \\ \mathcal{V}_n = \mathcal{V}_d & \forall x \in \Gamma_d \end{cases}$$



$\partial\Omega = \Gamma_d \cup \Gamma_i$  et  $\Gamma_d \cap \Gamma_i = \emptyset$ . **No boundary condition is given on the part  $\Gamma_i$  !**

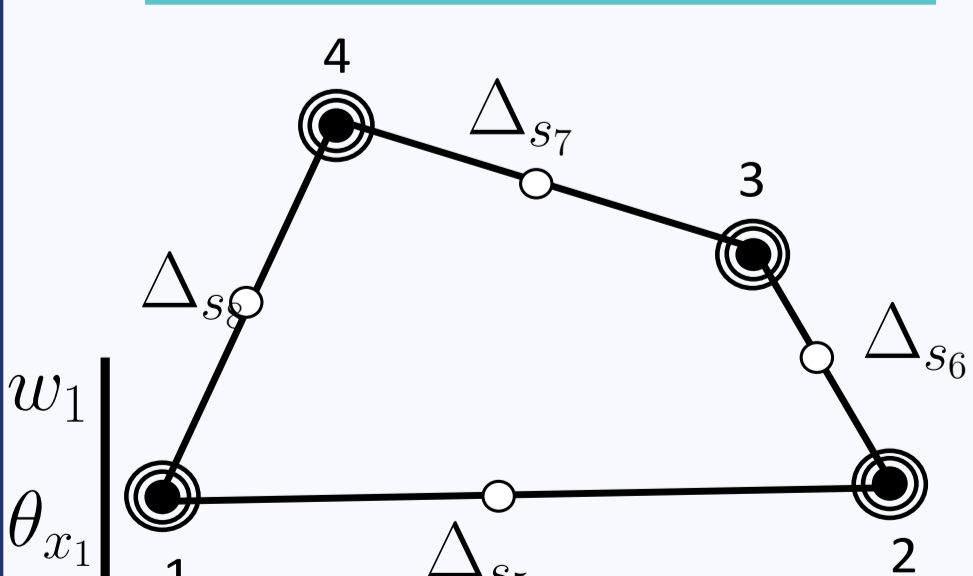
## Discrete Kirchhoff plate element



The stiffness matrix of the DKQ element is defined as

$$\mathbf{K} = \int_{\Omega} (\mathbf{L}\nabla w)^t [\mathbf{D}] (\mathbf{L}\nabla w) \, d\Omega$$

$$\text{where } \mathbf{L}\nabla w = \begin{bmatrix} \frac{\partial \theta_y}{\partial x} \\ \frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_x}{\partial x} + \frac{\partial \theta_y}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}$$



3 degrees of freedom per node :

$$\underline{w} = \mathbf{N} \underline{d}, \quad \underline{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}, \quad d_i = \begin{bmatrix} w_i \\ \theta_{x_i} \\ \theta_{y_i} \end{bmatrix} = \begin{bmatrix} w_i \\ -w_{,x_i} \\ -w_{,y_i} \end{bmatrix}$$

The corresponding finite element formulation is :  $\mathcal{E}(\underline{\mathbf{V}}) := \mathbf{K} \underline{\mathbf{d}} - \mathbf{F} \underline{\mathbf{b}} = 0$ , such as  $\underline{\mathbf{b}} = (\underline{\mathcal{M}}_n, \underline{\mathcal{V}}_n, \underline{R})$ .

## The fading regularization method

• **Idea :** Find the solution of the equilibrium equation that best fits the boundary data while maintaining independence of the regularization parameter and stability towards noisy data

$$\mathbf{V}^{k+1} = \underset{\mathbf{V} \in H(\Gamma)}{\text{Argmin}} \left\{ \|\mathbf{V} - \Phi_d\|_{H(\Gamma_d)}^2 + c \|\mathbf{V} - \mathbf{V}^k\|_{H(\Gamma)}^2 \right\}$$

