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Conic Nonholonomic Constraints on Surfaces and Control Systems

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Abstract

This paper addresses the equivalence problem of conic submanifolds in the tangent bundle of a smooth 2-dimensional manifold. Those are given by a quadratic relation between the velocities and are treated as nonholonomic constraints whose admissible curves are trajectories of the corresponding control systems, called quadratic systems. We deal with the problem of characterising and classifying conic submanifolds under the prism of feedback equivalence of control systems, both control-affine and fully nonlinear. The first main result of this work is a complete description of non-degenerate conic submanifolds via a characterisation under feedback transformations of the novel class of quadratic control-affine systems. This characterisation can explicitly be tested on structure functions defined for any control-affine system and gives a normal form of quadratisable systems and of conic submanifolds. Then, we consider the classification problem of regular conic submanifolds (ellipses, hyperbolas, and parabolas), which is treated via feedback classification of quadratic control-nonlinear systems. Our classification includes several normal forms of quadratic systems (in particular, normal forms not containing functional parameters as well as those containing neither functional nor real parameters) and, as a consequence, gives a classification of regular conic submanifolds.

Keywords nonlinear control system · feedback equivalence · conic submanifolds · nonholonomic constraint · normal forms · pseudo-Riemannian geometry

Mathematics Subject Classification (2020) 93B27 · 37N35 · 93A10 · 93B52 · 53B20 · 53B30

1	Introduction	2
2	Preliminaries	5
3	Quadratisable control-affine systems	12
3.1	Characterisation of quadratisable control-affine systems	15
3.2	Normal form of quadratisable control-affine systems	18
4	Classification of quadratic systems	21
4.1	Classification of elliptic and hyperbolic systems	24
4.2	Classification of parabolic systems	28
5	Conclusions	33
A	Resolution of equation (3)	35
B	Detailed computation of the proof of Theorem 3.5	36
B.1	Resolution of equation (12)	36
B.2	Smooth form of h'	36
C	Details of the computations of Lemma 4.3	36
D	Gaussian curvature for metric given in terms of vector fields	37

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1 Introduction

Let \mathcal{X} be a smooth connected manifold of dimension $n = 2$ (a surface), equipped with local coordinates x . In the tangent bundle $T\mathcal{X}$ of \mathcal{X} , we consider a smooth 3-dimensional submanifold \mathcal{S} , a hypersurface, given by

$$\mathcal{S} = \{(x, \dot{x}) \in T\mathcal{X}, S(x, \dot{x}) = 0\},$$

where $S : T\mathcal{X} \rightarrow \mathbb{R}$ is a smooth scalar function satisfying $\text{rk} \frac{\partial S}{\partial \dot{x}}(x, \dot{x}) = 1$ for all $(x, \dot{x}) \in \mathcal{S}$. Two submanifolds $\mathcal{S} \subset T\mathcal{X}$ and $\tilde{\mathcal{S}} \subset T\tilde{\mathcal{X}}$ are said to be equivalent if there exists a diffeomorphism $\phi : \mathcal{X} \rightarrow \tilde{\mathcal{X}}$ such that $\phi_*(\mathcal{S}) = \tilde{\mathcal{S}}$, where the tangent map $\phi_* : T\mathcal{X} \rightarrow T\tilde{\mathcal{X}}$ is given by $\phi_*(x, \dot{x}) = (\phi(x), D\phi(x)\dot{x})$ and $D\phi$ is the derivative of ϕ . If $\mathcal{S} = \{S(x, \dot{x}) = 0\}$ and $\tilde{\mathcal{S}} = \{\tilde{S}(\tilde{x}, \tilde{\dot{x}}) = 0\}$, then the above definition amounts to saying that there exists a smooth nonvanishing scalar function $\delta : T\mathcal{X} \rightarrow \mathbb{R}$ such that

$$\tilde{S}(\phi(x), D\phi(x)\dot{x}) = \delta(x, \dot{x}) S(x, \dot{x}).$$

The last equality implies that for all $(x, \dot{x}) \in \mathcal{S}$ we have $(\tilde{x}, \tilde{\dot{x}}) = (\phi(x), D\phi(x)\dot{x}) \in \tilde{\mathcal{S}}$, hence the map ϕ_* sends the graph of $S^{-1}(0)$ into that of $\tilde{S}^{-1}(0)$. Equivalence of submanifolds $\mathcal{S} = \{S(x, \dot{x}) = 0\}$ and $\tilde{\mathcal{S}} = \{\tilde{S}(\tilde{x}, \tilde{\dot{x}}) = 0\}$ means simply that the implicit underdetermined ordinary differential equations $S(x, \dot{x}) = 0$ and $\tilde{S}(\tilde{x}, \tilde{\dot{x}}) = 0$ are equivalent; see e.g. [4, Definition 2].

It is natural to ask how to characterise and classify submanifolds $\mathcal{S} \subset T\mathcal{X}$ (with \mathcal{X} and \mathcal{S} of arbitrary dimensions) of certain particular classes, for instance the class of linear submanifolds given by $S_{lin}(x, \dot{x}) = \omega(x)\dot{x} = 0$ or the class of affine submanifolds given by $S_{aff}(x, \dot{x}) = \omega(x)\dot{x} + h(x) = 0$, where ω is a smooth vector-valued differential 1-form on \mathcal{X} and h is a smooth vector-valued function on \mathcal{X} . Those questions have been widely studied under the prism of Pfaffian equations (linear and affine) and go back to Pfaff, Darboux, Cartan [20, 8, 9]. Although the problem of classification of Pfaffian equations is still open in its full generality, many important results have been obtained for various classes of linear Pfaffian equations (contact and quasi-contact case, Martinet case, singularities, see [32, 31, 18, 14, 11]) and of affine Pfaffian equations (dimension two [13], three [22, 21], and arbitrary [33, 11]).

We will call \mathcal{S}_q a quadratic, or a conic, submanifold of $T\mathcal{X}$ if it is given by $\mathcal{S}_q = \{S_q(x, \dot{x}) = 0\}$, where $S_q : T\mathcal{X} \rightarrow \mathbb{R}$ is a quadratic map of the form

$$S_q(x, \dot{x}) = \dot{x}^t \mathbf{g}(x) \dot{x} + 2\omega(x)\dot{x} + h(x),$$

with all involved objects being smooth, i.e. for each point $x \in \mathcal{X}$, the set $\mathcal{S}_q(x)$ forms a conic curve in $T_x\mathcal{X}$ (recall that we consider the case of \mathcal{X} being a surface). If \mathcal{S}_q is a conic submanifold, then so is any submanifold equivalent to it. Indeed, if $\mathcal{S}_q = \{S(x, \dot{x}) = 0\}$, then $\tilde{\mathcal{S}} = \{\tilde{S}(\tilde{x}, \tilde{\dot{x}}) = 0\}$, where $\tilde{S}(\tilde{x}, \tilde{\dot{x}}) = S_q(\phi^{-1}(\tilde{x}), D\phi^{-1}(\tilde{x})\tilde{\dot{x}})$ is quadratic because $D\phi$ acts linearly on fibres. Moreover, we actually have $\tilde{S}(\tilde{x}, \tilde{\dot{x}}) = \tilde{x}^t \tilde{\mathbf{g}} \tilde{\dot{x}} + 2\tilde{\omega}\tilde{\dot{x}} + \tilde{h}$ with $\mathbf{g} = \phi^* \tilde{\mathbf{g}}$, $\omega = \phi^* \tilde{\omega}$, and $h = \phi^* \tilde{h}$. On the other hand, a map S defining a quadratic submanifold $\mathcal{S}_q = \{S(x, \dot{x}) = 0\}$ need not be quadratic because $S(x, \dot{x})$ is of the form $\delta(x, \dot{x})S_q(x, \dot{x})$ and, in general, the latter will not be quadratic (unless δ depends on x only); see [Example 2.1](#) below. The first purpose of this work is to provide a local characterisation of submanifolds \mathcal{S} that are equivalent to quadratic submanifolds \mathcal{S}_q . In particular, we will identify submanifolds that define, in each $T_x\mathcal{X}$, elliptic, hyperbolic, and parabolic conics, given respectively by

$$\begin{aligned} \mathcal{S}_E &= \{a^2(\dot{z} - c_0)^2 + b^2(\dot{y} - c_1)^2 = 1\}, & \mathcal{S}_H &= \{a^2(\dot{z} - c_0)^2 - b^2(\dot{y} - c_1)^2 = 1\}, \\ \text{and } \mathcal{S}_P &= \{a\dot{y}^2 - \dot{z} + b\dot{y} + c = 0\}, \end{aligned}$$

where a, b, c_0, c_1 , and c are smooth functions of $x = (z, y)$, satisfying $a(\cdot) \neq 0$ and $b(\cdot) \neq 0$ in the elliptic and hyperbolic cases and $a(\cdot) \neq 0$ in the parabolic case; we choose the order (z, y) to be consistent with some normal forms existing in the literature (e.g. see [1], where quadratic submanifolds are used as canonical models which admit a parabolic Lie group of symmetries). We will also discuss the case of passing smoothly from one type to another. We call them elliptic, hyperbolic, and parabolic submanifolds and, to state and discuss general facts about them, we set $\mathcal{S}_Q = \{\mathcal{S}_E, \mathcal{S}_H, \mathcal{S}_P\}$.

The second goal of this work is to provide a classification of elliptic \mathcal{S}_E , hyperbolic \mathcal{S}_H , and parabolic \mathcal{S}_P submanifolds. In the elliptic and hyperbolic cases, we will first describe the submanifolds with the functions $a(x) = b(x)$ and call them *conformally-flat*; second, if, additionally, $a = b = 1$, then we call them *flat* elliptic or hyperbolic submanifolds. Finally, we will describe the forms for which we also have $c_0, c_1 \in \mathbb{R}$ (in the same coordinate system as the one for which we have $a = b = 1$), called a *constant-form* elliptic and hyperbolic submanifolds; the particular case of $c_0 = c_1 = 0$ is called a *null-form* to emphasis the absence of any (functional, continuous, or discrete) parameter. In the parabolic case, we will describe the submanifolds with $a = 1$, called *weakly-flat*, and characterise those with, additionally, $b = 0$ (called *strongly-flat*) and, moreover, with $c \in \mathbb{R}$ (called *constant-form*); in the particular case where $c = 0$ we call the parabolic submanifold a *null-form*. Our classification is summarised by [Table 1](#) below.

Our analysis of the equivalence problem of submanifolds will be based on attaching to a submanifold $\mathcal{S} = \{S(x, \dot{x}) = 0\} \subset T\mathcal{X}$ two control systems. First,

$$\Xi_{\mathcal{S}} : \dot{x} = F(x, w), \quad x \in \mathcal{X}, \quad w \in \mathcal{W} \subset \mathbb{R},$$

where $\dot{x} - F(x, w) = 0$ is a regular parametric representation of the submanifold \mathcal{S} , that is, for all values of the parameter $w \in \mathcal{W}$ (interpreted as a scalar control) we have $S(x, F(x, w)) = 0$ and $\text{rk} \frac{\partial F}{\partial w}(x, w) = 1$, and second,

$$\Sigma_{\mathcal{S}} : \begin{cases} \dot{x} &= F(x, w) \\ \dot{w} &= u \end{cases}, \quad (x, w) \in \mathcal{X} \times \mathcal{W}, \quad u \in \mathbb{R},$$

called, respectively, a first and second extension of \mathcal{S} . Notice that for $\Xi_{\mathcal{S}}$ the control w enters in a nonlinear way, whereas for $\Sigma_{\mathcal{S}}$ the control u enters in an affine way, but for the price of augmenting the dimension of the state space. Observe that, since $S(x, \dot{x}) = 0$, defining \mathcal{S} , relates the positions x with the velocities \dot{x} , it describes a nonholonomic constraint. We say that a smooth curve $x(t) \in \mathcal{X}$ satisfies the nonholonomic constraint given by \mathcal{S} if we have $(x(t), \dot{x}(t)) \in \mathcal{S}$. Clearly, $x(t)$ satisfies the nonholonomic constraint described by \mathcal{S} (equivalently, satisfies the implicit differential equation $S(x(t), \dot{x}(t)) = 0$) if and only if $x(t)$ is a trajectory of $\Xi_{\mathcal{S}}$ for a certain smooth control $w(t)$ or, equivalently, $(x(t), w(t))$ is a trajectory of $\Sigma_{\mathcal{S}}$ for a smooth control $u(t)$. A crucial observation that links studying submanifolds $\mathcal{S} \subset T\mathcal{X}$ and their extensions $\Xi_{\mathcal{S}}$ and $\Sigma_{\mathcal{S}}$ is that the equivalence of submanifolds corresponds to the equivalence of control systems $\Xi_{\mathcal{S}}$ and $\Sigma_{\mathcal{S}}$ via feedback transformations, general for $\Xi_{\mathcal{S}}$ and control-affine for $\Sigma_{\mathcal{S}}$, as ensured by [Proposition 2.4](#).

Organisation of the paper. In the next section, we will recall some definitions of control theory and we will show that the problem of characterising and classifying submanifolds of $T\mathcal{X}$ can be replaced by that of characterising and classifying their first and second extensions $\Xi_{\mathcal{S}}$ and $\Sigma_{\mathcal{S}}$ under feedback transformations (see [Proposition 2.4](#)). Moreover, we will give a first rough classification of non-degenerate conic submanifolds, introducing elliptic \mathcal{S}_E , hyperbolic \mathcal{S}_H , and parabolic \mathcal{S}_P subclasses (see [Lemma 2.5](#)). In [Section 3](#), we will define a general second extension of a conic submanifold \mathcal{S}_q , called a quadratic system Σ_q . Just as we do for conic submanifolds, we will identify elliptic Σ_E , hyperbolic Σ_H , and parabolic Σ_P systems as particular cases of Σ_q , see [Definition 3.1](#) and [Proposition 3.2](#). In [Theorem 3.3](#), we will fully characterise the class of quadratic systems by means of a checkable relation between well-defined structure functions attached to any control-affine system. The conditions obtained in that theorem allow to give a normal form for all quadratisable control-affine

Elliptic and hyperbolic submanifolds classification		Parabolic submanifolds classification	
$a = b$	conformally-flat	$a = 1$	weakly-flat
$a = b = 1$	flat	$a = 1, b = 0$	strongly-flat
$a = b = 1, (c_0, c_1) \in \mathbb{R}^2$	constant-form	$a = 1, b = 0, c \in \mathbb{R}$	constant-form
$a = b = 1, c_0 = c_1 = 0$	null-form	$a = 1, b = 0, \text{ and } c = 0$	null-form

Table 1: Classification of elliptic, hyperbolic, and parabolic submanifolds.

systems (i.e. systems Σ that are feedback equivalent to Σ_q), see [Theorem 3.5](#), which in turn leads to a normal form for all non-degenerate conic submanifolds \mathcal{S}_q . We will also show how our characterisation and the normal form apply to the classes of elliptic, hyperbolic, and parabolic control-affine systems, which gives us a deeper insight into our conditions and in our normal form (see [Corollary 3.4](#) and [Corollary 3.6](#)). Finally, in [Section 4](#), we will be interested in the classification of elliptic, hyperbolic, and parabolic submanifolds as presented in [Table 1](#). This problem is dealt with using the classification of their first extensions (treated as control-nonlinear systems) under feedback transformations. We first show that the classification of elliptic, hyperbolic, and parabolic submanifolds presented in [Table 1](#) is reflected by properties of a triple of vector fields attached to their parametrisations; see [Lemma 4.1](#) statements (i) to (iv) for the elliptic and hyperbolic cases and statements (v) to (viii) for the parabolic case. To every quadratic control-nonlinear system Ξ_E , Ξ_H , or Ξ_P (first extensions of elliptic, hyperbolic, and parabolic submanifolds, respectively), we will attach a frame of the tangent bundle (see the paragraph before [Proposition 4.2](#)) and we give conditions for that frame to be commutative: it turns out that in the elliptic and hyperbolic cases this requires that a certain pseudo-Riemannian metric is flat (see [Proposition 4.5](#)), whereas in the parabolic case this problem can be solved without any extra assumptions (see [Proposition 4.7](#)). Then we show how we can additionally normalize the systems while preserving the commutativity of that frame. Our classifications include several normal and canonical forms, given by [Proposition 4.6](#) for elliptic and hyperbolic systems and by [Theorem 4.9](#) for parabolic systems. We summarise the structure of the paper in [Figure 1](#).

