

# Conic nonholonomic constraints on surfaces and control systems

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- 1 Definitions and Motivations
- 2 Characterisation of conic submanifolds
- 3 Classification of conic submanifolds
- 4 Conclusion and perspectives

- 1 *smooth*: means  $C^\infty$  smooth,
- 2 We consider a *smooth* manifold  $\mathcal{X}$  of dimension 2, since all results are local, we can imagine  $\mathcal{X}$  an open subset of  $\mathbb{R}^2$ , equipped with coordinates  $x = (z, y)$ ,
- 3  $T\mathcal{X}$ : the tangent bundle of  $\mathcal{X}$ , with coordinates  $(x, \dot{x})$ ,

We consider a smooth sub-manifold  $\mathcal{S} \subset T\mathcal{X}$ , locally given by:

$$S(x, \dot{x}) = 0. \quad (\mathcal{S})$$

We assume that  $S$  is smooth and that  $\frac{\partial S}{\partial \dot{x}} \neq 0$ .

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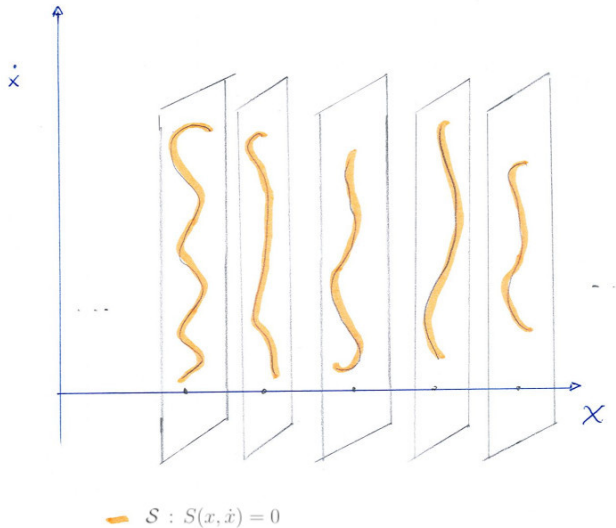
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# What is it?

A submanifold  $\mathcal{S}$  is a nonholonomic constraint.

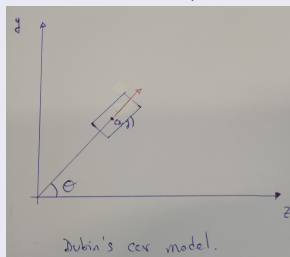


# Motivations - 1

## Dubin's car

Back to [D, 1957]. Consider the simple model of a car,

$$\begin{cases} \dot{z} = \cos(\theta) \\ \dot{y} = \sin(\theta) \\ \dot{\theta} = u \end{cases},$$



where  $(z, y)$  is the centre of mass, and  $\theta$  is the orientation. Then Dubin's car describes an elliptic nonholonomic constraint given by

$$(\dot{z})^2 + (\dot{y})^2 = 1.$$



## Quadratic submanifolds in Physics are common

From [B, 1991]. Consider the *attitude control problem* for a rigid spacecraft governed by gas jets. Let  $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$  be the orientation of the satellite and  $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$  be the angular velocity measured in a specific frame attached to the satellite. The control problem is,

$$\begin{cases} \dot{\theta}_1 = \omega_1 & \dot{\omega}_1 = a_1 \omega_2 \omega_3 \\ \dot{\theta}_2 = \omega_2 & \dot{\omega}_2 = u_2 \\ \dot{\theta}_3 = \omega_3 & \dot{\omega}_3 = u_3 \end{cases}$$

which is the quadratic submanifold (in  $T\mathbb{R}^3$ ) given by  $\dot{\omega}_1 = a_1 \dot{\theta}_2 \dot{\theta}_3$

## From the mathematical point of view-1

Also in [B, 1991], Bonnard started a classification of quadratic control systems (not of submanifolds), he left very interesting questions to answer.