• The fading regularization algorithm:

$$\begin{cases} \mathbf{V}^{k+1} = \underset{\mathbf{V} \in \mathbb{R}^{5N}}{\text{Argmin}} J_c^{k+1}(\mathbf{V}) \\ \text{with } \mathbf{V} = (d, \mathcal{M}_n, \mathcal{V}_n) = (W, \theta_x, \theta_y, \mathcal{M}_n, \mathcal{V}_n) \\ \text{under the equality constraints } \mathcal{E}(\mathbf{V}) = 0 \end{cases}$$

where

$$J_c^{k+1}(\mathbf{V}) = \|W|_{\Gamma_d} - \phi_d\|_{L^2(\Gamma_d)}^2 + \|n_y \theta_{,x} + n_x \theta_{,y}|_{\Gamma_d} - \mu_d\|_{L^2(\Gamma_d)}^2 + \|\mathcal{M}_n|_{\Gamma_d} - \mathcal{M}_d\|_{L^2(\Gamma_d)}^2 + \|\mathcal{V}_n|_{\Gamma_d} - \mathcal{V}_d\|_{L^2(\Gamma_d)}^2 + c \left( \|W - W^k\|_{L^2(\Gamma)}^2 + \|\theta_{,x} - \theta_{,x}^k\|_{L^2(\Gamma)}^2 + \|\theta_{,y} - \theta_{,y}^k\|_{L^2(\Gamma)}^2 + \|\mathcal{M}_n - \mathcal{M}_n^k\|_{L^2(\Gamma)}^2 + \|\mathcal{V}_n - \mathcal{V}_n^k\|_{L^2(\Gamma)}^2 \right)$$

## Numerical simulations

We consider the biharmonic analytical solution:

$$u_{an}(x,y) = \frac{1}{2} x (\sin x \cosh y - \cos x \sinh y), \quad (x,y) \in \Omega,$$

where  $\Omega$  is a two-dimensional domain such as Cauchy data are given only on a part  $\Gamma_d$  of its boundary  $\Gamma$ . Two example of domains are considered :

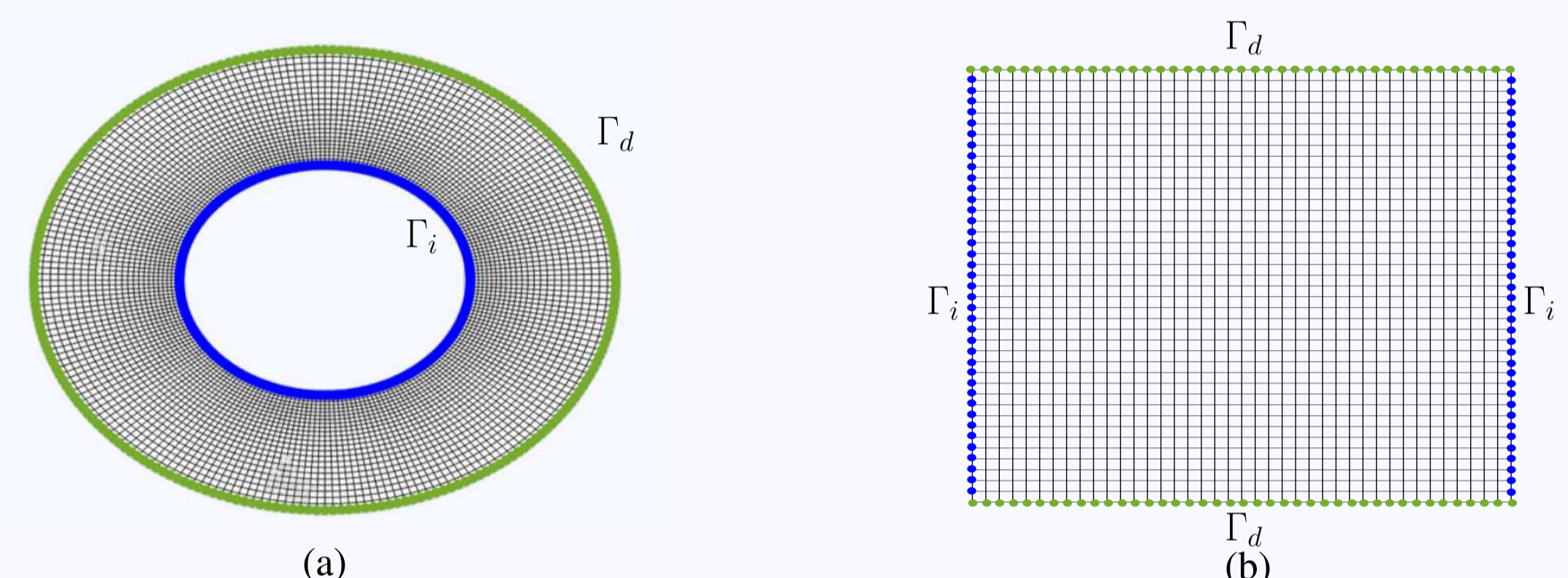


Figure 2: (a) An annular shaped domain and (b) a square domain such as data are given only on the part  $\Gamma_d$  of the boundary.