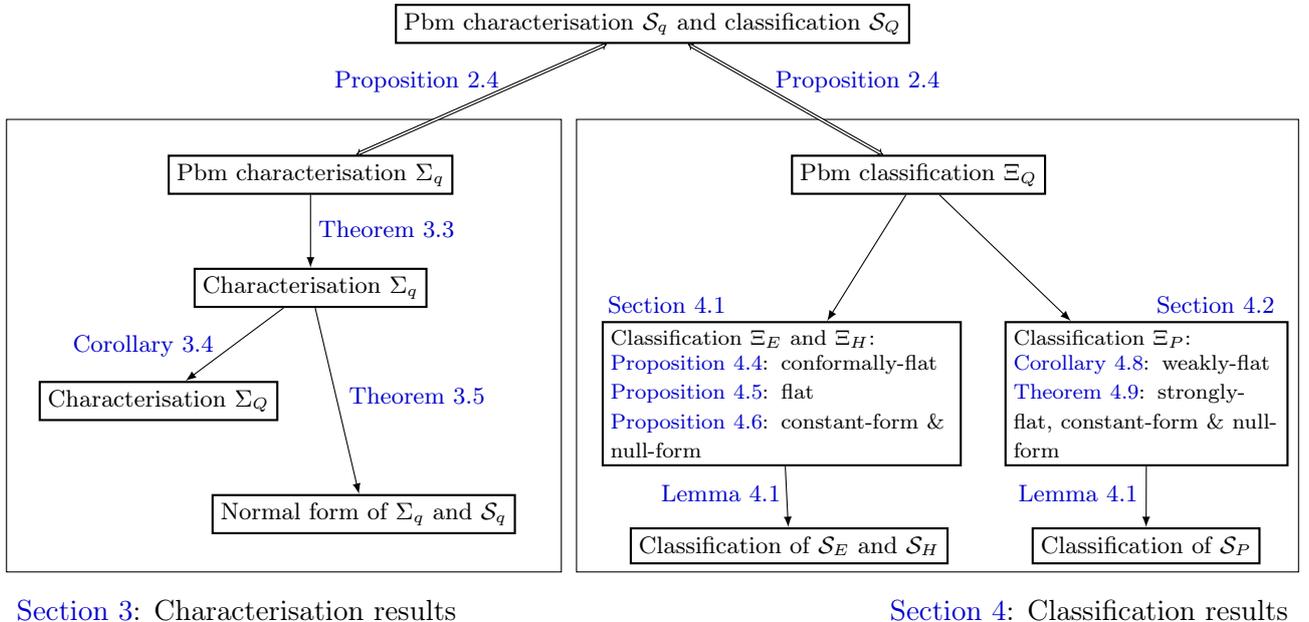


Figure 1: Walk-through the paper with a separation between characterisation and classification results

Related works. A classification of quadratic control systems was initiated by Bonnard in [5]. His work differs from our as he considered homogeneous systems of degree 2 with respect to all state variables. Hence, his class of quadratic control systems is a subclass of our parabolic systems (where we require that only one variable enters quadratically) but he considers the general dimension n while our results concern 3-dimensional systems only. In [16], Krener and Kang studied the problem of equivalence, via feedback, to polynomial systems of degree 2 modulo higher order terms. This work was continued in [15] and [28] for any degree but all those results are given for formal classification only. In [12], particularly in Example 2.2, Jakubczyk deals with a general elliptic system and studies its microlocal equivalence via symbols of its critical Hamiltonian. Examples of control systems subject to conic nonholonomic constraints appear in various domains of physics and engineering applications. In the next section, we will discuss Dubins' car [10] which is a simple model of a vehicle, as well as its hyperbolic counterpart [19]. In [23], the same elliptic model, as that of Dubins' car, is studied to minimize the energy of its trajectories, which is the famous Euler's elastica problem. We also mention

[34], where the planar tilting manoeuvre problem is considered under small angle assumption, the studied control system is elliptic with respect to the states.

2 Preliminaries

Main notations.

$\mathcal{X}, T\mathcal{X}, x = (z, y)$	A smooth 2d manifold, its tangent bundle, and its local coordinates.
$\mathcal{M}, \xi = (x, w) = (z, y, w)$	A smooth 3d manifold and its local coordinates.
ϕ, ϕ_*	A diffeomorphism and its tangent map.
\mathcal{S}	Smooth submanifold of $T\mathcal{X}$ given by an equation of the form $S(x, \dot{x}) = 0$, with $\text{rk} \frac{\partial S}{\partial \dot{x}}(x, \dot{x}) = 1$.
\mathcal{S}_q	Quadratic (conic) submanifold described by $S_q(x, \dot{x}) = \dot{x}^t g(x) \dot{x} + 2\omega(x) \dot{x} + h(x) = 0$.
$\mathcal{S}_Q = \{\mathcal{S}_E, \mathcal{S}_H, \mathcal{S}_P\}$	Set of elliptic, hyperbolic, and parabolic submanifolds.
$\Xi_{\mathcal{S}}$	Regular parametrisation of a submanifold \mathcal{S} , called a first extension, seen as a control-nonlinear system.
$\Sigma_{\mathcal{S}}$	Extension of $\Xi_{\mathcal{S}}$, called a second extension of a submanifold \mathcal{S} , seen as a control-affine system on a three dimensional manifold \mathcal{M} .

In this section, we introduce all tools and concepts that we will need in this paper. First, we define in full generality the notions of submanifolds of the tangent bundle and of control-nonlinear systems. Second, we recall some notions on the equivalence of submanifolds (of the tangent bundle) and of control systems. Third, most importantly, we show that the problem of equivalence of submanifolds can be replaced by that of equivalence of their first and second extensions, see [Proposition 2.4](#). Finally, we present quadratic submanifolds and we discuss the subclasses of elliptic, hyperbolic, and parabolic submanifolds. In the paper, the word *smooth* will always mean C^∞ -smooth and throughout all systems, functions, manifolds and submanifolds are assumed to be smooth.

Basic notions. We introduce basic notions following the approach proposed in [27], see also [7]. Let \mathcal{X} be a smooth n -dimensional manifold. A *field of admissible velocities* or a *velocities constraint* on \mathcal{X} is a fibred manifold $\pi_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{X}$, where \mathcal{S} is an $(m+n)$ -dimensional submanifold of $T\mathcal{X}$, and thus the following diagram is commutative

$$\begin{array}{ccc}
 \mathcal{S} & \xrightarrow{\iota} & T\mathcal{X} \\
 \pi_{\mathcal{S}} \downarrow & & \swarrow \pi_{T\mathcal{X}} \\
 \mathcal{X} & &
 \end{array}$$

In the diagram, the map $\pi_{\mathcal{S}}$ is a surjective submersion, $\pi_{T\mathcal{X}}$ is the canonical projection, and $\iota : \mathcal{S} \hookrightarrow T\mathcal{X}$ is the inclusion map attaching to any $s \in \mathcal{S}$ the same point s considered as a point of $T\mathcal{X}$. Denote by $\mathcal{S}_x = \pi_{\mathcal{S}}^{-1}(x) \subset T_x\mathcal{X}$ the fibers of \mathcal{S} . Any \mathcal{S}_x is an m -dimensional submanifold of $T_x\mathcal{X}$ that consists of all velocities admissible at $x \in \mathcal{X}$. If a field of admissible velocities $\pi_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{X}$ is a fiber bundle, with a typical fiber being a manifold \mathcal{W} of dimension m (in particular, if $\mathcal{S} = \mathcal{X} \times \mathcal{W}$), then the sets \mathcal{S}_x of admissible velocities at any $x \in \mathcal{X}$ are diffeomorphic to each other and they all are diffeomorphic to \mathcal{W} . Notice, however, that we do not assume \mathcal{S} to be a fiber bundle (just a fibred manifold only). In this case, the fibers \mathcal{S}_x need not be diffeomorphic meaning that the sets of admissible velocities \mathcal{S}_x may change completely when passing from one $x \in \mathcal{X}$ to another. We will see this phenomenon to appear for quadratic constraints.

A *control system* $\Xi = (F, \pi_{\mathcal{V}})$ on \mathcal{X} consists of a fibred manifold $\pi_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{X}$, called the control bundle, where $\pi_{\mathcal{V}}$ is a surjective submersion, and a map $F : \mathcal{V} \rightarrow T\mathcal{X}$ such that the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{V} & \xrightarrow{F} & T\mathcal{X} \\
\pi_{\mathcal{V}} \downarrow & & \swarrow \pi_{T\mathcal{X}} \\
\mathcal{X} & &
\end{array}$$

where $\pi_{T\mathcal{X}}$ denotes the canonical projection. If the control bundle $\pi_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{X}$ is the Cartesian product $\mathcal{V} = \mathcal{X} \times \mathcal{W}$ of the state space manifold \mathcal{X} and the control space manifold \mathcal{W} of dimension m , then the control system Ξ takes the usual form

$$\Xi : \dot{x} = F(x, w), \quad x \in \mathcal{X}, \quad w \in \mathcal{W}.$$

If \mathcal{V} is a nontrivial fibred manifold, then Ξ takes the form $\dot{x} = F(x, w)$ in local coordinates (x, w) adapted to the fibred manifold structure of \mathcal{V} .

There is a perfect correspondence between fields of admissible velocities and regular control systems. First, to any field of admissible velocities $\pi_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{X}$ we can associate a control system. Namely, we define the control bundle $\pi_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{X}$ by choosing $\mathcal{V} = \mathcal{S}$, $\pi_{\mathcal{V}} = \pi_{\mathcal{S}}$ and $F = \iota$. If the inclusion ι is an injective immersion (resp. an embedding), then $F = \iota$ is, obviously, an injective immersion (resp. embedding). Conversely, to any control system $(F, \pi_{\mathcal{V}})$ we can define point-wise the set of admissible velocities at $x \in \mathcal{X}$ by $\mathcal{S}_x = F(\pi_{\mathcal{V}}^{-1}(x))$. If $\pi_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{X}$ is a fiber bundle, with a typical fiber being a manifold \mathcal{W} (in particular, if $\mathcal{V} = \mathcal{X} \times \mathcal{W}$), then $\mathcal{S}_x = F(x, \mathcal{W})$. If for the control system Ξ the map F is an injective immersion (resp. an embedding), then \mathcal{S} , defined point-wise by \mathcal{S}_x , is an immersed (resp. embedded) submanifold of $T\mathcal{X}$ and \mathcal{S}_x are immersed (resp. embedded) submanifolds of the corresponding $T_x\mathcal{X}$. The reason is that the fibres $\pi_{\mathcal{V}}^{-1}(x)$ are embedded submanifolds because $\pi_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{X}$ is a fibred manifold.

Equivalence notions. From now on we will assume that \mathcal{X} is a surface, that is $n = 2$, and that the fibred submanifold \mathcal{S} , defining a velocities constraint, is a hypersurface in $T\mathcal{X}$. Observe, however, that the notions introduced below, and in particular [Proposition 2.4](#), are also valid in the general dimension setting. In this paper, we will be mostly working locally and, in all local statements, we will add the adjective "local" instead of using the notion of germs (which would be more proper but less frequently used in control theory). In local coordinates (x, \dot{x}) , a submanifold \mathcal{S} of $T\mathcal{X}$ can be expressed as the zero level set $\mathcal{S} = \{S(x, \dot{x}) = 0\}$ of a local map $S : T\mathcal{X} \rightarrow \mathbb{R}$ defined in a neighbourhood of (x_0, \dot{x}_0) . Since \mathcal{S} is a fibred manifold, it follows that $\text{rk} \frac{\partial S}{\partial \dot{x}}(x, \dot{x}) = 1$, which we thus assume throughout.

We say that two velocities constraints $\pi_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{X}$ and $\pi_{\tilde{\mathcal{S}}} : \tilde{\mathcal{S}} \rightarrow \tilde{\mathcal{X}}$ are (locally) equivalent if there exists a (local) diffeomorphism $\phi : \mathcal{X} \rightarrow \tilde{\mathcal{X}}$ whose tangent map $\phi_* : T\mathcal{X} \rightarrow T\tilde{\mathcal{X}}$ sends \mathcal{S} onto $\tilde{\mathcal{S}}$, that is $\phi_*(\mathcal{S}) = \tilde{\mathcal{S}}$. We say that two maps $S : T\mathcal{X} \rightarrow \mathbb{R}$ and $\tilde{S} : T\tilde{\mathcal{X}} \rightarrow \mathbb{R}$ are locally V -equivalent at (x_0, \dot{x}_0) and $(\tilde{x}_0, \tilde{\dot{x}}_0)$, respectively, if there exists a local diffeomorphism $\phi : \mathcal{X} \rightarrow \tilde{\mathcal{X}}$, satisfying $\phi_*(x_0, \dot{x}_0) = (\tilde{x}_0, \tilde{\dot{x}}_0)$, and a non-vanishing function $\delta : T\mathcal{X} \rightarrow \mathbb{R}^*$ such that

$$\tilde{S}(\phi(x), D\phi(x)\dot{x}) = \delta(x, \dot{x})S(x, \dot{x})$$

holds locally around (x_0, \dot{x}_0) . This notion is a natural adaptation (to the case of maps defined on $T\mathcal{X}$) of the notion of V -equivalence (V referring to a variety) of germs of maps $(\mathbb{R}^q, 0) \rightarrow (\mathbb{R}^p, 0)$ as defined in [2]. The following three examples illustrate the notion of V -equivalence of maps and its correspondence with the equivalence of submanifolds (that correspondence will be proven with full generality in [Proposition 2.4](#) below).

Example 2.1. The submanifolds \mathcal{S} and $\tilde{\mathcal{S}}$ given by $S(x, \dot{x}) = \dot{z} - (-1 + \sqrt{1 + \dot{y}})^2 = 0$, around $(x_0, \dot{x}_0) = (0, 0)$, and $\tilde{S}(\tilde{x}, \tilde{\dot{x}}) = \tilde{\dot{z}} - \left(\frac{\tilde{\dot{y}}}{2}\right)^2 = 0$, around $(\tilde{x}_0, \tilde{\dot{x}}_0) = (0, 0)$, respectively, are locally equivalent by the diffeomorphism $\phi(x) = (z, y - z)$ because the maps S and \tilde{S} are locally V -equivalent via

$$(\tilde{z}, \tilde{y}) = \phi(x) = (z, y - z) \quad \text{and} \quad \delta(x, \dot{x}) = -\frac{1}{4} \left(\dot{z} - \dot{y} - 2 - 2\sqrt{1 + \dot{y}} \right).$$

Example 2.2. Since on a 2-dimensional manifold all metrics are locally conformally flat (see [26, Addendum 1 of Chapter 9] for the Riemannian case and [25, Theorem 7.2] for the Lorentzian one), it follows that the submanifold given by $S(x, \dot{x}) = a(x)\dot{z}^2 + 2b(x)\dot{z}\dot{y} + c(x)\dot{y}^2 - 1 = 0$ (where $ac - b^2 \neq 0$) is locally equivalent to that given by $\tilde{S}(\tilde{x}, \dot{\tilde{x}}) = (\dot{\tilde{z}}^2 \pm \dot{\tilde{y}}^2) - r(\tilde{x})^2 = 0$.

Example 2.3. Since $\text{rk} \frac{\partial S}{\partial \dot{x}}(x, \dot{x}) = 1$, assume that $\frac{\partial S}{\partial \dot{z}}(x_0, \dot{x}_0) \neq 0$; recall that $x = (z, y)$. By the implicit function theorem, we can locally write $S(x, \dot{x}) = \delta(x, \dot{x})(\dot{z} - s(x, \dot{y}))$, where $\delta(x_0, \dot{x}_0) \neq 0$. So, a submanifold \mathcal{S} is always locally equivalent to the one given by $\dot{z} - s(x, \dot{y}) = 0$.

Two control systems $\Xi = (F, \pi_{\mathcal{V}})$ and $\tilde{\Xi} = (\tilde{F}, \pi_{\tilde{\mathcal{V}}})$ are *feedback equivalent* if there exists a diffeomorphism $\phi : \mathcal{X} \rightarrow \tilde{\mathcal{X}}$ and a fiber preserving lift $\Phi : \mathcal{V} \rightarrow \tilde{\mathcal{V}}$ of ϕ , i.e. $\phi(\pi_{\mathcal{V}}) = \pi_{\tilde{\mathcal{V}}}(\Phi)$, such that

$$D\phi(x)F(v) = \tilde{F}(\Phi(v)), \quad (1)$$

for any $v \in \mathcal{V}$. Namely, the following diagram is commutative

$$\begin{array}{ccccc} \mathcal{X} & \xleftarrow{\pi_{\mathcal{V}}} & \mathcal{V} & \xrightarrow{F} & T\mathcal{X} \\ \downarrow \phi & & \downarrow \Phi & & \downarrow \phi_* \\ \tilde{\mathcal{X}} & \xleftarrow{\pi_{\tilde{\mathcal{V}}}} & \tilde{\mathcal{V}} & \xrightarrow{\tilde{F}} & T\tilde{\mathcal{X}} \end{array}$$

Locally, in adapted coordinates (x, w) of \mathcal{V} , the fiber preserving lift Φ takes the form $\Phi(x, w) = (\phi(x), \psi(x, w))$ and (1) becomes the usual feedback equivalence

$$D\phi(x)F(x, w) = \tilde{F}(\phi(x), \psi(x, w)).$$

The map ϕ plays the role of a coordinates change in the state space \mathcal{X} , and the map ψ is called a feedback transformation as it changes the parametrisation by control w in a way that depends on the state x . If the diffeomorphism Φ is defined in a neighbourhood of (x_0, w_0) only, and $\Phi(x_0, w_0) = (\tilde{x}_0, \tilde{w}_0)$, then we say that the two systems Ξ and $\tilde{\Xi}$ are *locally feedback equivalent* at (x_0, w_0) and $(\tilde{x}_0, \tilde{w}_0)$, respectively. If $\pi_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{X}$ is a fiber bundle, with a typical fiber \mathcal{W} , and the diffeomorphism Φ is defined on the product $\mathcal{X}_0 \times \mathcal{W}$, that is, it is global with respect to w , where \mathcal{X}_0 is a neighbourhood of a state x_0 , and $\phi(x_0) = \tilde{x}_0$, then we say that Ξ and $\tilde{\Xi}$ are locally feedback equivalent at x_0 and \tilde{x}_0 , respectively. The latter local feedback equivalence will be especially useful for the class of control-affine systems

$$\Sigma : \dot{\xi} = f(\xi) + g(\xi)u, \quad u \in \mathbb{R},$$

where $\xi \in \mathcal{M}$, f and g are smooth vector fields on \mathcal{M} ; below the state space \mathcal{M} will be of dimension 3 so \mathcal{M} should not be confused with \mathcal{X} , which is of dimension 2. For control-affine systems we will restrict the feedback transformations to the control-affine transformations

$$\psi(\xi, u) = \alpha(\xi) + \beta(\xi)u,$$

where $\alpha(\xi)$ and $\beta(\xi)$ are smooth functions satisfying $\beta(\cdot) \neq 0$. In that case, we denote the feedback transformation by the triple (ϕ, α, β) and if $\phi = \text{id}$, then this action is called a *pure feedback* transformation and is denoted (α, β) . Observe that a general control system of the form $\Xi : \dot{x} = F(x, w)$, where $(x, w) \in \mathcal{X} \times \mathcal{W}$, can be extended to a control-affine system Ξ^e by augmenting the state space with the control w and introducing the new control $u = \dot{w}$, which gives

$$\Xi^e : \begin{cases} \dot{x} & = F(x, w) \\ \dot{w} & = u \end{cases}, \quad u \in \mathbb{R}.$$

Notice that Ξ^e lives on the manifold $\mathcal{M} = \mathcal{X} \times \mathcal{W}$ of dimension $n = 3$.