## From the mathematical point of view-2

The following sub-manifold appear in [A-N-N, 2015] as infinitesimal realisation of the simple Lie Algebra  $\mathfrak{so}(r+2, s+2)$  (where  $(r, s)$  is the signature of  $k$ ).

$$\dot{z} = \frac{1}{2} \sum_{ij}^m k_{ij} \dot{y}^i \dot{y}^j.$$

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## Definition (Equivalence of submanifolds)

We say that two sub-manifold  $\mathcal{S}$  and  $\tilde{\mathcal{S}}$ , given by  $\{S(x, \dot{x}) = 0\}$  and  $\{\tilde{S}(\tilde{x}, \dot{\tilde{x}}) = 0\}$ , are (locally)-equivalent if there exists a (local) diffeomorphism  $\tilde{x} = \phi(x)$  and a nonvanishing function  $\delta(x, \dot{x})$  such that

$$\tilde{S}(\phi(x), D\phi(x)\dot{x}) = \delta(x, \dot{x})S(x, \dot{x}).$$

## Conic submanifolds

In the tangent bundle  $T\mathcal{X}$  of a smooth 2D manifold.

- 1 Characterise the submanifolds  $\mathcal{S}$  that are *equivalent* to a conic submanifold  $\mathcal{S}_q$  given by

$$S_q = \dot{x}^T g(x)\dot{x} + 2\omega(x)\dot{x} + h(x) = 0. \quad (\mathcal{S}_q)$$

- 2 Classify normal forms of those submanifolds.

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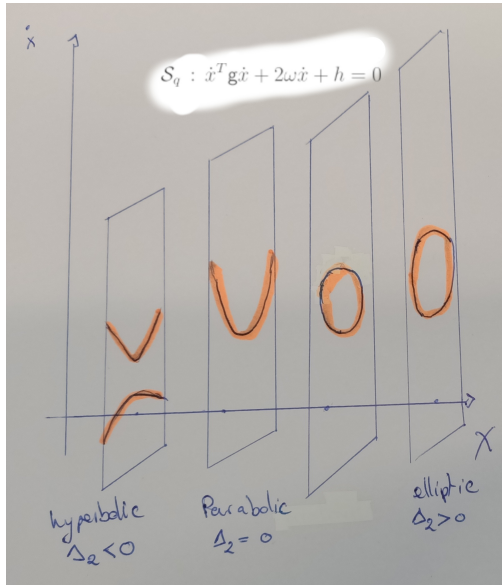
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# Conic submanifolds



We assume that  $S_q$  is non-degenerate, that is  $\Delta_1(x_0) = \det \begin{pmatrix} g(x_0) & \omega(x_0)^T \\ \omega(x_0) & h(x_0) \end{pmatrix} \neq 0$ .

Then conic submanifold can be classified using

$$\Delta_2(x) = \det(g(x)).$$

Using classical results in differential geometry we can easily find a normal form when  $\Delta_2 \neq 0$  and when  $\Delta_2 \equiv 0$ .

# Solving the equivalence problem by feedback equivalence of control-affine systems

The idea: prolong a parametrization of the submanifold  $\mathcal{S} = \{S(x, \dot{x}) = 0\}$  and then see it as a control-affine system.

$$\begin{aligned} \mathcal{S} &\iff \dot{x} = F(x, w), \quad S(x, F(x, w)) = 0 \quad \forall w \in \mathcal{U} \\ &\iff \begin{cases} \dot{x} &= F(x, w) \\ \dot{w} &= u \end{cases}, \end{aligned} \quad (\Sigma_{\mathcal{S}})$$

where  $u \in \mathbb{R}$  is the *control*, and  $\xi = (z, y, w) \in \mathcal{X} \times \mathbb{R}$  is the extended coordinate system.

## Example (Dubin's car)

Starting from the submanifold  $(\dot{z})^2 + (\dot{y})^2 = 1$  we have

$$\begin{cases} \dot{z} &= \cos(w) \\ \dot{y} &= \sin(w) \end{cases} \longrightarrow \begin{cases} \dot{z} &= \cos(w) \\ \dot{y} &= \sin(w) \\ \dot{w} &= u \end{cases}$$

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# Transformations diagram

What is the notion of equivalence for affine systems that make this diagram commute ?

$$\begin{array}{ccc} \mathcal{S} & \xleftrightarrow{(\phi, \delta)} & \mathcal{S}_q \\ \updownarrow & & \updownarrow \\ \Sigma_{\mathcal{S}} & \xleftrightarrow{?} & \Sigma_{\mathcal{S}_q} \end{array}$$

# Feedback equivalence

We consider  $\Sigma^i : \dot{\xi} = f^i(\xi) + g^i(\xi)u^i$  with  $u^i \in \mathbb{R}$ ,  $i = 1, 2$ .