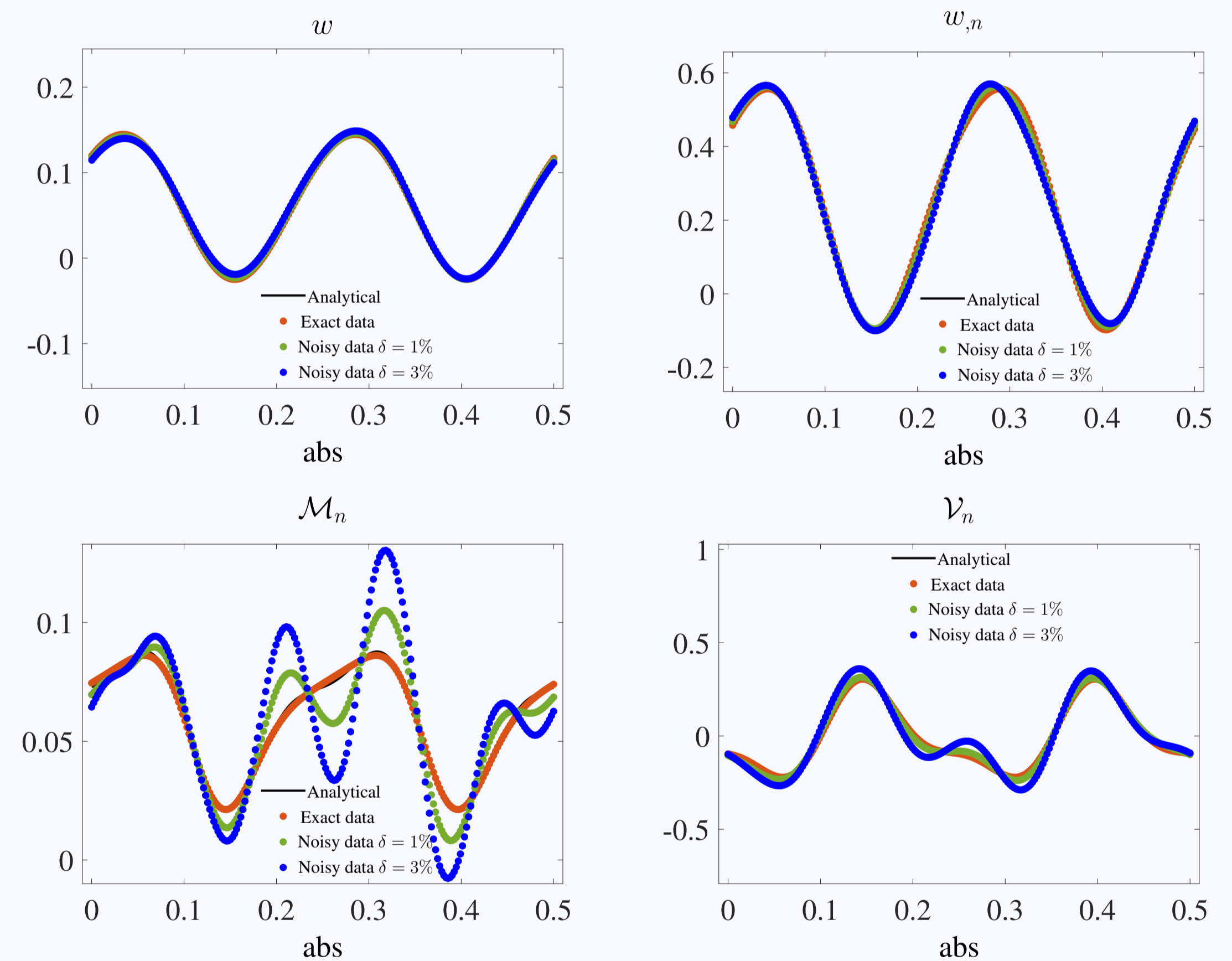


Figure 3: Reconstructions of the boundary conditions on the inner boundary of an annular domain (Fig. 2a) such as data are given on the outer boundary.