Recall that $\mathcal{S} = \{S(x, \dot{x}) = 0\}$ is a fibred hypersurface of $T\mathcal{X}$, and that $S : T\mathcal{X} \rightarrow \mathbb{R}$ satisfies $\text{rk} \frac{\partial S}{\partial \dot{x}}(x, \dot{x}) = 1$ for all $(x, \dot{x}) \in \mathcal{S}$. Therefore, as already discussed in Introduction, we can locally

attach to \mathcal{S} its regular *parametrisation* $\dot{x} - F(x, w) = 0$, satisfying $S(x, F(x, w)) = 0$, thus defining a control-nonlinear system $\Xi_{\mathcal{S}} : \dot{x} = F(x, w)$, for which $\text{rk} \frac{\partial F}{\partial w}(x, w) = 1$. Since constructing $\Xi_{\mathcal{S}}$ requires to introduce an extra variable w (a control), we will also call it a *first extension* of \mathcal{S} . Then, we can attach to $\Xi_{\mathcal{S}}$ its extension $\Xi_{\mathcal{S}}^e$, called also a *prolongation*, denoted $\Sigma_{\mathcal{S}}$ and given by

$$\Sigma_{\mathcal{S}} : \begin{cases} \dot{x} &= F(x, w) \\ \dot{w} &= u \end{cases}, \quad u \in \mathbb{R},$$

and called a *second extension* of \mathcal{S} . To distinguish different control systems attached to the submanifolds \mathcal{S} and $\tilde{\mathcal{S}}$, we will denote $\Xi_{\mathcal{S}}$ (resp. $\Sigma_{\mathcal{S}}$) by $\tilde{\Xi}_{\mathcal{S}}$ (resp. by $\tilde{\Sigma}_{\mathcal{S}}$). The following proposition shows that the problem of equivalence of submanifolds corresponds to the equivalence under feedback transformations of their corresponding first and second extensions.

Proposition 2.4 (Equivalence of equivalence notions). *Consider submanifolds $\mathcal{S} = \{S(x, \dot{x}) = 0\}$ of $T\mathcal{X}$ and $\tilde{\mathcal{S}} = \{\tilde{S}(\tilde{x}, \dot{\tilde{x}}) = 0\}$ of $T\tilde{\mathcal{X}}$. The following statements are locally equivalent:*

- (i) *The submanifolds \mathcal{S} and $\tilde{\mathcal{S}}$ are equivalent via a diffeomorphism $\phi(x)$.*
- (ii) *The scalar valued maps S and \tilde{S} are V -equivalent via $\phi(x)$ and $\delta(x, \dot{x})$.*
- (iii) *Their regular parametrisations (first extensions) $\Xi_{\mathcal{S}}$ and $\tilde{\Xi}_{\tilde{\mathcal{S}}}$ are feedback equivalent via $\phi(x)$ and $\psi(x, w)$.*
- (iv) *Their second extensions $\Sigma_{\mathcal{S}}$ and $\tilde{\Sigma}_{\tilde{\mathcal{S}}}$ are feedback equivalent via $\varphi(x, w) = (\phi(x), \psi(x, w))$ and (α, β) .*

That is, the following diagram is commutative:

$$\begin{array}{ccccccc} \mathcal{S} & \xrightarrow{\text{representation}} & \mathcal{S} & \xrightarrow{\text{parametrisation}} & \Xi_{\mathcal{S}} & \xrightarrow{\text{prolongation}} & \Sigma_{\mathcal{S}} \\ \phi \downarrow & & (\phi, \delta) \downarrow & & (\phi, \psi) \downarrow & & (\varphi, \alpha, \beta) \downarrow \\ \tilde{\mathcal{S}} & \xrightarrow{\text{representation}} & \tilde{\mathcal{S}} & \xrightarrow{\text{parametrisation}} & \tilde{\Xi}_{\tilde{\mathcal{S}}} & \xrightarrow{\text{prolongation}} & \tilde{\Sigma}_{\tilde{\mathcal{S}}} \end{array}$$

Proof. The proof of (i) \Leftrightarrow (ii) follows immediately from [2, Section 6.5].

(ii) \Rightarrow (iii). To show that the V -equivalence of maps (defining submanifolds) implies the equivalence under feedback of their first extensions, we will first use a diffeomorphism (of the state space) to analyse both first extensions in the same coordinate system. Then, we will show that the parameters (controls) of those two first extensions are related by a pure feedback transformation. To obtain invertibility of the feedback transformation, our argument will strongly rely on the regularity of the submanifold, $\text{rk} \frac{\partial S}{\partial \dot{x}} = 1$, and of its first extension, $\text{rk} \frac{\partial F}{\partial w} = 1$. Assume that the maps $S : T\mathcal{X} \rightarrow \mathbb{R}$ and $\tilde{S} : T\tilde{\mathcal{X}} \rightarrow \mathbb{R}$ are V -equivalent via a diffeomorphism $\tilde{x} = \phi(x)$ and a nonvanishing function $\delta(x, \dot{x})$, that is, $\tilde{S}(\phi(x), D\phi(x)\dot{x}) = \delta(x, \dot{x})S(x, \dot{x})$. Consider $\Xi_{\mathcal{S}} : \dot{x} = F(x, w)$ and $\tilde{\Xi}_{\tilde{\mathcal{S}}} : \dot{\tilde{x}} = \tilde{F}(\tilde{x}, \tilde{w})$, two regular parametrisations of the corresponding submanifolds \mathcal{S} and $\tilde{\mathcal{S}}$. Then, using $0 = \tilde{S}(\tilde{x}, \dot{\tilde{x}}) = \tilde{S}(\tilde{x}, \tilde{F}(\tilde{x}, \tilde{w}))$, we have

$$\tilde{S}(\phi(x), \tilde{F}(\phi(x), \tilde{w})) = \delta(x, (D\phi(x))^{-1} \tilde{F}(\phi(x), \tilde{w}))S(x, (D\phi(x))^{-1} \tilde{F}(\phi(x), \tilde{w})), \quad \forall \tilde{w} \in \tilde{\mathcal{W}},$$

implying $S(x, \hat{F}(x, \tilde{w})) = 0$, where $\hat{F}(x, \tilde{w}) = (D\phi(x))^{-1} \tilde{F}(\phi(x), \tilde{w})$. Therefore, $\dot{x} = F(x, w)$ and $\dot{x} = \hat{F}(x, \tilde{w})$ are two regular parametrisations of the same submanifold \mathcal{S} . We will prove that $F(x, w)$ and $\hat{F}(x, \tilde{w})$ are related by an invertible (pure) feedback transformation of the form $\tilde{w} = \psi(x, w)$.

Since $\text{rk} \frac{\partial S}{\partial \dot{x}}(x_0, \dot{x}_0) = 1$, we may assume that $\frac{\partial S}{\partial \dot{z}}(x_0, \dot{x}_0) \neq 0$, where $x = (z, y)$ and, by the implicit function theorem, we have $S(x, \dot{x}) = \delta(x, \dot{x})(\dot{z} - s(x, \dot{y})) = 0$. Let $\dot{z} = F_1(x, w)$, $\dot{y} = F_2(x, w)$ be a regular parametrisation of \mathcal{S} , it follows that $\frac{\partial F_2}{\partial w}(x_0, w_0) \neq 0$. Indeed, if $\frac{\partial F_2}{\partial w}(x_0, w_0) = 0$, then $\dot{z} - s(x, \dot{y}) = 0$ (recall that $\delta \neq 0$), hence $F_1 - s(x, F_2) = 0$ and thus $\frac{\partial F_1}{\partial w} - \frac{\partial s}{\partial y} \frac{\partial F_2}{\partial w} = 0$ implying that $\frac{\partial F_1}{\partial w}(x_0, w_0) = 0$; contradiction.

Hence, the two regular parametrisations of $\mathcal{S} = \{S(x, \dot{x}) = 0\}$ given by

$$\Xi_{\mathcal{S}} : \begin{cases} \dot{z} = F_1(x, w) \\ \dot{y} = F_2(x, w) \end{cases} \quad \text{and} \quad \hat{\Xi}_{\mathcal{S}} : \begin{cases} \dot{z} = \hat{F}_1(x, \tilde{w}) \\ \dot{y} = \hat{F}_2(x, \tilde{w}) \end{cases}$$

satisfy $\frac{\partial F_2}{\partial w}(x_0, w_0) \neq 0$ and $\frac{\partial \hat{F}_2}{\partial \tilde{w}}(x_0, \tilde{w}_0) \neq 0$. Therefore, $\tilde{w} = \hat{F}_2^{-1}(x, \dot{y}) = \hat{F}_2^{-1}(x, F_2(x, w))$, where \hat{F}_2^{-1} is the inverse with respect to the second argument. And using $\dot{z} = s(x, \dot{y})$ we obtain $\dot{z} = F_1(x, w) = s(x, \dot{y}) = \hat{F}_1(x, \tilde{w})$. Thus, $\Xi_{\mathcal{S}}$ and $\hat{\Xi}_{\mathcal{S}}$ are feedback equivalent via $\tilde{w} = \psi(x, w)$ with $\psi(x, w) = \hat{F}_2^{-1}(x, F_2(x, w))$ and the systems $\Xi_{\mathcal{S}}$ and $\tilde{\Xi}_{\mathcal{S}}$ are feedback equivalent since $\tilde{\Xi}_{\mathcal{S}}$ is the system $\hat{\Xi}_{\mathcal{S}}$ mapped via the diffeomorphism $\tilde{x} = \phi(x)$.

(iii) \Rightarrow (ii). Assume that the two regular parametrisations $\Xi_{\mathcal{S}} : \dot{x} = F(x, w)$ and $\tilde{\Xi}_{\mathcal{S}} : \dot{\tilde{x}} = \tilde{F}(\tilde{x}, \tilde{w})$ of \mathcal{S} and $\tilde{\mathcal{S}}$, respectively, are feedback equivalent via $\tilde{x} = \phi(x)$ and $\tilde{w} = \psi(x, w)$. Denote $\tilde{\phi} = \phi^{-1}$ and apply the diffeomorphism $x = \tilde{\phi}(\tilde{x})$ to $\tilde{\Xi}_{\mathcal{S}}$ to obtain a new vector field, parametrized by \tilde{w} , $\hat{F}(x, \tilde{w}) = D\tilde{\phi}(\phi(x))\tilde{F}(\phi(x), \tilde{w})$ related to $F(x, w)$ by the pure feedback transformation $\tilde{w} = \psi(x, w)$. Denote $F = (F_1, F_2)^t$ and $\hat{F} = (\hat{F}_1, \hat{F}_2)^t$ in the $x = (z, y)$ coordinates. Without loss of generality we can assume that $\frac{\partial F_2}{\partial w}(x_0, w_0) \neq 0$ and $\frac{\partial \hat{F}_2}{\partial \tilde{w}}(x_0, \tilde{w}_0) \neq 0$.

Now, apply to $\tilde{\mathcal{S}} = \{\tilde{S}(\tilde{x}, \dot{\tilde{x}}) = 0\}$ the same diffeomorphism $x = \tilde{\phi}(\tilde{x})$, whose inverse is denoted by $\tilde{x} = \phi(x)$, and set $\hat{S}(x, \dot{x}) = \tilde{S}(\phi(x), D\phi(x)\dot{x})$. Since by definition of $\tilde{\Xi}_{\mathcal{S}}$ we have $\tilde{S}(\tilde{x}, \tilde{F}(\tilde{x}, \tilde{w})) = 0$, we conclude that $\hat{S}(x, \hat{F}(x, \tilde{w})) = 0$. Then we claim that $\frac{\partial \hat{S}}{\partial \tilde{z}}(x_0, \dot{x}_0) \neq 0$; indeed, if $\frac{\partial \hat{S}}{\partial \tilde{z}}(x_0, \dot{x}_0) = 0$, then $\frac{\partial \hat{S}}{\partial w}(x, F(x, w)) = 0$ yields $\frac{\partial \hat{S}}{\partial \tilde{z}} \frac{\partial F_1}{\partial w} + \frac{\partial \hat{S}}{\partial \tilde{y}} \frac{\partial F_2}{\partial w} = 0$ and thus $\frac{\partial \hat{S}}{\partial \tilde{y}}(x_0, \dot{x}_0) = 0$ giving a contradiction. The same observation holds for $\frac{\partial \hat{S}}{\partial \tilde{z}}(x_0, \dot{x}_0)$. Thus, by the implicit function theorem we have $S(x, \dot{x}) = \delta(x, \dot{x})(\dot{z} - s(x, \dot{y}))$ and $\hat{S}(x, \dot{x}) = \hat{\delta}(x, \dot{x})(\dot{z} - \hat{s}(x, \dot{y}))$, with $\delta \neq 0$ and $\hat{\delta} \neq 0$.

Using $\dot{z} = s(x, \dot{y})$ and $\dot{z} = \hat{s}(x, \dot{y})$, we obtain $s(x, \dot{y}) = F_1(x, w) = \hat{F}_1(x, \tilde{w}) = \hat{s}(x, \dot{y})$, hence we have

$$S(x, \dot{x}) = \delta(x, \dot{x})(\dot{z} - s(x, \dot{y})) = \delta(x, \dot{x})(\dot{z} - \hat{s}(x, \dot{y})) = \frac{\delta}{\hat{\delta}} \tilde{S}(\phi(x), D\phi(x)\dot{x})$$

establishing the V -equivalence between S and \tilde{S} .

(iii) \Rightarrow (iv). If $\Xi_{\mathcal{S}}$ and $\tilde{\Xi}_{\mathcal{S}}$ are feedback equivalent, then

$$D\phi(x)F(x, w) = \tilde{F}(\phi(x), \psi(x, w)).$$

Thus the diffeomorphism $(\phi(x), \psi(x, w))$, of the augmented state space (x, w) , together with the control-affine feedback

$$\tilde{u} = \frac{\partial \psi}{\partial x} F(x, w) + \frac{\partial \psi}{\partial w} u,$$

transform $\Sigma_{\mathcal{S}}$ into $\tilde{\Sigma}_{\tilde{\mathcal{S}}}$.

(iv) \Rightarrow (iii). Assume that $\Sigma_{\mathcal{S}}$ and $\tilde{\Sigma}_{\tilde{\mathcal{S}}}$ are feedback equivalent via $(\tilde{x}, \tilde{w}) = \varphi(x, w)$ and $\tilde{u} = \alpha(x, w) + \beta(x, w)u$. Since the distribution $\text{span}\{\frac{\partial}{\partial w}\}$ is sent by φ_* into $\text{span}\{\frac{\partial}{\partial \tilde{w}}\}$, it follows that φ has the triangular form $\varphi(x, w) = (\phi(x), \psi(x, w))$. Therefore, feedback equivalence of the systems $\Xi_{\mathcal{S}}$ and $\tilde{\Xi}_{\tilde{\mathcal{S}}}$ is established via the diffeomorphism $\tilde{x} = \phi(x)$ and the reparametrisation $\tilde{w} = \psi(x, w)$. \square

Remark. The use of extensions of control-nonlinear systems was introduced in [30] and used to study controllability and observability of nonlinear systems, and then to analyse linearisability and decoupling [29]. Moreover, notice that the same proof as that of (iii) \Leftrightarrow (iv) shows that any two control-nonlinear systems Ξ and $\tilde{\Xi}$ (which need not be regular parametrisations of submanifolds) are feedback equivalent if and only if their extensions Ξ^e and $\tilde{\Xi}^e$ are equivalent via control-affine feedback; see [13].