## Definition (Affine Feedback Equivalence)

We say that two control-affine systems  $\Sigma^1$  and  $\Sigma^2$  are feedback equivalent if and only if there exist smooth functions  $\alpha(x)$  and  $\beta(x)$ ,  $\beta(\cdot) \neq 0$ , and a diffeomorphism  $\phi$  of  $\mathcal{X}$  such that:

$$f^2 = \frac{\partial \phi}{\partial x} (f^1 + \alpha g^1),$$
$$g^2 = \frac{\partial \phi}{\partial x} (g^1 \beta).$$

It is like taking the control  $u^1 = \alpha + \beta u^2$ . Geometrically, it is the equivalence of affine distributions  $\mathcal{A}^2 = \phi_* \mathcal{A}^1$  where  $\mathcal{A}^i = f^i + \text{span} \{g^i\}$ .

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# Main result - 1

## Definition (Parametrisation of conic submanifold.)

A control-affine system  $\Sigma$  is *quadratzable* if it is feedback equivalent to

$$\Sigma_q : \begin{cases} \dot{z} &= f^1(z, y, w) \\ \dot{y} &= f^2(z, y, w) \\ \dot{w} &= u \end{cases}, \quad \text{where} \quad \frac{\partial^3 f^i}{\partial w^3} = \tau(z, y) \frac{\partial f^i}{\partial w}$$

and  $\left( \frac{\partial^2 f}{\partial w^2} \wedge \frac{\partial f}{\partial w} \right) (x_0, w_0) \neq 0$ .

## Properties of quadratzable systems

$\Sigma_q$  is a parametrisation of a conic nonholonomic constraint  $\mathcal{S}_q$ , for which we have  $\Delta_2 = -\tau$ . If  $\tau \equiv 0$ , resp.  $\tau < 0$ , resp.  $\tau > 0$ , then we have

$$f_P = A(x)w^2 + B(x)w + C(x), \quad f_E = A(x)\cos(w) + B(x)\sin(w) + C(x), \\ f_H = A(x)\cosh(w) + B(x)\sinh(w) + C(x).$$

# Main result - 1

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## Theorem (Affine feedback quadratization)

Let  $\Sigma$  be an control-affine system on a 3-dimensional smooth manifold with 1 control.  $\Sigma$  is locally around  $\xi_0$  feedback equivalent to  $\Sigma_q$  if, and only if,

- 1  $g \wedge ad_g f \wedge ad_g^2 f(\xi_0) \neq 0$ ,
- 2 The structure functions  $\rho$  and  $\tau$  in the decomposition  $ad_g^3 f = \rho ad_g^2 f + \tau ad_g f \pmod{\text{span}\{g\}}$  satisfy

$$L_g \chi - \frac{2}{3} \rho \chi = 0,$$

where  $\chi = 3L_g \rho - 2\rho^2 - 9\tau$ .

These conditions are checkable by algebraic operations and derivations.

# Idea of the proof

- 1 Check that  $\Sigma_q$  satisfies the conditions (easy),
- 2 Check that the conditions are invariant under feedback transformations,
- 3 Given  $\Sigma$  with  $\rho$  and  $\tau$ , find a feedback  $(\alpha, \beta)$  such that  $\tilde{\rho} \equiv 0$ , then applying a diffeomorphism  $\phi$  satisfying  $\phi_*g = \frac{\partial}{\partial w}$  we obtain  $\Sigma_q$ .

## Corrolary

We can check that  $\text{sgn}(\chi)$  is an invariant of feedback transformations thus, we have the following corollary (under the same assumptions),

- 1  $\Sigma$  is feedback equivalent to  $\Sigma_P$  iff  $\chi \equiv 0$ ,
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# Normal forms of conic systems and submanifolds - 1

If  $g \wedge \text{ad}_g f(\xi_0) \neq 0$ , then a control-affine system is feedback equivalent to,

$$\Sigma_h : \begin{cases} \dot{z} &= h(x, y, w) \\ \dot{y} &= w \\ \dot{w} &= u \end{cases}, \quad f = \begin{pmatrix} h \\ w \\ 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

for which we have  $\rho = \frac{h^{(3)}}{h^{(2)}}$  and  $\tau \equiv 0$ . Then the conditions of the theorem read

$$3\rho'' - 6\rho\rho' + \frac{4}{3}\rho^3 = 0.$$

If  $\Sigma_h$  is feedback equivalent to  $\Sigma_q$ , then locally, we have

$$h(x, w) = 2a \left( \frac{w^2}{(\sqrt{dw^2 + ew + 1} + 1)^2 - dw^2} \right) + bw + c,$$

where  $a, b, c, d, e$  are any smooth function of  $x$ , such that  $a(0) \neq 0$ .