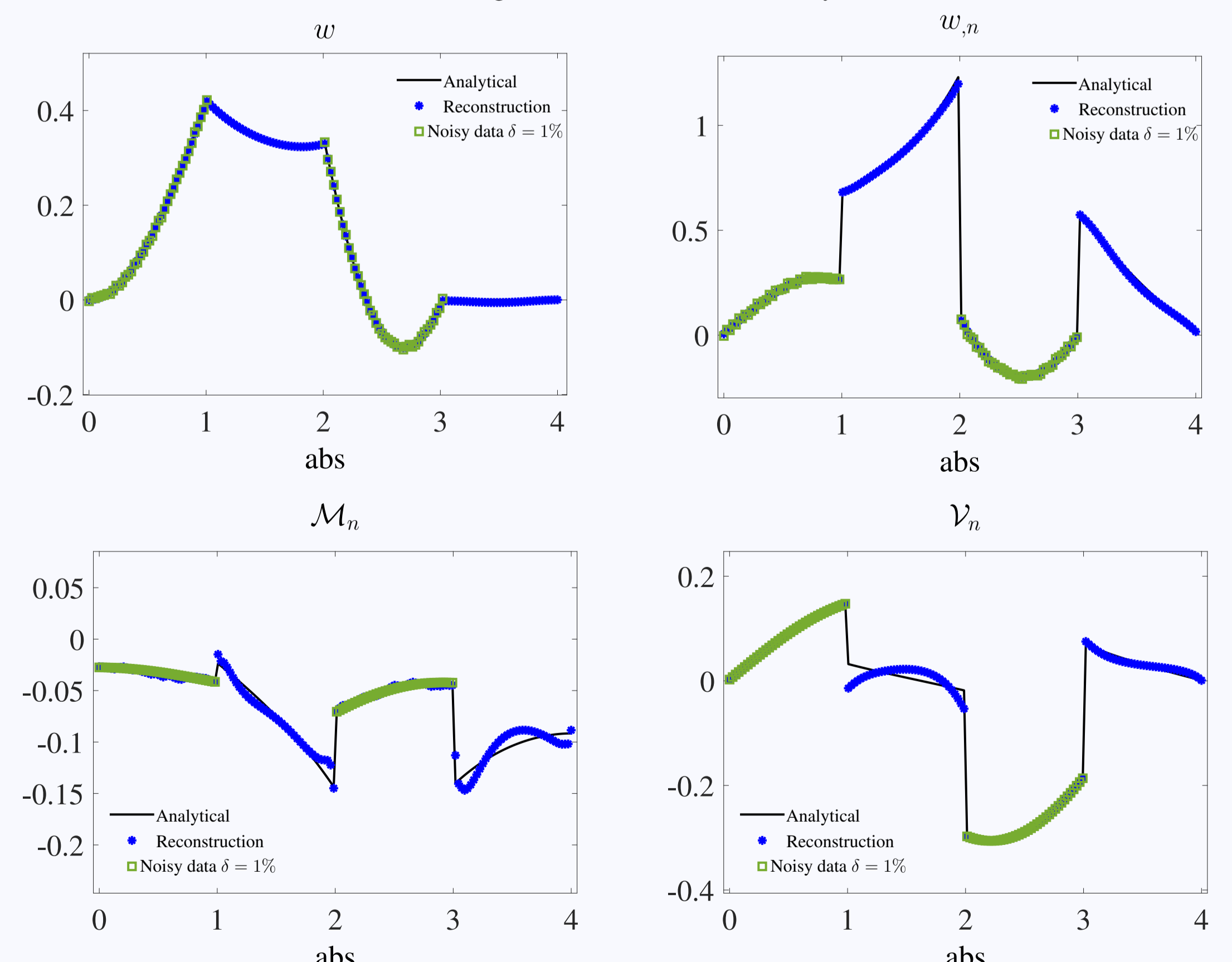


Figure 4: Reconstructions of the boundary conditions on the boundary of a square domain (Fig. 2b) such as data are given on two opposite sides.

## Nomenclature

$w$  vertical displacement of the plate  
 $q$  transversal load  
 $D$  the flexural rigidity of the plate  
 $c$  regularization parameter

$R_i$  concentrated forces at corner nodes  
 $n$  outer normal  
 $H(\Gamma)$  space of the operator solutions  
 $\delta$  level of noise