Remark. The equivalence (ii) \Leftrightarrow (iii) does not hold, in general, if the parametrisation $\Xi_{\mathcal{S}} : \dot{x} = F(x, w)$ does not satisfy the regularity condition $\frac{\partial F}{\partial w}(x, w) \neq 0$ that we assume. To see that, consider the submanifold \mathcal{S} given by $\dot{z} - \dot{y}^2 = 0$. The parametrisations of \mathcal{S}

$$\Xi_{\mathcal{S}} : \begin{cases} \dot{z} = w^2 \\ \dot{y} = w \end{cases} \quad \text{and} \quad \tilde{\Xi}_{\mathcal{S}} : \begin{cases} \dot{z} = \tilde{w}^6 \\ \dot{y} = \tilde{w}^3 \end{cases}$$

are not feedback equivalent around $w_0 = 0$, and the reason is that $\tilde{\Xi}_S$ fails to satisfy $\frac{\partial \tilde{F}}{\partial \tilde{w}}(\tilde{w}_0) \neq 0$ at $\tilde{w}_0 = 0$.

According to the previous proposition, in order to deal with the problem of equivalence of submanifolds of the tangent bundle $T\mathcal{X}$ it is interesting, first, to study the problem of equivalence of general control-nonlinear systems Ξ and, second, the problem of equivalence of general control-affine systems Σ .

Conic submanifolds. A map $S : T\mathcal{X} \rightarrow \mathbb{R}$ will be called quadratic if it is of the form

$$S_q(x, \dot{x}) = \dot{x}^t \mathbf{g}(x) \dot{x} + 2\omega(x) \dot{x} + h(x),$$

that is, S_q is a smooth polynomial of degree two in \dot{x} . A velocities constraint \mathcal{S} is called quadratic (or conic) if there exists a quadratic map S_q such that $\mathcal{S} = \{S_q(x, \dot{x}) = 0\}$. As we already observed in the Introduction, if \mathcal{S} is quadratic, then any $\tilde{\mathcal{S}}$, equivalent to \mathcal{S} , is also quadratic. On the other hand, if a map S_q is quadratic, then a map \tilde{S} that is V -equivalent to S_q need not be quadratic (although they define equivalent quadratic velocities constraints).

The map S_q can be represented by the triple $S_q = (\mathbf{g}, \omega, h)$, where \mathbf{g} is a symmetric $(0, 2)$ -tensor (possibly degenerated), ω is a one-form and h is a function. Clearly, two conics \mathcal{S}_q of $T\mathcal{X}$ and $\tilde{\mathcal{S}}_q$ of $T\tilde{\mathcal{X}}$, given by (\mathbf{g}, ω, h) and $(\tilde{\mathbf{g}}, \tilde{\omega}, \tilde{h})$, respectively, are equivalent if and only if there exists a diffeomorphism $\tilde{x} = \phi(x)$ and a non-vanishing function $\delta = \delta(x)$ on \mathcal{X} such that $\delta \mathbf{g} = \phi^* \tilde{\mathbf{g}}$, $\delta \omega = \phi^* \tilde{\omega}$, and $\delta h = \phi^* \tilde{h}$. In particular, observe that the tensors \mathbf{g} and $\tilde{\mathbf{g}}$, which are (pseudo-)Riemannian metrics (possibly degenerated), are conformally equivalent.

It is well-known in affine geometry that such conic equations can be classified by the signature of the matrix $M_q(x) = \begin{pmatrix} \mathbf{g}(x) & \omega(x)^t \\ \omega(x) & h(x) \end{pmatrix}$ and that of $\mathbf{g}(x)$. We will use the following two determinants

$$\Delta_1(x) = \det(M_q(x)) \quad \text{and} \quad \Delta_2(x) = \det(\mathbf{g}(x)). \quad (2)$$

Of course, Δ_1 and Δ_2 depend on the choice of coordinates, however, since a diffeomorphism $\tilde{x} = \phi(x)$ transforms them as $\Delta_i = \theta^2 \phi^*(\tilde{\Delta}_i)$, for $i = 1, 2$, where $\theta(x) = \det D\phi(x)$, their zero-level sets $\{\Delta_i(x) = 0\}$ are invariantly related to the submanifold \mathcal{S}_q . In this work we will characterise non-degenerate conics, that is, non-empty and satisfying $\Delta_1 \neq 0$; notice that non-empty conics at points of degeneration $\Delta_1(x) = 0$ form in fact (pairs of) linear subspaces of $T_x\mathcal{X}$. Excluding empty \mathcal{S}_q is needed when considering elliptic submanifolds (see lemma below) and it implies that M_q is indefinite. The non-degeneracy assumption $\Delta_1(x) \neq 0$ implies that $\frac{\partial S_q}{\partial \dot{x}}(x, \dot{x}) \neq 0$ (but the converse does not hold in general) and $\text{rk } \mathbf{g}(x) \geq 1$.

If $\Delta_2(x_0) \neq 0$ or if $\Delta_2 \equiv 0$ in a neighbourhood of x_0 , then we can describe three particular types of conics given by the classification lemma below. Notice, however, that this lemma does not describe conics for which we pass smoothly from one type to another (see remark below the proof of [Theorem 3.5](#) for that case).

Lemma 2.5 (Classification of non-degenerate conics). *Consider a nonempty conic \mathcal{S}_q , given by (\mathbf{g}, ω, h) , and assume $\Delta_1(x_0) \neq 0$. Then, locally around x_0 , we have*

(i) *If $\Delta_2(x_0) > 0$, then \mathcal{S}_q is equivalent to $\mathcal{S}_E = \{a^2(\dot{z} - c_0)^2 + b^2(\dot{y} - c_1)^2 = 1\}$,*

(ii) *If $\Delta_2(x_0) < 0$, then \mathcal{S}_q is equivalent to $\mathcal{S}_H = \{a^2(\dot{z} - c_0)^2 - b^2(\dot{y} - c_1)^2 = 1\}$,*

(iii) *If $\Delta_2 \equiv 0$, then \mathcal{S}_q is equivalent to $\mathcal{S}_P = \{a\dot{y}^2 - \dot{z} + b\dot{y} + c = 0\}$,*

where a, b, c_0, c_1 , and c are smooth functions satisfying $a \neq 0$ and $b \neq 0$ in the elliptic and hyperbolic cases, and $a \neq 0$ in the parabolic case.

We call \mathcal{S}_E , resp. \mathcal{S}_H , resp. \mathcal{S}_P , an elliptic, resp. a hyperbolic, resp. a parabolic, submanifold and we will use the notation \mathcal{S}_Q to denote the set $\{\mathcal{S}_E, \mathcal{S}_H, \mathcal{S}_P\}$ of those three particular forms. Observe that for the parabolic form \mathcal{S}_P , the non-degeneracy assumption $\Delta_1 \neq 0$ implies $\text{rk } \mathbf{g} \equiv 1$ and the

existence of a nonvanishing one-form $\omega = -dz + b dy$ satisfying $\omega \notin \text{ann}(\ker \mathbf{g})$, whereas in the elliptic and hyperbolic cases this one-form, given by $\omega = -(c_0 dz \pm c_1 dy)$, can vanish at some points. Those three classes of submanifolds are related to the signature of the metric \mathbf{g} ; indeed if $\text{sgn}(\mathbf{g})$ is constant in a neighbourhood of x_0 , then \mathcal{S}_E , resp. \mathcal{S}_H , resp. \mathcal{S}_P , corresponds to $\text{sgn}(\mathbf{g}) = (+, +)$, resp. $\text{sgn}(\mathbf{g}) = (+, -)$, resp. $\text{sgn}(\mathbf{g}) = (+, 0)$; notice that we can always assume that there is at least one positive eigenvalue, otherwise take the equivalent submanifold defined by $\tilde{S} = -S$.

Proof. Consider a submanifold \mathcal{S}_q given in local coordinates by $S_q(x, \dot{x}) = 0$, with

$$S_q(x, \dot{x}) = \dot{x}^t \begin{pmatrix} \mathbf{g}_{11} & \mathbf{g}_{12} \\ \mathbf{g}_{12} & \mathbf{g}_{22} \end{pmatrix} \dot{x} + 2(\omega_1, \omega_2) \dot{x} + h,$$

where all functions \mathbf{g}_{ij} , ω_1 , ω_2 , and h depend smoothly on $x \in \mathcal{X}$.

(i)-(ii) We deal with $\Delta_2(x_0) \neq 0$, that is, the elliptic and hyperbolic cases together. In those cases, \mathbf{g} is a non-degenerate symmetric $(0, 2)$ -tensor, therefore it can be interpreted as a pseudo-Riemannian metric. Since on 2-dimensional manifolds all metrics are smoothly diagonalisable (actually, all metrics are conformally flat; see [26, Addendum 1 of Chapter 9] for the Riemannian case and [25, Theorem 7.2] for the Lorentzian one), introduce coordinates $\tilde{x} = \phi(x) = (z, y)$ such that in those coordinates, S_q can be written (we drop the tildes for more readability) as

$$S_q = \lambda_1^2(x) \dot{z}^2 \pm \lambda_2^2(x) \dot{y}^2 + 2\omega(x) \dot{x} + h(x)$$

or, equivalently (since $\lambda_1(\cdot) \neq 0$ and $\lambda_2(\cdot) \neq 0$), as

$$S_q = \lambda_1^2 \left(\dot{z} + \frac{\omega_1}{\lambda_1^2} \right)^2 \pm \lambda_2^2 \left(\dot{y} \pm \frac{\omega_2}{\lambda_2^2} \right)^2 + h - \lambda_1^2 \left(\frac{\omega_1}{\lambda_1^2} \right)^2 \mp \lambda_2^2 \left(\frac{\omega_2}{\lambda_2^2} \right)^2.$$

Notice that for this form we have $\Delta_1 = \pm \lambda_1^2 \lambda_2^2 \left(h - \frac{(\omega_1)^2}{\lambda_1^2} \mp \frac{(\omega_2)^2}{\lambda_2^2} \right)$ which, by our assumption, does not vanish. Denote $c_0 = -\frac{\omega_1}{\lambda_1^2}$, $c_1 = \mp \frac{\omega_2}{\lambda_2^2}$, and divide by $\tilde{h} = -h + \lambda_1^2 \left(\frac{\omega_1}{\lambda_1^2} \right)^2 \pm \lambda_2^2 \left(\frac{\omega_2}{\lambda_2^2} \right)^2$, observe that $\tilde{h} \neq 0$ as $\tilde{h} = \mp \frac{1}{\lambda_1^2 \lambda_2^2} \Delta_1$, to obtain

$$S_q = \frac{\lambda_1^4 \lambda_2^2}{\mp \Delta_1} (\dot{z} - c_0)^2 \pm \frac{\lambda_1^2 \lambda_2^4}{\mp \Delta_1} (\dot{y} - c_1)^2 - 1,$$

where the upper, resp. lower, sign corresponds to the elliptic, resp. hyperbolic, case. In the elliptic case, if $\Delta_1 > 0$ the conic is empty which is excluded by assumption, therefore $\Delta_1 < 0$ and set $a^2 = \frac{\lambda_1^4 \lambda_2^2}{-\Delta_1}$ and $b^2 = \frac{\lambda_1^2 \lambda_2^4}{-\Delta_1}$ to obtain \mathcal{S}_E . In the hyperbolic case, if $\Delta_1 < 0$, then permute the variables (z, y) and we can thus always obtain a conic defined by S_q in the above form with $\Delta_1 > 0$. Then, set $a^2 = \frac{\lambda_1^4 \lambda_2^2}{\Delta_1}$ and $b^2 = \frac{\lambda_1^2 \lambda_2^4}{\Delta_1}$ to obtain \mathcal{S}_H .

(iii) Assume $\Delta_2 \equiv 0$. Since $\Delta_1 \neq 0$, we have $\text{rk } \mathbf{g}(x) = 1$ in a neighbourhood of x_0 , which implies that $\mathbf{g}_{11}(x_0) \mathbf{g}_{22}(x_0) \neq 0$ and thus, without loss of generality, we can assume that $\mathbf{g}_{22}(x_0) \neq 0$. Then, by $\text{rk } \mathbf{g}(x) = 1$, the distribution $\ker \mathbf{g} = \text{span} \left\{ \mathbf{g}_{22}(x) \frac{\partial}{\partial z} - \mathbf{g}_{12}(x) \frac{\partial}{\partial y} \right\}$ is locally of constant rank and thus we introduce coordinates $\tilde{x} = (\tilde{z}, \tilde{y})$ such that $\ker \mathbf{g} = \text{span} \left\{ \frac{\partial}{\partial \tilde{z}} \right\}$ in which we have

$$\tilde{S}_q = \tilde{a}(\tilde{x}) \dot{\tilde{y}}^2 + 2\tilde{\omega}(\tilde{x}) \dot{\tilde{x}} + \tilde{h}(\tilde{x}),$$

whose determinant $\Delta_1 = -\tilde{a}(\tilde{\omega}_1)^2 \neq 0$ implies that $\tilde{\omega}_1 \neq 0$. Dividing \tilde{S}_q by $-2\tilde{\omega}_1$ we obtain the desired form \mathcal{S}_P with $a = \frac{\tilde{a}}{-2\tilde{\omega}_1}$, $b = \frac{\tilde{\omega}_2}{-2\tilde{\omega}_1}$, and $c = \frac{\tilde{h}}{-2\tilde{\omega}_1}$.

□

Example 2.6. *There is a well known example of a control-nonlinear system subject to an elliptic constraint \mathcal{S}_E , namely Dubins' car [10]. The state of the system is the centre of mass of the vehicle $(z, y) \in \mathbb{R}^2$, and we control the orientation of the vehicle (with respect to the z -axis) via $w \in \mathbb{S}^1$. Then, the dynamics of Dubins' car reads*

$$\begin{cases} \dot{z} &= r \cos(w) \\ \dot{y} &= r \sin(w) \end{cases}, \quad r \in \mathbb{R}^*,$$

which clearly is a first extension, i.e. a regular parametrisation, of the elliptic submanifold $z^2 + y^2 = r^2$.

We have established all notions necessary for our characterisation and classification of conic submanifolds \mathcal{S}_q by studying the feedback equivalence of their first and second extensions $\Xi_{\mathcal{S}_q}$ and $\Sigma_{\mathcal{S}_q}$.

3 Quadratisable control-affine systems

Notations.

$\Sigma = (f, g)$	A control-affine system on a 3-dimensional manifold and with scalar control.
(ϕ, α, β)	Feedback transformations acting on control-affine systems.
$[g, f], \text{ad}_g^k f, L_g()$	Lie bracket, iterated Lie bracket, Lie derivative.
\mathcal{G}	Distribution spanned by the vector field g .
$\Sigma_q = (f_q, g_q)$	Quadratic control-affine system given, in coordinates (x, w) , by the vector fields $f_q = f_q \frac{\partial}{\partial x}$ and $g_q = \frac{\partial}{\partial w}$, which satisfy Definition 3.1 .
(ρ, τ)	Structure functions attached to a control-affine system $\Sigma = (f, g)$, see condition (C2) of Theorem 3.3 .
$\Sigma_Q = \{\Sigma_E, \Sigma_H, \Sigma_P\}$	Set of elliptic, hyperbolic, and parabolic subclasses of quadratic systems Σ_q .
Σ_h	Second extension of a submanifold $\mathcal{S} = \{\dot{z} = h(x, y)\}$.

In this section, we introduce the novel class of quadratic control-affine systems Σ_q that describes second extensions of quadratic submanifolds \mathcal{S}_q given by

$$S_q(x, \dot{x}) = \dot{x}^t \mathbf{g}(x) \dot{x} + 2\omega(x) \dot{x} + h(x) = 0$$

and satisfying $\Delta_1 \neq 0$. Next, we address the equivalence problem of a control-affine system Σ to a quadratic control-affine system Σ_q , and in that way, due to [Proposition 2.4](#), we provide a characterisation of quadratic submanifolds \mathcal{S}_q . As a corollary, we will give a characterisation of the elliptic, hyperbolic, and parabolic submanifolds via a characterisation of corresponding subclasses of quadratic control-affine systems. Moreover, by studying our conditions, we will give a normal form of control-affine systems that are feedback equivalent to a quadratic one and as a consequence, this will give us a normal form of quadratic submanifolds that smoothly passes from the elliptic to the hyperbolic classes.

On a 3-dimensional manifold \mathcal{M} , equipped with local coordinates ξ , we consider the control-affine system

$$\Sigma : \dot{\xi} = f(\xi) + g(\xi)u,$$

with a scalar control $u \in \mathbb{R}$ and smooth vector fields f and g . A control-affine system Σ is denoted by the pair $\Sigma = (f, g)$, and we set $\mathcal{G} = \text{span}\{g\}$ the distribution spanned by the vector field g . Moreover, we will use the following notations: given two vector fields g and f on \mathcal{M} , by $[g, f]$ we denote the Lie bracket of g and f , in coordinates we have $[g, f] = \frac{\partial f}{\partial \xi} g - \frac{\partial g}{\partial \xi} f$, and $\text{ad}_g^k f = [g, \text{ad}_g^{k-1} f]$ stands for the iterated Lie bracket, with the convention $\text{ad}_g^0 f = f$.

Definition 3.1 (Quadratic and quadratisable systems). *We say that a control-affine system $\Sigma = (f, g)$ is quadratisable if it is feedback equivalent to*

$$\Sigma_q : \begin{cases} \dot{x} &= f_q(x, w) \\ \dot{w} &= u \end{cases},$$

where $x \in \mathcal{X}$, a 2-dimensional manifold, $w \in \mathcal{W} \subset \mathbb{R}$, and $f_q(x, w)$ satisfies $\left(\frac{\partial^2 f_q}{\partial w^2} \wedge \frac{\partial f_q}{\partial w}\right)(x, w) \neq 0$, and

$$\frac{\partial^3 f_q}{\partial w^3} = \tau_q(x) \frac{\partial f_q}{\partial w}, \quad (3)$$

with τ_q a smooth function of the indicated variable $x \in \mathcal{X}$. A system Σ_q of the above form is called quadratic.