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## Normal forms of conic systems and submanifolds - 2

A normal form of conic submanifolds  $\mathcal{S}_q$  is given by

$$\mathcal{S}_q = (e(\dot{z} - c) - 2a\dot{y})^2 - 4d(\dot{z} - c)^2 - 8a(\dot{z} - c) = 0$$

for which  $\Delta_2 = -16a^2d$  can smoothly go through 0.

(<https://www.geogebra.org/m/qzkmfzsf>)

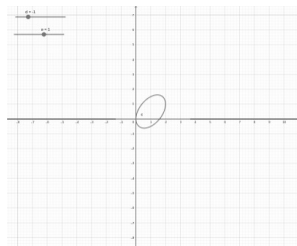


Figure:  $d = -1$

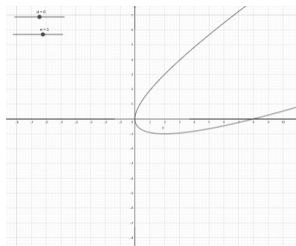


Figure:  $d = 0$

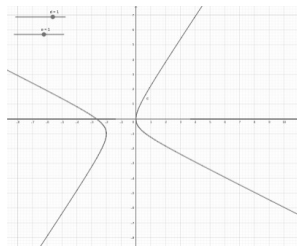


Figure:  $d = 1$

# Classification of conic submanifolds

We will now be working within the following three classes of conic submanifolds:

$$\Xi_P : \dot{x} = A(x)w^2 + B(x)w + C(x),$$

$$\Xi_E : \dot{x} = A(x)\cos(w) + B(x)\sin(w) + C(x),$$

$$\Xi_H : \dot{x} = A(x)\cosh(w) + B(x)\sinh(w) + C(x),$$

seen as nonlinear control systems, satisfying  $A \wedge B \neq 0$ . The group of transformations is a pure feedback  $w = \alpha(x) + \beta(x)\tilde{w}$  and a diffeomorphism  $\tilde{x} = \phi(x)$ . We try to characterise the following normal forms:

$$A = \frac{\partial}{\partial z}, \quad \text{and} \quad B = \frac{\partial}{\partial y}, \quad \text{and, additionally, } C = \text{const.}$$

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# Classification of elliptic/hyperbolic submanifolds - 1

We focus on

$$\Xi_E : \dot{x} = A(x) \cos(w) + B(x) \sin(w) + C(x),$$

$$\Xi_H : \dot{x} = A(x) \cosh(w) + B(x) \sinh(w) + C(x).$$

We associate a (pseudo-)Riemannian metric  $g_{\pm}$  to the pair  $(A, B)$ :

$$g_{\pm}(A, A) = 1, \quad g_{\pm}(B, B) = \pm 1, \quad g_{\pm}(A, B) = 0.$$

## Theorem (Normal form of elliptic/hyperbolic systems)

$\Xi_E$ , resp.  $\Xi_H$ , is feedback equivalent to

$$\Xi'_E : \begin{cases} \dot{z} = \cos(w) + c_0(x) \\ \dot{y} = \sin(w) + c_1(x) \end{cases}, \quad \Xi'_H : \begin{cases} \dot{z} = \cosh(w) + c_0(x) \\ \dot{y} = \sinh(w) + c_1(x) \end{cases},$$

if and only the Gaussian curvature of  $g_{\pm}$  vanishes.

# Classification of elliptic/hyperbolic submanifolds - 2

Theorem (Canonical form of flat elliptic systems with constant  $C$ )

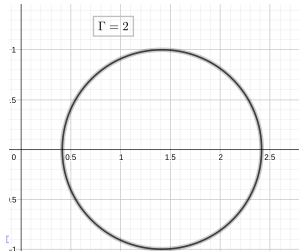
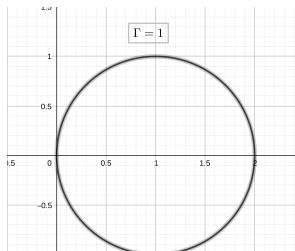
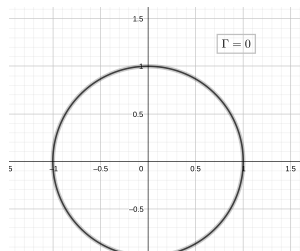
A system  $\Xi'_E$  with  $(c_0, c_1) \in \mathbb{R}^2$  is equivalent to the canonical form

$$\Xi_E^{\Gamma_E} : \begin{cases} \dot{z} &= \cos(w) + \sqrt{\Gamma_E} \\ \dot{y} &= \sin(w) \end{cases}$$

where  $\Gamma_E = (c_0)^2 + (c_1)^2$  is an invariant.

For elliptic submanifolds we have the following family of canonical forms

$$\mathcal{S}_E^{\Gamma_E} = \left\{ (\dot{z} - \sqrt{\Gamma_E})^2 + (\dot{y})^2 = 1 \right\}$$



# Classification of elliptic/hyperbolic submanifolds - 2

Theorem (Canonical form of flat elliptic systems with constant  $C$ )

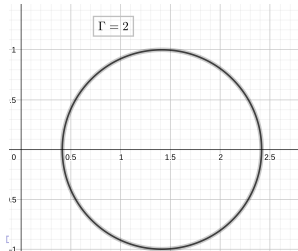
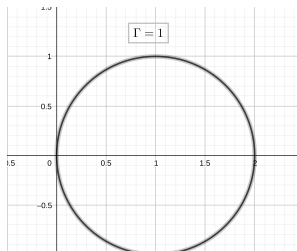
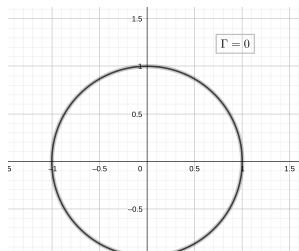
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# Classification of elliptic/hyperbolic submanifolds - 2

## Theorem (Canonical form of flat hyperbolic systems with constant $C$ )

A system  $\Xi'_H$  with  $(c_0, c_1) \in \mathbb{R}^2$  is equivalent to one of the canonical forms

$$\begin{aligned} \Xi_H^{\Gamma_H, \varepsilon} &: \begin{cases} \dot{z} = \cosh(w) + \varepsilon\sqrt{\Gamma_H} \\ \dot{y} = \sinh(w) \end{cases}, \text{ or } \Xi_H^{0,0} : \begin{cases} \dot{z} = \cosh(w) \\ \dot{y} = \sinh(w) \end{cases} \\ \text{or } \Xi_H^{-\Gamma_H} &: \begin{cases} \dot{z} = \cosh(w) \\ \dot{y} = \sinh(w) + \sqrt{-\Gamma_H} \end{cases}, \text{ or } \Xi_H^{0,\varepsilon} : \begin{cases} \dot{z} = \cosh(w) + \varepsilon \\ \dot{y} = \sinh(w) + 1 \end{cases}, \end{aligned}$$

depending on  $\Gamma_H = (c_0)^2 - (c_1)^2$  satisfying  $\Gamma_H > 0$ ,  $\Gamma_H < 0$ , or  $\Gamma_H = 0$ , and where  $\varepsilon = \text{sgn}(c_0) = \pm 1$ . Moreover  $(\Gamma_H, \varepsilon)$  is a complete invariant.

For hyperbolic submanifolds, we have the following families of canonical forms

$$\begin{aligned} \mathcal{S}_H^{\Gamma_H} &= \{(\dot{z} - \sqrt{\Gamma_H})^2 - (\dot{y})^2 = 1\}, & \mathcal{S}_H^{0,0} &= \{(\dot{z})^2 - (\dot{y})^2 = 1\} \\ \mathcal{S}_H^{-\Gamma_H} &= \{(\dot{z})^2 - (\dot{y} - \sqrt{-\Gamma_H})^2 = 1\}, & \mathcal{S}_H^{0,1} &= \{(\dot{z} - 1)^2 - (\dot{y} - 1)^2 = 1\}, \end{aligned}$$

# Classification of elliptic/hyperbolic submanifolds - 2

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# Classification of elliptic/hyperbolic submanifolds - 2