For a quadratic system $\Sigma_q = (f_q, g_q)$, where $f_q = f_q \frac{\partial}{\partial x}$ and $g_q = \frac{\partial}{\partial w}$, we show by applying [Appendix A](#) with $\zeta = f_q^1$ and $\zeta = f_q^2$, that any smooth f_q , satisfying relation (3) locally around (x_0, w_0) , can be written as

$$f_q(x, w) = A(x) \sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k+2}}{(2k+2)!} \tau_q^k(x) + B(x) \sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k+1}}{(2k+1)!} \tau_q^k(x) + C(x), \quad (4)$$

where A, B, C are smooth. Since $\mathcal{G} = \text{span}\{g_q\} = \text{span}\left\{\frac{\partial}{\partial w}\right\}$ is involutive and of constant rank one, it is integrable and its integral curves can be identified with the points x of \mathcal{X} and A, B , and C can be seen as smooth vector fields on \mathcal{X} , for which we have $A \wedge B \neq 0$. The following proposition shows that Σ_q is a second extension of a quadratic submanifold \mathcal{S}_q , thus justifies to call Σ_q a quadratic system, and identifies the elliptic, hyperbolic, and parabolic subclasses by describing three normal forms of $f_q = f_q \frac{\partial}{\partial x}$ given for $\tau_q \neq 0$ and $\tau_q \equiv 0$.

Proposition 3.2. *Locally around ξ_0 belonging to \mathcal{M} , the following statements hold*

- (i) Σ_q is a second extension of a conic submanifold \mathcal{S}_q and, conversely, any second extension $\Sigma_{\mathcal{S}_q}$ of a conic submanifold \mathcal{S}_q is feedback equivalent to a system of the form Σ_q .
- (ii) If $\tau_q(\xi_0) < 0$, resp. $\tau_q(\xi_0) > 0$, resp. $\tau_q \equiv 0$, then Σ_q is locally feedback equivalent to Σ_E , resp. Σ_H , resp. Σ_P , given by f_q of, respectively, the form

$$\begin{aligned} f_E &= A(x) \cos(\tilde{w}) + B(x) \sin(\tilde{w}) + C(x), & f_H &= A(x) \cosh(\tilde{w}) + B(x) \sinh(\tilde{w}) + C(x), \\ f_P &= A(x) w^2 + B(x) w + C(x), \end{aligned}$$

where in all three cases $A \wedge B \neq 0$.

- (iii) Σ_E , resp. Σ_H , resp. Σ_P , is a second extension of a conic submanifold \mathcal{S}_q satisfying $\Delta_2 > 0$, resp. $\Delta_2 < 0$, resp. $\Delta_2 \equiv 0$.

The conic submanifold \mathcal{S}_q of item (iii) is, by [Lemma 2.5](#), equivalent to \mathcal{S}_E (if $\Delta_2 > 0$), resp. \mathcal{S}_H (if $\Delta_2 < 0$), resp. \mathcal{S}_P (if $\Delta_2 \equiv 0$). So it is natural to call Σ_E an elliptic system, Σ_H a hyperbolic system, and Σ_P a parabolic system. We will denote by Q the set $\{E, H, P\}$ and, consequently, $f_Q = \{f_E, f_H, f_P\}$, $g_Q = \{g_E, g_H, g_P\}$, $f_Q = \{f_E, f_H, f_P\}$, and $\Sigma_Q = \{\Sigma_E, \Sigma_H, \Sigma_P\}$.

Proof.

- (i) Consider f_q given by (4), for simplicity of the notations we assume $w_0 = 0$. Notice that the two rank 1 distributions $\mathcal{A} = \text{span}\{A\}$ and $\mathcal{B} = \text{span}\{B\}$ satisfy $\mathcal{A}(x_0) \oplus \mathcal{B}(x_0) = T_{x_0} \mathbb{R}^2$. Thus, we can locally choose two independent smooth functions ϕ and ψ satisfying $d\phi(x_0) \neq 0$, $d\psi(x_0) \neq 0$, $d\phi \in \text{ann}(\mathcal{B})$, and $d\psi \in \text{ann}(\mathcal{A})$. In the $(z, y) = (\phi, \psi)$ -coordinates, we have $\mathcal{A} = \text{span}\left\{\frac{\partial}{\partial z}\right\}$ and $\mathcal{B} = \text{span}\left\{\frac{\partial}{\partial y}\right\}$ and, consequently, $A = a \frac{\partial}{\partial z}$ and $B = b \frac{\partial}{\partial y}$, where a and b are smooth functions satisfying $a(x_0)b(x_0) \neq 0$. We set $C = c_0 \frac{\partial}{\partial z} + c_1 \frac{\partial}{\partial y}$, where c_0 and c_1 are smooth functions of x .

Using Cauchy products we compute

$$\begin{aligned}
\left(\frac{\dot{z} - c_0}{a}\right)^2 &= \left(\sum_{k=0}^{+\infty} \frac{w^{2k+2}}{(2k+2)!} \tau_q^k\right)^2 = \sum_{k=0}^{+\infty} \frac{(8 \cdot 4^k - 2)w^{2k+4} \tau_q^k}{(2k+4)!}, \\
\left(\frac{\dot{y} - c_1}{b}\right)^2 &= \left(\sum_{k=0}^{+\infty} \frac{w^{2k+1}}{(2k+1)!} \tau_q^k\right)^2 \\
&= \sum_{k=0}^{+\infty} \frac{2 \cdot 4^k w^{2k+2} \tau_q^k}{(2k+2)!} \\
&= w^2 + \sum_{k=1}^{+\infty} \frac{2 \cdot 4^k w^{2k+2} \tau_q^k}{(2k+2)!} \\
&= w^2 + \sum_{k=0}^{+\infty} \frac{2 \cdot 4^{k+1} w^{2k+4} \tau_q^{k+1}}{(2k+4)!} \\
&= w^2 + \tau_q \sum_{k=0}^{+\infty} \frac{8 \cdot 4^k w^{2k+4} \tau_q^k}{(2k+4)!} \\
&= w^2 + \tau_q \left(\frac{\dot{z} - c_0}{a}\right)^2 + \tau_q \sum_{k=0}^{+\infty} \frac{2w^{2k+4} \tau_q^k}{(2k+4)!}, \\
\left(\frac{\dot{y} - c_1}{b}\right)^2 - \tau_q \left(\frac{\dot{z} - c_0}{a}\right)^2 &= w^2 + \sum_{k=0}^{+\infty} \frac{2w^{2k+4} \tau_q^{k+1}}{(2k+4)!} = \sum_{k=-1}^{+\infty} \frac{2w^{2k+4} \tau_q^{k+1}}{(2k+4)!} \\
&= \sum_{k=0}^{+\infty} \frac{2w^{2k+2} \tau_q^k}{(2k+2)!} = 2 \left(\frac{\dot{z} - c_0}{a}\right).
\end{aligned}$$

Hence, Σ_q is a second extension of the submanifold \mathcal{S}_q given by

$$\left(\frac{\dot{y} - c_1}{b}\right)^2 - \tau_q \left(\frac{\dot{z} - c_0}{a}\right)^2 - 2 \left(\frac{\dot{z} - c_0}{a}\right) = 0, \quad (5)$$

and for which we have $\Delta_1 = -\frac{1}{a^2 b^2}$ and $\Delta_2 = -\frac{\tau_q}{a^2 b^2}$.

To prove that any second extension of a conic submanifold \mathcal{S}_q is feedback equivalent to a system of the form Σ_q , we show that any conic submanifold is equivalent via a diffeomorphism to a conic submanifold of the form given by (5). To this end, consider $S_q = \dot{x}^t \mathbf{g} \dot{x} + 2\omega \dot{x} + h = a\dot{z}^2 + 2b\dot{z}\dot{y} + c\dot{y}^2 + 2d\dot{z} + 2e\dot{y} + h = 0$, where all functions a, b, c, d, e, h depend smoothly on $x = (z, y)$. By $\Delta_1(x_0) \neq 0$, we have $\text{rk } \mathbf{g}(x_0) \geq 1$, where $\mathbf{g} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. If $\text{rk } \mathbf{g}(x_0) = 2$, then we can apply the results of [Lemma 2.5](#) to get \mathcal{S}_q in the form of (5). If $\text{rk } \mathbf{g}(x_0) = 1$, then we may assume that $c(x_0) \neq 0$. Indeed, if $a(x_0) = c(x_0) = 0$, then $b(x_0) \neq 0$ implying that $\text{rk } \mathbf{g}(x_0) = 2$ so either $a(x_0) \neq 0$ or $c(x_0) \neq 0$ and we can always suppose $c(x_0) \neq 0$ by permuting y and z , if necessary. Dividing $S_q = 0$ by c we obtain (we keep the same names for all remaining functions):

$$S_q = a\dot{z}^2 + 2b\dot{z}\dot{y} + \dot{y}^2 + 2d\dot{z} + 2e\dot{y} + h = (a - b^2)\dot{z}^2 + (b\dot{z} + \dot{y})^2 + 2d\dot{z} + 2e\dot{y} + h = 0.$$

Choose local coordinates $(\tilde{z}, \tilde{y}) = (z, \psi(z, y))$, with $\frac{\partial \psi}{\partial y} \neq 0$, that rectify the line-distribution $\text{span} \left\{ \frac{\partial}{\partial z} - b \frac{\partial}{\partial y} \right\}$, i.e. $\frac{\partial \psi}{\partial z} - b \frac{\partial \psi}{\partial y} = 0$. Therefore, $\dot{\tilde{z}} = \dot{z}$ and $\dot{\tilde{y}} = \frac{\partial \psi}{\partial z} \dot{z} + \frac{\partial \psi}{\partial y} \dot{y} = \frac{\partial \psi}{\partial y} (b\dot{z} + \dot{y})$. Thus $\tilde{\mathcal{S}}_q$, which is \mathcal{S}_q in (\tilde{z}, \tilde{y}) -coordinates, reads

$$\tilde{\mathcal{S}}_q = \frac{1}{\tilde{b}^2} \dot{\tilde{y}}^2 + \tilde{a} \dot{\tilde{z}}^2 + 2\tilde{d} \dot{\tilde{z}} + 2\tilde{e} \dot{\tilde{y}} + \tilde{h} = 0,$$

where $\tilde{b} = \frac{\partial \psi}{\partial y}$ and $\tilde{a}(\tilde{x}_0) = 0$ because $\text{rk } \tilde{\mathbf{g}}(\tilde{x}_0) = 1$. By $\tilde{\Delta}_1(\tilde{x}_0) \neq 0$ and $\tilde{a}(\tilde{x}_0) = 0$, we conclude that $\tilde{d}(\tilde{x}_0) \neq 0$ and thus dividing by \tilde{d} (and removing the "tildes" from the coordinates) we get

$$\bar{S}_q = \frac{\tilde{S}_q}{\tilde{d}} = \frac{1}{\bar{b}^2} \dot{y}^2 + \bar{A} \dot{z}^2 + 2\dot{z} + 2\bar{e} \dot{y} + \bar{h} = \left(\frac{\dot{y} - \bar{c}_1}{\bar{b}} \right)^2 + \bar{A} \dot{z}^2 + 2\dot{z} + \bar{H} = 0,$$

with $\bar{b} = \tilde{b} \sqrt{\tilde{d}}$ (if $\tilde{d} < 0$, then we take $z = -\tilde{z}$ to get $\tilde{d} > 0$), $\bar{A} = \frac{\bar{a}}{\tilde{d}}$ (satisfying $\bar{A}(x_0) = 0$), $\bar{c}_1 = -\bar{e} \bar{b}^2$, and $\bar{H} = \bar{h} - \bar{e}^2 \bar{b}^2$ (see also [6] for another proof for the smooth diagonalisation of a symmetric $(0, 2)$ -tensor in the case of rank deficiency). To get the form (5), we will now prove that there exists, locally around x_0 , functions $\bar{\tau}_q$, $\bar{a} \neq 0$, and \bar{c}_0 such that $\bar{A} \dot{z}^2 + 2\dot{z} + \bar{H} = -\bar{\tau}_q \left(\frac{\dot{z} - \bar{c}_0}{\bar{a}} \right)^2 - 2 \left(\frac{\dot{z} - \bar{c}_0}{\bar{a}} \right)$. We obtain

$$\frac{-\bar{\tau}_q}{\bar{a}^2} = \bar{A}, \quad \frac{\bar{\tau}_q \bar{c}_0}{\bar{a}^2} - \frac{1}{\bar{a}} = 1, \quad \text{and} \quad -\frac{\bar{\tau}_q (\bar{c}_0)^2}{\bar{a}^2} + 2 \frac{\bar{c}_0}{\bar{a}} = \bar{H}.$$

Hence, from the second equation, $\frac{\bar{c}_0}{\bar{a}} = -\bar{A}(\bar{c}_0)^2 - \bar{c}_0$ implying that $\bar{A}(\bar{c}_0)^2 + 2\bar{c}_0 + \bar{H} = 0$. The last equation possesses a smooth local solution $c_0(z, y) = \frac{-\bar{H}}{1 + \sqrt{1 - \bar{A}\bar{H}}}$; recall that $\bar{A}(x_0) = 0$. So

$$\bar{S}_q = \left(\frac{\dot{y} - \bar{c}_1}{\bar{b}} \right)^2 - \bar{\tau}_q \left(\frac{\dot{z} - \bar{c}_0}{\bar{a}} \right)^2 - 2 \left(\frac{\dot{z} - \bar{c}_0}{\bar{a}} \right) = 0,$$

where $\bar{a} = \frac{-1}{1 + \bar{A}\bar{c}_0}$, which is well defined since $\bar{A}(x_0) = 0$, and $\bar{\tau}_q = -\bar{A}\bar{a}^2$, proving that \bar{S}_q is of the desired form (5). From (5) we go to Σ_q following the passage already presented in the first part of the proof.

(ii) If $\tau_q < 0$, then $\frac{\partial^3 f_q}{\partial w^3} = \tau_q \frac{\partial f_q}{\partial w}$ implies that $f_q = A \cos(\sqrt{-\tau_q} w) + B \sin(\sqrt{-\tau_q} w) + C$, where A , B , and C depend on x and, clearly, $\frac{\partial^2 f_q}{\partial w^2} \wedge \frac{\partial f_q}{\partial w} \neq 0$ implies $A \wedge B \neq 0$ (for this case, as well as for the next two cases). Via the change of coordinate $\tilde{w} = \sqrt{-\tau_q} w$, we get $f_q = f_E$. If $\tau_q > 0$, then $\frac{\partial^3 f_q}{\partial w^3} = \tau_q \frac{\partial f_q}{\partial w}$ implies that $f_q = A \cosh(\sqrt{\tau_q} w) + B \sinh(\sqrt{\tau_q} w) + C$ which, via the change of coordinate $\tilde{w} = \sqrt{\tau_q} w$, gives $f_q = f_H$. Finally, if $\tau_q \equiv 0$, then $\frac{\partial^3 f_q}{\partial w^3} \equiv 0$ implies that $f_q = f_P = Aw^2 + Bw + C$.

(iii) By item (i), Σ_q is a second extension of a conic submanifold \mathcal{S}_q for which we have $\Delta_2 = -\frac{\tau_q}{a^2 b^2}$. Thus, $\Delta_2 > 0$, resp. $\Delta_2 < 0$, resp. $\Delta_2 \equiv 0$ if and only if $\tau_q < 0$, resp. $\tau_q > 0$, resp. $\tau_q \equiv 0$, which, by item (ii), correspond to Σ_E , resp. Σ_H , resp. Σ_P .

□

Notice that τ_q plays for Σ_q an analogous role to that played by Δ_2 for \mathcal{S}_q ; indeed, we have $\text{sgn}(\Delta_2(x)) = -\text{sgn}(\tau_q(x))$. In particular, the sign of τ_q identifies the subclasses of elliptic, hyperbolic, and parabolic control-affine systems. Moreover, statement (i) shows that every second extension of a quadratic submanifold \mathcal{S}_q is feedback equivalent to a quadratic system Σ_q and the other way around, therefore to obtain a characterisation of quadratic submanifolds, it is crucial to characterise the class of quadratisable systems.

The remaining part of this section is organised as follows. First, we will state our main theorem giving necessary and sufficient conditions characterising the class of quadratic control-affine systems Σ_q . Second, by carefully studying the conditions of that theorem, we will give a normal form of all quadratisable systems.

3.1 Characterisation of quadratisable control-affine systems

We now focus on the feedback equivalence of a general control-affine system $\Sigma : \dot{\xi} = f(\xi) + g(\xi)u$ with a quadratic control-affine system of the form Σ_q . The theorem below gives checkable necessary and

sufficient conditions in terms of the vector fields f and g of Σ for the existence of a smooth feedback transformation (ϕ, α, β) that locally brings Σ into a quadratic system Σ_q . Equivalence to particular subcases of Σ_q , namely elliptic Σ_E , hyperbolic Σ_H , and parabolic Σ_P , is provided by [Corollary 3.4](#) below.

Theorem 3.3 (Feedback quadratisation). *Let $\Sigma = (f, g)$ be a control-affine system on a 3-dimensional smooth manifold with a scalar control. The system Σ is, locally around $\xi_0 \in \mathcal{M}$, feedback equivalent to a quadratic system Σ_q if and only if*

$$(C1) \quad g \wedge \text{ad}_g f \wedge \text{ad}_g^2 f (\xi_0) \neq 0,$$

$$(C2) \quad \text{The structure functions } \rho \text{ and } \tau \text{ in the decomposition } \text{ad}_g^3 f = \rho \text{ad}_g^2 f + \tau \text{ad}_g f \pmod{\mathcal{G}} \text{ satisfy, locally around } \xi_0,$$

$$L_g(\chi) - \frac{2}{3}\rho\chi = 0, \tag{6}$$

$$\text{where } \chi = 3L_g(\rho) - 2\rho^2 - 9\tau.$$

Condition (C1) is a regularity condition, it ensures that the vector fields $g, \text{ad}_g f$, and $\text{ad}_g^2 f$ are locally linearly independent and thus that they form a local frame, hence the structure functions (ρ, τ) of (C2) are well defined. The main idea behind this theorem is to observe that if for Σ we have $\text{ad}_g^3 f = \tau(x) \text{ad}_g f$, modulo $\mathcal{G} = \text{span}\{g\}$, i.e. the third Lie derivative of f along g is proportional to the first Lie derivative of f along g , modulo \mathcal{G} , then with the help of a diffeomorphism we can obtain the form Σ_q , see the sufficiency part of the proof for details. Thus condition (C2) shows how that relation changes when we allow for feedback transformations (α, β) .

Proof. Necessity. Consider the affine control system Σ given by two smooth vector fields f and g and recall that \mathcal{G} is the distribution $\mathcal{G} = \text{span}\{g\}$. Let (α, β) form a control-affine feedback and let ϕ be a diffeomorphism such that Σ is, locally, transformed into Σ_q via ϕ and (α, β) . In coordinates $\tilde{\xi} = \phi(\xi) = (\tilde{x}, \tilde{w}) = (\tilde{z}, \tilde{y}, \tilde{w})$, we denote $\tilde{f}_q = \tilde{f}_q \frac{\partial}{\partial \tilde{x}}$ and $\tilde{g}_q = \frac{\partial}{\partial \tilde{w}}$ the vector fields of Σ_q , $\tilde{\mathcal{G}}$ the distribution spanned by \tilde{g}_q , and $(\tilde{\rho}, \tilde{\tau})$ the structure functions of Σ_q , defined as in (C2). By definition of feedback equivalence the following relations between (f, g) and $(\tilde{f}_q, \tilde{g}_q)$ hold: $\tilde{f}_q = \phi_*(f + \alpha g)$ and $\tilde{g}_q = \phi_*(g\beta)$. The system Σ_q is quadratic, so by [Definition 3.1](#), we have $\frac{\partial^2 \tilde{f}_q}{\partial \tilde{w}^2} \wedge \frac{\partial \tilde{f}_q}{\partial \tilde{w}}(\tilde{x}_0, \tilde{w}_0) \neq 0$, which implies that (C1) holds for Σ_q , and we also have $\frac{\partial^3 \tilde{f}_q}{\partial \tilde{w}^3} = \tilde{\tau}_q \frac{\partial \tilde{f}_q}{\partial \tilde{w}}$, thus we get $\tilde{\rho} = 0$ and $\tilde{\tau} = \tilde{\tau}_q(\tilde{z}, \tilde{y})$. Therefore for Σ_q we have $\tilde{\chi}(\tilde{z}, \tilde{y}) = -9\tilde{\tau}(\tilde{z}, \tilde{y})$ implying $L_{\tilde{g}_q}(\tilde{\chi}) - \frac{2}{3}\tilde{\rho}\tilde{\chi} = \frac{\partial \tilde{\chi}}{\partial \tilde{w}} = 0$. Hence Σ_q satisfies (C1) and (C2) and we will now prove that those conditions are invariant under diffeomorphisms ϕ and feedback transformations (α, β) .

Clearly, (C1) is invariant under diffeomorphisms (as $[\phi_* f, \phi_* g] = \phi_* [f, g]$) and under feedback (α, β) since $\beta \neq 0$. We have checked that (C2) holds for $\Sigma_q = (f_q, \tilde{g}_q)$ and, clearly, (C2) is invariant under diffeomorphisms since they conjugate structure functions. Moreover (C2) is invariant under the transformation $\tilde{f}_q \mapsto \tilde{f}_q + \alpha \tilde{g}_q$, since the expression of $\text{ad}_{\tilde{g}_q}^3 \tilde{f}_q$ is considered modulo the distribution $\tilde{\mathcal{G}}$. Finally, under the action of β the brackets, with $\tilde{g}_q = g\beta$, are transformed by

$$\begin{aligned} \text{ad}_{\tilde{g}_q} f_q &= \beta \text{ad}_g f \pmod{\tilde{\mathcal{G}}}, \\ \text{ad}_{\tilde{g}_q}^2 f_q &= \beta^2 \text{ad}_g^2 f + \beta L_g(\beta) \text{ad}_g f \pmod{\tilde{\mathcal{G}}}, \\ \text{ad}_{\tilde{g}_q}^3 f_q &= (\beta^3 \rho + 3\beta^2 L_g(\beta)) \text{ad}_g^2 f + (\beta^3 \tau + \beta L_g(\beta L_g(\beta))) \text{ad}_g f \pmod{\tilde{\mathcal{G}}}, \\ &= (\rho\beta + 3L_g(\beta)) \text{ad}_{\tilde{g}_q}^2 f + \left(\tau\beta^2 + L_g(\beta L_g(\beta)) - \rho\beta L_g(\beta) - 3(L_g(\beta))^2 \right) \text{ad}_{\tilde{g}_q} f \pmod{\tilde{\mathcal{G}}}. \end{aligned}$$

This implies that the structure functions $\tilde{\rho}$ and $\tilde{\tau}$ of Σ_q defined by $\text{ad}_{\tilde{g}_q}^3 f_q = \tilde{\rho} \text{ad}_{\tilde{g}_q}^2 f_q + \tilde{\tau} \text{ad}_{\tilde{g}_q} f_q \pmod{\tilde{\mathcal{G}}}$ are given in terms of the feedback transformation β and the structure functions ρ and τ of Σ by

$$\tilde{\rho} = \rho\beta + 3L_g(\beta), \quad \tilde{\tau} = \tau\beta^2 + L_g(\beta L_g(\beta)) - \rho\beta L_g(\beta) - 3(L_g(\beta))^2. \tag{7}$$

Since for Σ_q the structure function $\tilde{\rho} = 0$, we have the relation $L_g(\beta) = -\frac{\beta\rho}{3}$ and thus $\tilde{\chi} = -9\tilde{\tau}$, which is equal to

$$\begin{aligned}\tilde{\chi} &= -9 \left(\tau\beta^2 + L_g(\beta L_g(\beta)) \right) = -9 \left(\tau\beta^2 + L_g\left(\frac{-\beta^2\rho}{3}\right) \right), \\ &= -9 \left(\tau\beta^2 - \frac{1}{3}(\rho L_g(\beta^2) + \beta^2 L_g(\rho)) \right) = -9\beta^2 \left(\tau - \frac{1}{3}L_g(\rho) + \frac{2}{9}\rho^2 \right) = \beta^2\chi.\end{aligned}$$

And finally,

$$L_{\tilde{g}_q}(\tilde{\chi}) = \beta L_g(\beta^2\chi) = \beta^3 L_g(\chi) + 2\beta^2\chi L_g(\beta) = \beta^3 L_g(\chi) - \frac{2}{3}\beta^3\chi\rho = 0,$$

showing the necessity of relation (6) and concludes the necessity part of the proof.

Sufficiency. There are two steps in the sufficiency part. The first one consists in building a vector field \tilde{g} such that $\text{ad}_{\tilde{g}}^3 f = \tau \text{ad}_{\tilde{g}} f \pmod{\mathcal{G}}$ with $L_{\tilde{g}}(\tau) = 0$. Then we will construct a diffeomorphism ϕ that brings Σ into the form Σ_q .

Consider the system $\Sigma : \dot{\xi} = f + gu$, for which we assume $g \wedge \text{ad}_g f \wedge \text{ad}_g^2 f(\xi_0) \neq 0$ and suppose that relation (6) holds for the structure functions ρ and τ of Σ . Choose a function $\beta \neq 0$ satisfying $L_g(\beta) = -\frac{\beta\rho}{3}$, which exists locally since $g \neq 0$ by condition (C1); to guarantee that $\beta \neq 0$, we actually may solve the equation $L_g(\ln(\beta)) = -\frac{\rho}{3}$. Define the system $\tilde{\Sigma} : \dot{\xi} = \tilde{f} + \tilde{g}u$, where $\tilde{g} = g\beta$ and $\tilde{f} = f$, then by (7) the structure function $\tilde{\rho}$ of $\tilde{\Sigma}$ vanishes. Therefore, we have $\tilde{\chi} = -9\tilde{\tau}$ and thus relation (6) implies that $L_{\tilde{g}}(\tilde{\tau}) = 0$.

Since $\tilde{g} \neq 0$, we apply a diffeomorphism $(z, y, w) = \phi(\xi)$ such that $\phi_*\tilde{g} = g_q = \frac{\partial}{\partial w}$ and denote $f_q = \phi_*\tilde{f}$, and $\tau_q \circ \phi = \tilde{\tau}$. Therefore, the decomposition $\text{ad}_{g_q}^3 f_q = \tau_q \text{ad}_{g_q} f_q \pmod{\mathcal{G}}$ implies that $f_q = f_q^1 \frac{\partial}{\partial z} + f_q^2 \frac{\partial}{\partial y} + f_q^3 \frac{\partial}{\partial w}$ satisfies

$$\frac{\partial^3 f_q^i}{\partial w^3} = \tau(z, y) \frac{\partial f_q^i}{\partial w}, \quad (8)$$

for $i = 1, 2$. Applying the feedback $u = f_q^3(z, y, w) + \tilde{u}$ we obtain the form Σ_q with $f_q = f_q^1 \frac{\partial}{\partial z} + f_q^2 \frac{\partial}{\partial y}$ and $g_q = \frac{\partial}{\partial w}$. The condition $\frac{\partial^2 f_q}{\partial w^2} \wedge \frac{\partial f_q}{\partial w} \neq 0$ follows from (C1) and feedback invariance of the latter. \square

The following corollary shows that we can test on the structure functions of Σ if the equivalent quadratic system Σ_q will be of elliptic, hyperbolic, or parabolic type.

Corollary 3.4. *Under conditions (C1) and (C2) of the previous theorem we have, locally around ξ_0 ,*

(i) Σ is feedback equivalent to Σ_E if and only if $\chi(\xi_0) > 0$,

(ii) Σ is feedback equivalent to Σ_H if and only if $\chi(\xi_0) < 0$,

(iii) Σ is feedback equivalent to Σ_P if and only if $\chi \equiv 0$ in a neighbourhood of ξ_0 ,

where $\chi = 3L_g(\rho) - 2\rho^2 - 9\tau$.

Notice that Σ is locally feedback equivalent to Σ_P if and only if it satisfies (C1) and $\chi \equiv 0$, condition (C2) being satisfied automatically.

Proof. From the necessity part of the proof of Theorem 3.3 we know that for Σ_q , with structure functions $\rho = 0$ and $\tau = \tau_q$, we have $\chi = -9\tau_q$ and we observed that under pure feedback transformations (α, β) we have $\tilde{\chi} = \beta^2\chi$, thus the sign of χ is invariant as well as the locus where it vanishes. Moreover, by statement (ii) of Proposition 3.2, Σ_q is elliptic if $\tau_q > 0$, equivalently $\chi < 0$, Σ_q is hyperbolic if $\tau_q < 0$, equivalently $\chi > 0$, and Σ_q is parabolic if $\tau_q \equiv 0$, equivalently $\chi \equiv 0$. Hence the necessity of the stated conditions is established.

Conversely, in the sufficiency part of the proof of Theorem 3.3 we obtained Σ with structure functions $(\tilde{\rho}, \tilde{\tau}) = (0, \tilde{\tau}(z, y))$ via a suitable feedback transformations. Since $\tilde{\chi} = \beta^2\chi$, we have $-9\tilde{\tau} = \beta^2\chi$ and thus we get $\text{sgn}(\tilde{\tau}) = -\text{sgn}(\chi)$ and the conclusion follows by statement (ii) of Proposition 3.2. \square

3.2 Normal form of quadratisable control-affine systems

Any control-affine system Σ , under the regularity assumption $g \wedge \text{ad}_g f(\xi_0) \neq 0$, can be written (after applying a suitable feedback transformation) locally around $0 \in \mathbb{R}^3$ as

$$\Sigma_h : \begin{cases} \dot{z} &= h(z, y, w) \\ \dot{y} &= w + \varepsilon \\ \dot{w} &= u \end{cases},$$

with h a smooth function and $\varepsilon = 0$ or 1 . The parameter ε , which is invariant under local feedback transformations, is $\varepsilon = 0$ if either $(f \wedge g \wedge \text{ad}_g f)(\xi_0) \neq 0$ or $(f \wedge g)(\xi_0) = 0$ and $\varepsilon = 1$ otherwise, i.e. $(f \wedge g)(\xi_0) \neq 0$ but $(f \wedge g \wedge \text{ad}_g f)(\xi_0) = 0$. By applying [Theorem 3.3](#), we will give in this subsection a normal form of all smooth functions $h(z, y, w)$ that describe quadratisable systems, that is, control-affine systems feedback equivalent to Σ_q . In what follows, we assume to work locally around $0 \in \mathbb{R}^3$ and all derivatives are taken with respect to w and denoted by prime, double prime, etc. Whenever we apply $\ln(a)$, we assume that $a > 0$ (if not, we take the absolute value).

Theorem 3.5 (Normal form of quadratisable control-affine systems). *The following statements are equivalent, locally around $0 \in \mathbb{R}^3$:*

- (i) Σ_h is feedback equivalent to a quadratic system Σ_q ;
- (ii) The function h satisfies $h''(0) \neq 0$ and, in a neighbourhood, it holds

$$9h^{(5)}(h'')^2 - 45h^{(4)}h^{(3)}h'' + 40(h^{(3)})^3 = 0, \quad (9)$$

recall that the derivatives are taken with respect to w ;

- (iii) The second derivative of h is of the following form

$$h''(x, w) = a(dw^2 + ew + 1)^{-3/2}, \quad (10)$$

where $a = a(x)$, $d = d(x)$, and $e = e(x)$ are smooth functions satisfying $a(0) \neq 0$;

- (iv) The function h is given by

$$h(x, w) = 2a \left(\frac{w^2}{(\sqrt{dw^2 + ew + 1} + 1)^2 - dw^2} \right) + bw + c, \quad (11)$$

where a, b, c, d, e are any smooth functions of x such that $a(0) \neq 0$.

Proof. (i) \Rightarrow (ii). It is a straightforward application of the conditions of [Theorem 3.3](#) with the structure functions of Σ_h given by $\rho = \frac{h^{(3)}}{h''}$ and $\tau = 0$ yielding $\chi = 3\rho' - 2\rho^2$. By (C1), we have $h''(0) \neq 0$ and then condition (C2) reads

$$\chi' - \frac{2}{3}\rho\chi = 3\rho'' - 6\rho\rho' + \frac{4}{3}\rho^3 = 0, \quad (12)$$

which, by plugging $\rho = \frac{h^{(3)}}{h''}$ into the last equation, gives (9).

(ii) \Rightarrow (iii). Assume that h satisfies $h''(0) \neq 0$ and (9). Set $\rho = \frac{h^{(3)}}{h''}$, then $\rho = \rho(x, w)$ fulfils $3\rho'' - 6\rho\rho' + \frac{4}{3}\rho^3 = 0$, namely, the second equation of (12). By a change of variable, it is easy to obtain that the solutions of (12) are of the following form (see [Appendix B.1](#) for the proof)

$$\rho(x, w) = -\frac{3}{2} \frac{2d(x)w + e(x)}{d(x)w^2 + e(x)w + 1}. \quad (13)$$

This form can be integrated, using $\rho = \frac{h^{(3)}}{h''} = (\ln h'')$, into $h''(x, w) = a(x) (d(x)w^2 + e(x)w + 1)^{-3/2}$ with a, d , and e any smooth functions such that $a(0) \neq 0$.