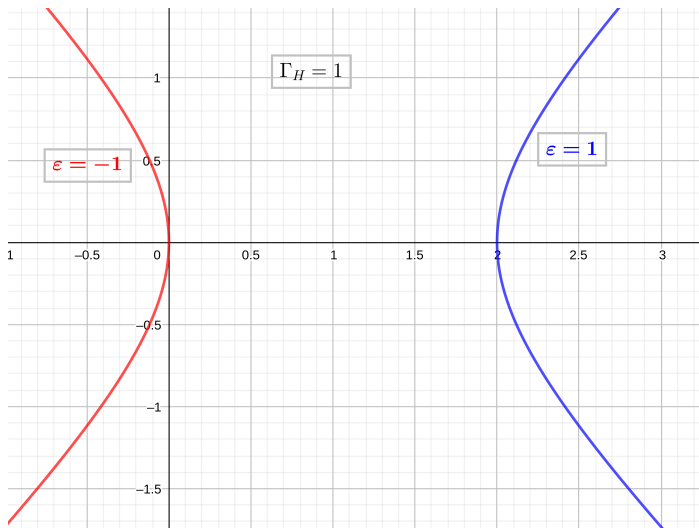


Figure: Canonical forms  $\Xi_H^{\Gamma_H, \epsilon}$

# Classification of parabolic submanifolds - 1

We now focus on parabolic systems

$$\Xi_P : \dot{x} = A(x)w^2 + B(x)w + C(x).$$

## Proposition

A parabolic system always admits the following normal forms

$$\Xi'_P : \begin{cases} \dot{z} = w^2 + c_0(x) \\ \dot{y} = w + c_1(x) \end{cases} \quad \text{or, equivalently,}$$

$$\Xi''_P : \begin{cases} \dot{z} = w^2 + b(x)w + c_0(x) \\ \dot{y} = w \end{cases}$$

That is, for parabolic systems we can always obtain  $A = \frac{\partial}{\partial z}$  and  $B = \frac{\partial}{\partial y}$ .

Moreover if  $\frac{\partial^2 c_1}{\partial z^2} = 0$  we have

$$\Xi'''_P : \begin{cases} \dot{z} = w^2 + c_0(x) \\ \dot{y} = w \end{cases} .$$

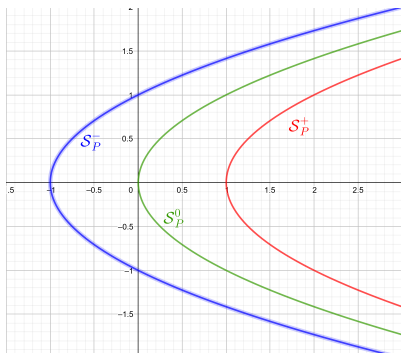
## Classification of parabolic submanifolds - 2

If  $c_0 \in \mathbb{R}$ , we have the following canonical forms

$$\Xi_{\mathcal{P}}^{\pm} : \begin{cases} \dot{z} = w^2 \pm 1 \\ \dot{y} = w \end{cases}, \quad \Xi_{\mathcal{P}}^0 : \begin{cases} \dot{z} = w^2 \\ \dot{y} = w \end{cases}.$$

i.e. we have only 3 parabolic submanifolds with constant coefficients:

$$\mathcal{S}_{\mathcal{P}}^+ = \{ \dot{z} = (\dot{y})^2 + 1 \}, \quad \mathcal{S}_{\mathcal{P}}^- = \{ \dot{z} = (\dot{y})^2 - 1 \}, \quad \mathcal{S}_{\mathcal{P}}^0 = \{ \dot{z} = (\dot{y})^2 \}.$$



# Conclusion and perspectives

We presented an introduction to the equivalence and classification problem of *conic* submanifolds in  $\mathbb{TR}^2$ .

We propose a method that directly characterises the conic form without parameters i.e.

$$\left\{ \begin{array}{l} \dot{z} = \cos(w) \\ \dot{y} = \sin(w) \end{array} \right\}, \quad \left\{ \begin{array}{l} \dot{z} = \cosh(w) \\ \dot{y} = \sinh(w) \end{array} \right\}, \quad \left\{ \begin{array}{l} \dot{z} = w^2 \\ \dot{y} = w \end{array} \right\}.$$

It is done by the study of the Lie algebra of infinitesimal symmetries, and is easily generalisable to higher dimensions (however when  $m > 2$ , checkability of the conditions is hard).

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# Conclusion and perspectives

We have results for the equivalence and classification of quadric submanifolds in  $T\mathbb{R}^3$ , especially,

$$\dot{z} = a(x)(\dot{y}_1^2 + \epsilon \dot{y}_2^2) + b_1(x)\dot{y}_1 + b_2(x)\dot{y}_2 + c(x).$$

The case when  $\epsilon = -1$  is called p-hyperbolic and is *easier* to solve (the geometry is nice). The case when  $\epsilon = 1$ , called p-elliptic, is a bit more trickier to deal with.



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