(iii) \Rightarrow (iv). To show (11), we integrate twice the second derivative of h given by (10). Denote $p =$

$p(x, w) = d(x)w^2 + e(x)w + 1$ and $\Delta = \Delta(x) = e(x)^2 - 4d(x)$. First, we obtain (see [Appendix B.2](#) for details)

$$h'(x, w) = \frac{2aw(\sqrt{p} + 1)}{\sqrt{p}(ew + 2 + 2\sqrt{p})} + b,$$

with b an arbitrary smooth function of x . Integrate once more to get

$$\begin{aligned} h(x, w) &= \frac{2a}{\Delta\sqrt{p}}(ew\sqrt{p} - 2p) + \frac{4a}{\Delta} + bw + c \\ &= \frac{2a}{\Delta}(ew + 2 - 2\sqrt{p}) + bw + c \\ &= \frac{2aw^2}{ew + 2 + 2\sqrt{p}} + bw + c \\ &= \frac{2aw^2}{(\sqrt{p} + 1)^2 - dw^2} + bw + c. \end{aligned}$$

(iv) \Rightarrow (i). Given Σ_h with h defined by (11) we will construct a feedback transformation that brings the system into Σ_q . First, we introduce coordinates, centred at $0 \in \mathbb{R}^2$, $(\tilde{z}, \tilde{y}) = \phi(z, y)$, where $\tilde{y} = y$, such that $\phi_* \left(b \frac{\partial}{\partial z} + \frac{\partial}{\partial y} \right) = \frac{\partial}{\partial \tilde{y}}$. Those coordinates transform the system Σ_h into

$$\begin{cases} \dot{\tilde{z}} &= 2\tilde{a} \frac{w^2}{(\sqrt{\tilde{p}+1})^2 - \tilde{d}w^2} + \tilde{c} \\ \dot{\tilde{y}} &= w + \varepsilon \\ \dot{w} &= u \end{cases},$$

where \tilde{a} , \tilde{c} , \tilde{d} , and \tilde{p} are new functions satisfying $\tilde{a}(0) \neq 0$ and $d = \tilde{d} \circ \phi$ and $p = \tilde{p} \circ \phi$. Second, we set $\tilde{w}^2 = \frac{w^2}{(\sqrt{\tilde{p}+1})^2 - \tilde{d}w^2}$ or, equivalently, $w = \tilde{w} \left(\tilde{c}\tilde{w} \pm 2\sqrt{\tilde{d}\tilde{w}^2 + 1} \right)$, which brings the above system into (after applying a suitable feedback along the last component)

$$\tilde{\Sigma}_h : \begin{cases} \dot{\tilde{z}} &= 2\tilde{a}\tilde{w}^2 + \tilde{c} \\ \dot{\tilde{y}} &= \tilde{c}\tilde{w}^2 \pm 2\tilde{w}\sqrt{\tilde{d}\tilde{w}^2 + 1} + \varepsilon \\ \dot{\tilde{w}} &= \tilde{u} \end{cases}. \quad (14)$$

The structure functions of $\tilde{\Sigma}_h$ are given by $\tilde{\rho} = \frac{-3\tilde{w}\tilde{d}}{\tilde{d}\tilde{w}^2 + 1}$ and $\tilde{\tau} = \frac{3\tilde{d}}{\tilde{d}\tilde{w}^2 + 1}$, then apply the feedback $\tilde{u} = \beta\bar{u}$, where $\beta(\tilde{x}, \tilde{w}) = \sqrt{\tilde{d}\tilde{w}^2 + 1}$ (it is a solution of the equation $\frac{\partial\beta}{\partial\tilde{w}} = -\frac{\tilde{p}\beta}{3}$), to obtain a new vector field $\bar{g} = \beta \frac{\partial}{\partial \tilde{w}}$ and structure functions of (\bar{f}, \bar{g}) , where \bar{f} is the drift of $\tilde{\Sigma}_h$, are given by $\bar{\rho} = 0$ and $\bar{\tau} = 4\tilde{d}(x)$. To complete the form, it remains to find a new $\bar{w} = \psi(\tilde{x}, \tilde{w})$ such that the diffeomorphism $(\bar{x}, \bar{w}) = \Psi(\tilde{x}, \tilde{w}) = (\tilde{x}, \psi(\tilde{x}, \tilde{w}))$ satisfies $\Psi_*\bar{g} = \Psi_*\beta \frac{\partial}{\partial \tilde{w}} = \frac{\partial}{\partial \bar{w}}$. In general, ψ is given in terms of the following integral

$$\bar{w} = \psi(\tilde{x}, \tilde{w}) = \int_0^{\tilde{w}} \frac{1}{\sqrt{\tilde{d}\tilde{w}^2 + 1}} d\tilde{w}$$

and, together with a suitable feedback along $\dot{\tilde{w}}$, thus provides a quadratic system Σ_q . \square

Remark. *This theorem provides a normal form of submanifolds $\mathcal{S} = \{\dot{z} = s(x, \dot{y})\}$ that are equivalent to a conic submanifold \mathcal{S}_q , namely, they are represented by $s(x, \dot{y}) = h(x, \dot{y})$ with h as in (11). Moreover, system (14) leads to a normal form for all conic submanifolds \mathcal{S}_q (even for those that smoothly pass through $\Delta_2(x) = 0$, i.e. from the elliptic to the hyperbolic submanifolds) and thus completes the characterisation of [Lemma 2.5](#). Indeed, for (14) we have (tildes have been removed for more readability)*

$$S_q(x, \dot{x}) = [e(\dot{z} - c) - 2a(\dot{y} - \varepsilon)]^2 - 4d(\dot{z} - c)^2 - 8a(\dot{z} - c),$$

for which $a \neq 0$ and we can compute $\Delta_1 = -64a^4 \neq 0$ and $\Delta_2 = -16a^2d$. Notice that the above expression of $S_q = 0$ can be normalised to the form (5) thus giving another proof of the latter. Indeed,

observe that the differential 1-form $\omega = edz - 2ady$ satisfies $d\omega \wedge \omega \equiv 0$, since $e = e(z, y)$ and $a = a(z, y)$, and thus there exists a nonvanishing function $\tilde{b}(z, y)$ such that $\tilde{\omega} = \frac{1}{\tilde{b}}\omega$ is exact. Thus, $\tilde{\omega} = d\tilde{y}$ for a smooth function $\tilde{y} = \psi(z, y)$, where $\frac{\partial \psi}{\partial y} \neq 0$ (since $a(x_0) \neq 0$). Therefore the first term of S_q reads $[e(\dot{z} - c) - 2a(\dot{y} - \varepsilon)]^2 = (\tilde{b}\dot{\tilde{y}} - \tilde{c})^2$, where $\tilde{c} = ec - 2a\varepsilon$, and we can normalise $S_q = 0$ to the form (5) (recall that $a \neq 0$ and $\tilde{b} \neq 0$).

In the last item of the proof we saw that the function d plays an import role for the shape of the transformation, in the following corollary we show that this function is the key of the normal form of quadratisable control systems.

Corollary 3.6. *Assume that Σ_h is given by h of the form (11). Then Σ is feedback equivalent to Σ_P , resp. Σ_E , resp. Σ_H if and only if $d \equiv 0$, resp. $d < 0$, resp. $d > 0$. Moreover, the normalizing feedback transformation is given by*

$$\begin{aligned} \bar{w} &= \frac{w}{1 + \sqrt{ew + 1}} \quad \text{for } \Sigma_P, \\ \text{resp. } \sin^2(\sqrt{-d}\bar{w}) &= \frac{-dw^2}{ew + 2 + 2\sqrt{p}} \quad \text{for } \Sigma_E, \\ \text{resp. } \sinh^2(\sqrt{d}\bar{w}) &= \frac{dw^2}{ew + 2 + 2\sqrt{p}} \quad \text{for } \Sigma_H, \end{aligned}$$

where $p = p(x, w) = d(x)w^2 + e(x)w + 1$.

Proof. First we show that Σ_h is feedback equivalent to Σ_P , resp. Σ_E , resp. Σ_H , if and only if $d \equiv 0$, resp. $d < 0$, and resp. $d > 0$. From Corollary 3.4 we know that we have to compute the sign of χ , which is given by $\chi = -9\frac{d}{p}$ for Σ_h (this can easily be deduced from the expression of ρ given by (13)). Since $p(0) = 1 > 0$ we have $\text{sgn}(\chi) = -\text{sgn}(d)$ and thus the conclusion follows.

We now show how to explicitly transform Σ_h into Σ_P , resp. Σ_E , resp. Σ_H . From the last part of the proof of the previous theorem we know that a suitable parametrisation \bar{w} is given by the following two steps

$$\bar{w}^2 = \frac{w^2}{ew + 2 + 2\sqrt{p}}, \quad \text{and} \quad \bar{w} = \int_0^{\bar{w}} \frac{1}{\sqrt{d\bar{w}^2 + 1}} d\bar{w}.$$

Assume $d \equiv 0$, then the procedure reduces to the first step only and thus $\bar{w}^2 = \tilde{w}^2 = \frac{w^2}{ew + 2 + 2\sqrt{ew + 1}} = \left(\frac{w}{1 + \sqrt{ew + 1}}\right)^2$ and we choose $\tilde{w} = \frac{w}{1 + \sqrt{ew + 1}}$. Assume $d < 0$, then the second step of the procedure leads to $\bar{w} = \frac{1}{\sqrt{-d}} \arcsin(\sqrt{-d}\tilde{w})$. Hence a reparametrisation is given by $\sin^2(\sqrt{-d}\bar{w}) = \frac{-d\tilde{w}^2}{ew + 2 + 2\sqrt{p}}$. Assume $d > 0$, then from the second step of the procedure we have $\bar{w} = \frac{1}{\sqrt{d}} \operatorname{arcsinh}(\sqrt{d}\tilde{w})$. Hence a reparametrisation is given by $\sinh^2(\sqrt{d}\bar{w}) = \frac{d\tilde{w}^2}{ew + 2 + 2\sqrt{p}}$. \square

Remark (Interpretation of parametrising functions). *In the normal form (11), there are 5 parametrising functions¹. However, only $d = d(x)$ and $e = e(x)$ play a significant role in the shape of the submanifold \mathcal{S}_q . Indeed, a is a scaling of the submanifold, c is the value of h at $w = 0$, and by an appropriate choice of coordinates (as in (14)) we can always assume that $b \equiv 0$. From the above corollary, the role of d is clear: its sign around $x_0 = 0 \in \mathbb{R}^2$ determines the nature of the submanifold, that is, whether the submanifold is elliptic, hyperbolic, or parabolic.*

The role of the function e is, however, more subtle. Clearly, h is well defined whenever $p > 0$ and, for a given d , the function e determines the region in which $p > 0$ (in particular, whether h is defined globally with respect to w or not). If $d \equiv 0$, then $p > 0$ holds everywhere (h is defined globally) if and only if $e \equiv 0$ that is, h is explicitly given by $h = 2aw^2 + bw + c$. If $d < 0$, then we have $p > 0$ only between its roots and the parametrisation is never global. Finally, if $d > 0$ then the parametrisation is global if and only if $\Delta < 0$ (where Δ is the discriminant of $p = 0$), that is $|e| < 2\sqrt{d}$.

¹On <https://www.geogebra.org/m/tyb4yggpb> the reader can play with those parameters (the functions a, b, c, d , and e become real numbers when fixing $x \in \mathcal{X}$).

4 Classification of quadratic systems

Notations.

$\Xi_Q = \{\Xi_{EH}, \Xi_P\}, \Xi_{EH} = \{\Xi_E, \Xi_H\}$	First extension of elliptic, hyperbolic, parabolic submanifolds, seen as a control-nonlinear system on \mathcal{X} .
$\Xi_Q = (A, B, C)$	Triple of vector fields attached to a first extension of a quadratic submanifold.
(α, β)	Reparametrisation (feedback) acting on quadratic nonlinear systems, given by Proposition 4.2 .
$(\mu_0, \mu_1), \gamma = (\gamma_0, \gamma_1)$	Structure functions attached to (A, B, C) by $[A, B] = \mu_0 A + \mu_1 B$ and $C = \gamma_0 A + \gamma_1 B$; see (18) .
\mathcal{L}	Notation for $\mathcal{L} = (\gamma_1, \mp \gamma_0)$, used in the elliptic and hyperbolic cases.
$\Gamma_E, \Gamma_H, \Gamma_P$	Functions given by $\Gamma_E = (\gamma_0)^2 + (\gamma_1)^2$, $\Gamma_H = (\gamma_0)^2 - (\gamma_1)^2$, and $\Gamma_P = \gamma_0 + (\gamma_1)^2$.
$\mathfrak{g}_\pm, \kappa_\pm$	A (pseudo-)Riemannian metric defined by $\mathfrak{g}_\pm(A, A) = 1$, $\mathfrak{g}_\pm(B, B) = \pm 1$, and $\mathfrak{g}_\pm(A, B) = 0$, and κ_\pm its Gaussian curvature; see (21) .

From [Theorem 3.3](#) we know how to characterise control-affine system equivalent to the quadratic form Σ_q and, in particular, we know how to characterise the subclasses of elliptic, hyperbolic, and parabolic systems (see [Corollary 3.4](#)). We are now interested in classifying, under feedback transformations, the systems inside those three subclasses. Indeed, due to [Proposition 2.4](#), the proposed classification of those classes provides an equivalent classification of the elliptic, hyperbolic, and parabolic submanifolds (see [Lemma 4.1](#) below). To this end, we consider the quadratic nonlinear system

$$\Xi_Q : \dot{x} = \mathfrak{f}_Q(x, w),$$

where $x \in \mathcal{X}$ is the 2-dimensional state, $w \in \mathbb{R}$ plays the role of a control that enters in a nonlinear way and \mathfrak{f}_Q is a w -parameterised vector field on \mathcal{X} given by either

$$\begin{aligned} \mathfrak{f}_E &= A(x) \cos(w) + B(x) \sin(w) + C(x), & \text{defining } \Xi_E, & \text{ or} \\ \mathfrak{f}_H &= A(x) \cosh(w) + B(x) \sinh(w) + C(x), & \text{defining } \Xi_H, & \text{ or} \\ \mathfrak{f}_P &= A(x)w^2 + B(x)w + C(x), & \text{defining } \Xi_P. & \end{aligned}$$

In each of the three cases, A , B , and C are smooth vector fields on \mathcal{X} satisfying $(A \wedge B)(x_0) \neq 0$. We call Ξ_E an elliptic system, Ξ_H a hyperbolic system, and Ξ_P a parabolic system, because in each fiber $T_x \mathcal{X}$, the system Ξ_E , resp. Ξ_H , resp. Ξ_P parametrises an ellipse, resp. a hyperbola, resp. a parabola. A quadratic nonlinear system Ξ_Q is then represented by the triple (A, B, C) of three smooth vector fields satisfying $A \wedge B \neq 0$. We call the pair (A, B) a Q -frame, and if additionally $[A, B] = 0$, then we call (A, B) a *commutative* Q -frame. We will denote by the index EH objects attached to either the elliptic or the hyperbolic case, as those two are treated in a similar manner.

For quadratic submanifolds, elliptic and hyperbolic, of the form $\mathcal{S}_{EH} = \{a^2(\dot{z} - c_0)^2 \pm b^2(\dot{y} - c_1)^2 = 1\}$ and parabolic of the form $\mathcal{S}_P = \{a\dot{y}^2 - \dot{z} + b\dot{y} + c = 0\}$, we distinguished specific classes (conformally-flat, flat, constant and null forms for elliptic and hyperbolic submanifolds; weakly and strongly flat, constant and null forms for parabolic submanifolds) that are presented in [Table 1](#) of the Introduction. Our goal is to characterise those types of quadratic submanifolds and we show in the next lemma that the classification of elliptic, hyperbolic, and parabolic submanifolds \mathcal{S}_Q presented in [Table 1](#) is reflected in properties of the control system $\Xi_Q = (A, B, C)$.

Lemma 4.1. *Consider a quadratic submanifold \mathcal{S}_Q together with its regular parametrisation $\Xi_Q = (A, B, C)$.*

- (i) \mathcal{S}_{EH} is locally equivalent to a conformally-flat elliptic/hyperbolic submanifold if and only if Ξ_{EH} is locally feedback equivalent to Ξ_{EH} , whose EH -frame (A, B) is given by $A = r(x) \frac{\partial}{\partial z}$ and $B = r(x) \frac{\partial}{\partial y}$ for some nonvanishing function $r(x)$.

- (ii) \mathcal{S}_{EH} is locally equivalent to a flat elliptic/hyperbolic submanifold if and only if Ξ_{EH} is locally feedback equivalent to Ξ_{EH} , whose frame (A, B) is commutative.
- (iii) \mathcal{S}_{EH} is locally equivalent to a constant-form elliptic/hyperbolic submanifold if and only if Ξ_{EH} is locally feedback equivalent to Ξ_{EH} , whose EH -frame (A, B) is commutative and, additionally, $[A, C] = [B, C] = 0$.
- (iv) \mathcal{S}_{EH} is locally equivalent to a null-form elliptic/hyperbolic submanifold if and only if Ξ_{EH} is locally feedback equivalent to Ξ_{EH} , whose EH -frame (A, B) is commutative and, additionally, $C = 0$.
- (v) \mathcal{S}_P is locally equivalent to a weakly-flat parabolic submanifold if and only if Ξ_P is locally feedback equivalent to Ξ_P , whose P -frame (A, B) is commutative.
- (vi) \mathcal{S}_P is locally equivalent to a strongly-flat parabolic submanifold if and only if Ξ_P is locally feedback equivalent to Ξ_P , whose P -frame (A, B) is commutative and, additionally, $A \wedge C = 0$.
- (vii) \mathcal{S}_P is locally equivalent to a constant-form parabolic submanifold if and only if Ξ_P is locally feedback equivalent to Ξ_P , whose P -frame (A, B) is commutative and, additionally, $[A, C] = [B, C] = 0$.
- (viii) \mathcal{S}_P is locally equivalent to a null-form parabolic submanifold if and only if Ξ_P is locally feedback equivalent to Ξ_P , whose P -frame (A, B) is commutative and, additionally, $C = 0$.

Recall that a general conic submanifold \mathcal{S}_q is equivalent to an elliptic \mathcal{S}_E , resp. a hyperbolic \mathcal{S}_H , resp. a parabolic \mathcal{S}_P , submanifold if and only the determinant Δ_2 satisfies $\Delta_2 > 0$, resp. $\Delta_2 < 0$, resp. $\Delta_2 \equiv 0$; see [Lemma 2.5](#). Therefore the above lemma allows to check equivalence of \mathcal{S}_q to a submanifold of any of the subclasses listed in [Table 1](#).

Proof. It is a straightforward computation to check that for the submanifolds \mathcal{S}_Q of the indicated forms, their first extensions $\Xi_Q = (A, B, C)$ have the triple (A, B, C) or the Q -frame (A, B) satisfying the stated conditions. Conversely, in local coordinates $x = (z, y)$ in which either $A = r(x) \frac{\partial}{\partial z}$ and $B = r(x) \frac{\partial}{\partial y}$, for (i), or $A = \frac{\partial}{\partial z}$ and $B = \frac{\partial}{\partial y}$, for (ii) to (viii), the system Ξ_Q is a regular parametrisation of \mathcal{S}_Q with the desired properties. Now, all items (i) to (viii) follow from [Proposition 2.4](#). \square

The above lemma asserts that to achieve the classification of elliptic, hyperbolic, and parabolic submanifolds presented in [Table 1](#), it is crucial to classify, under feedback transformations, quadratic control systems $\Xi_Q = (A, B, C)$ with the properties presented in [Table 2](#), which will be the goal of the remaining part of this section. In particular, we will show that the characterisation of item (i), resp. item (v), is always satisfied by any Ξ_{EH} and thus by the corresponding \mathcal{S}_{EH} , resp. by any Ξ_P and thus by the corresponding \mathcal{S}_P , while the characterisations of the remaining classes of systems, and thus of the corresponding submanifolds, require non-trivial conditions.

Although the systems of the form Ξ_Q are nonlinear with respect to the control w , the feedback transformations that preserve this class are not as general as possible. Indeed, feedback transformations which preserve the class of quadratic system Ξ_Q are *affine* (and even of Brockett type, in the case of elliptic and hyperbolic systems) with respect to the control w , as ensured by the next proposition, which also shows how feedback acts on the triple (A, B, C) .

Elliptic and hyperbolic classification		Parabolic classification	
conformally-flat	$A = r(x) \frac{\partial}{\partial z}$ and $B = r(x) \frac{\partial}{\partial y}$	weakly-flat	$[A, B] = 0$
flat	$[A, B] = 0$	strongly-flat	$[A, B] = 0$ and $A \wedge C = 0$
constant-form	$[A, B] = [A, C] = [B, C] = 0$,	constant-form	$[A, B] = [A, C] = [B, C] = 0$
null-form	$[A, B] = 0$ and $C = 0$	null-form	$[A, B] = 0$ and $C = 0$

Table 2: Reflection of the classification of elliptic, hyperbolic, and parabolic submanifolds in properties of the triple $\Xi_Q = (A, B, C)$.

Proposition 4.2 (Reparametrisation of quadratic nonlinear systems). *Consider two quadratic systems Ξ_Q and $\tilde{\Xi}_Q$ around (x_0, w_0) and $(\tilde{x}_0, \tilde{w}_0)$, respectively.*

- (i) *Two elliptic systems Ξ_E and $\tilde{\Xi}_E$ are locally feedback equivalent if and only if there exists a local diffeomorphism $\tilde{x} = \phi(x)$ and a reparametrisation (feedback) $w = \psi(x, \tilde{w})$, given by $\psi = \pm\tilde{w} + \alpha(x)$, satisfying*

$$\tilde{A} = \phi_*(A \cos \alpha + B \sin \alpha), \quad \tilde{B} = \pm\phi_*(-A \sin \alpha + B \cos \alpha), \quad \tilde{C} = \phi_*(C). \quad (15)$$

- (ii) *Two hyperbolic systems Ξ_H and $\tilde{\Xi}_H$ are locally feedback equivalent if and only if there exists a local diffeomorphism $\tilde{x} = \phi(x)$ and a reparametrisation (feedback) $w = \psi(x, \tilde{w})$, given by $\psi = \pm\tilde{w} + \alpha(x)$, satisfying*

$$\tilde{A} = \phi_*(A \cosh \alpha + B \sinh \alpha), \quad \tilde{B} = \pm\phi_*(A \sinh \alpha + B \cosh \alpha), \quad \tilde{C} = \phi_*(C). \quad (16)$$

- (iii) *Two parabolic systems Ξ_P and $\tilde{\Xi}_P$ are locally feedback equivalent if and only if there exists a local diffeomorphism $\tilde{x} = \phi(x)$ and an invertible reparametrisation (feedback) $w = \psi(x, \tilde{w})$, given by $\psi = \alpha(x) + \beta(x)\tilde{w}$ and $\beta(\cdot) \neq 0$, satisfying*

$$\tilde{A} = \phi_*(A\beta^2), \quad \tilde{B} = \phi_*(2A\alpha\beta + B\beta), \quad \tilde{C} = \phi_*(C + A\alpha^2 + B\alpha). \quad (17)$$

Proof. We show the necessity of each statement as the converse implications are immediate.

- (i) Assume that Ξ_E and $\tilde{\Xi}_E$ are locally equivalent via a diffeomorphism $\tilde{x} = \phi(x)$ and a reparametrisation $w = \psi(x, \tilde{w})$. Then we have the following relation $\phi_*f_E(x, \psi(x, \tilde{w})) = \tilde{f}_E(\tilde{x}, \tilde{w})$, which we differentiate 3 times with respect to \tilde{w} and using $\frac{\partial^3 f_E}{\partial \tilde{w}^3} = -\frac{\partial \tilde{f}_E}{\partial \tilde{w}}$, we conclude the relation $\phi_*\frac{\partial^3}{\partial \tilde{w}^3}f_E = -\phi_*\frac{\partial}{\partial \tilde{w}}f_E$, which translates into

$$\begin{aligned} & A(-\psi''' \sin(\psi) + (\psi')^3 \sin(\psi) - 3\psi'\psi'' \cos(\psi)) \\ & + B(\psi''' \cos(\psi) - (\psi')^3 \cos(\psi) - 3\psi'\psi'' \sin(\psi)) = A\psi' \sin(\psi) - B\psi' \cos(\psi), \end{aligned}$$

where the derivatives are taken with respect to \tilde{w} . Since the functions \cos and \sin are linearly independent, we obtain $\psi''' = 0$ and $(\psi')^2 = 1$. Thus $\psi(x, \tilde{w}) = \pm\tilde{w} + \alpha(x)$. Applying this reparametrisation to $\Xi_E = (A, B, C)$ we obtain the relations of (15).

- (ii) Exactly the same reasoning, using f_H and the fact $\phi_*\frac{\partial^3}{\partial \tilde{w}^3}f_H = \phi_*\frac{\partial}{\partial \tilde{w}}f_H$, implies that $\psi(x, \tilde{w}) = \pm\tilde{w} + \alpha(x)$. Applying $\phi(x)$ and $w = \psi(x, \tilde{w}) = \pm\tilde{w} + \alpha$ to $\Xi_H = (A, B, C)$ we obtain the relations of (16).

- (iii) We repeat again the same reasoning to f_P with the property $\phi_*\frac{\partial^3}{\partial \tilde{w}^3}f_P = 0$. However, this time we obtain the conditions $\psi''' = 0$ and $\psi\psi''' + 3\psi'\psi'' = 0$ on the reparametrisation ψ , which implies $\psi'' = 0$, that is, $\psi(x, \tilde{w}) = \beta(x)\tilde{w} + \alpha(x)$, with β satisfying $\beta(\cdot) \neq 0$. Applying this reparametrisation together with a diffeomorphism ϕ yields the relations of (17). □

Remark (Local character of the results). *Initially, Ξ_Q was considered locally around a point x_0 and a control w_0 , however, since Ξ_Q is defined globally with respect to w and, moreover, by the last proposition, the transformations $w = \psi(x, \tilde{w})$ are global with respect to w , so we will consider the systems Ξ_Q and their equivalence locally in x and globally with respect to w . All results below are stated assuming this structure.*

We will develop relations involving structure functions attached to any fixed triple (A, B, C) in a unique way and thus change accordingly with diffeomorphisms $\tilde{x} = \phi(x)$. So we will act on (A, B, C) by (α, β) only (β is ± 1 in the elliptic and hyperbolic cases) and we will denote by $(\tilde{A}, \tilde{B}, \tilde{C})$ the result of that action (given by (15), or (16), or (17), with $\phi = \text{id}$), called a reparametrisation.

Observe that the reparametrisations of Ξ_P depend on two smooth functions α and β while those of Ξ_E and Ξ_H depend on one smooth function α only. Therefore, we expect the classification of parabolic systems to be less rich (less parametrising functions) than the classification of elliptic and hyperbolic systems. In the following subsections we will first classify elliptic and hyperbolic systems as the procedures are similar, and then we will classify parabolic systems under reparametrisation actions.

4.1 Classification of elliptic and hyperbolic systems

In this subsection, we classify elliptic and hyperbolic systems under the action of reparametrisations. Recall, that our aim is to classify elliptic and hyperbolic nonholonomic constraints \mathcal{S}_E and \mathcal{S}_H (submanifolds of $T\mathcal{X}$), which are parameterised by systems of the form Ξ_E and Ξ_H , respectively. The classification of submanifolds given in [Table 1](#) of the Introduction is reflected in special properties of the vector fields (A, B, C) , attached to the control system Ξ_E and Ξ_H , that we list in [Table 2](#) above and summarise in [Lemma 4.1](#). Firstly, we will give a normal form for both types of systems Ξ_E and Ξ_H showing that they actually depend on three smooth functions, that normal form corresponds to conformally-flat elliptic and hyperbolic submanifolds. Secondly, we will further develop their classification, in particular we will give conditions for the existence of commutative frames (corresponding to flat elliptic and hyperbolic submanifolds) and a complete characterisation of forms without functional parameters, corresponding to constant-form (and, in particular, null-form) elliptic and hyperbolic submanifolds.

Notations. In order to simplify and unify notations, in the following formulae the upper sign always corresponds to the elliptic case and the lower sign to the hyperbolic case, e.g. we will use the symbol \pm to design similar objects attached to the elliptic (+ case) and to the hyperbolic (− case) systems and in the case of a \mp symbol we have − for elliptic systems and + for hyperbolic ones. We denote Ξ_{EH} elliptic and hyperbolic systems, and an EH-frame stands for an E-frame or an H-frame. To avoid unnecessary computations, we assume that EH-frames (A, B) and (\tilde{A}, \tilde{B}) of two equivalent systems have the same orientation (we will come back to this simplification in [Proposition 4.6](#)), therefore we restrict reparametrisations of the control to $w = \tilde{w} + \alpha(x)$ thus resulting in the "+" sign in [\(15\)](#) and [\(16\)](#). Denoting by $\bar{R}_{EH}(\alpha)$ the (trigonometric or hyperbolic) rotation matrix given by

$$\bar{R}_E(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \quad \text{and} \quad \bar{R}_H(\alpha) = \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{pmatrix},$$

respectively, we see from [\(15\)](#) and [\(16\)](#) that EH-frames are transformed by $(\tilde{A}, \tilde{B}) = (A, B)\bar{R}_{EH}(\pm\alpha)$ under reparametrisations of the form $w = \tilde{w} + \alpha$. Introduce structure functions (μ_0, μ_1) and (γ_0, γ_1) uniquely defined by

$$[A, B] = \mu_0 A + \mu_1 B \quad \text{and} \quad C = \gamma_0 A + \gamma_1 B, \quad (18)$$

respectively. We denote $\gamma = (\gamma_0, \gamma_1)$, and $\mathcal{L} = (\gamma_1, \mp\gamma_0)$, and set $\Gamma_{EH} = (\gamma_0)^2 \pm (\gamma_1)^2$.

We begin by a technical lemma showing how structure functions behave under reparametrisations of the control w .

Lemma 4.3 (Transformation of structure functions). *Consider an elliptic/hyperbolic system Ξ_{EH} with structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$. Then under the reparametrisation $w = \tilde{w} + \alpha(x)$ we have*

$$(\tilde{\mu}_0, \tilde{\mu}_1) = (\mu_0 \mp L_A(\alpha), \mu_1 - L_B(\alpha))\bar{R}_{EH}(\alpha), \quad \text{and} \quad \tilde{\gamma} = \gamma\bar{R}_{EH}(\alpha). \quad (19)$$

Proof. Details of the computations can be found in [Appendix C](#). □

Clearly, from [\(19\)](#), $\Gamma_{EH} = (\gamma_0)^2 \pm (\gamma_1)^2$ is invariant under reparametrisations, i.e. $\tilde{\Gamma}_{EH} = \Gamma_{EH}$.

Proposition 4.4 (Conformal form of elliptic and hyperbolic systems).

(i) *Any elliptic system Ξ_E , resp. hyperbolic system Ξ_H , always admits under a reparametrisation of the following conformal form, locally around x_0 ,*

$$\Xi_E^c : \dot{x} = r(x) \begin{pmatrix} \cos(w) \\ \sin(w) \end{pmatrix} + \begin{pmatrix} c_0(x) \\ c_1(x) \end{pmatrix}, \quad \text{resp.} \quad \Xi_H^c : \dot{x} = r(x) \begin{pmatrix} \cosh(w) \\ \sinh(w) \end{pmatrix} + \begin{pmatrix} c_0(x) \\ c_1(x) \end{pmatrix},$$

with r a smooth function satisfying $r > 0$.

(ii) Two conformal forms Ξ_E^c and $\tilde{\Xi}_E^c$, resp. Ξ_H^c and $\tilde{\Xi}_H^c$, are locally feedback equivalent if and only if there exists a local diffeomorphism $\tilde{x} = \phi(x) = (\phi_1(x), \phi_2(x))$, where $x = (z, y)$, satisfying

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial y}, \quad \frac{\partial \phi_1}{\partial y} = \mp \frac{\partial \phi_2}{\partial z}, \quad \left(\frac{\partial \phi_1}{\partial z} \right)^2 \pm \left(\frac{\partial \phi_1}{\partial y} \right)^2 = \left(\frac{\tilde{r}}{r} \right)^2, \quad \text{and} \quad \phi_* C = \tilde{C}, \quad (20)$$

where \tilde{r} is expressed in x -coordinates, i.e. we set $\tilde{r}(\phi(x))$.

We call Ξ_{EH}^c a conformal-form because the systems of that class parametrise elliptic and hyperbolic submanifolds for which the quadratic term, interpreted as a (pseudo-)Riemannian metric, is conformally flat.

Proof.

(i) For the system $\Xi_{EH} = (A, B, C)$, define a (pseudo-)Riemannian metric \mathbf{g}_\pm on \mathcal{X} by $\mathbf{g}_\pm(A, A) = 1$, $\mathbf{g}_\pm(B, B) = \pm 1$, and $\mathbf{g}_\pm(A, B) = 0$. It is known that any non-degenerate metric on a manifold of dimension two is conformally flat (see [3, pp 15-35] or [26, Addendum 1 of chapter 9] for the elliptic case and [25, theorem 7.2] for the hyperbolic case). Therefore, there exists a local diffeomorphism $(\tilde{z}, \tilde{y}) = \tilde{x} = \phi(x)$ such that $\mathbf{g}_\pm = \phi^* \tilde{\mathbf{g}}_\pm$, where $\tilde{\mathbf{g}}_\pm = \varrho(\tilde{x}) (d\tilde{z}^2 \pm d\tilde{y}^2)$, $\varrho > 0$. The vector fields $\tilde{A} = \phi_* A$ and $\tilde{B} = \phi_* B$ satisfy $\tilde{\mathbf{g}}_\pm(\tilde{A}, \tilde{A}) = 1$, $\tilde{\mathbf{g}}_\pm(\tilde{B}, \tilde{B}) = \pm 1$, and $\tilde{\mathbf{g}}_\pm(\tilde{A}, \tilde{B}) = 0$ which implies that (\tilde{A}, \tilde{B}) is a (pseudo-)orthonormal frame. Finally, using the feedback $w \mapsto \tilde{w} - \alpha$ we can smoothly rotate (\tilde{A}, \tilde{B}) into $(r \frac{\partial}{\partial \tilde{z}}, r \frac{\partial}{\partial \tilde{y}})$ with $r = \frac{1}{\sqrt{\varrho}}$, which gives the desired form $\Xi_{EH}^c = (\tilde{A}, \tilde{B}, \tilde{C})$, with $\tilde{C} = \phi_* C$.

(ii) By relations (15) and (16), the reparametrisations do not act on C and thus the relation $\tilde{C} = \phi_* C$ is necessary for the equivalence of conformal forms. Consider two elliptic conformal systems Ξ_E^c and $\tilde{\Xi}_E^c$, resp. two hyperbolic conformal systems Ξ_H^c and $\tilde{\Xi}_H^c$, with frames (A, B) and (\tilde{A}, \tilde{B}) and related by a feedback $w = \tilde{w} + \alpha$ and a diffeomorphism ϕ . In all computations below, \tilde{r} is expressed in x -coordinates, that is, we set $\tilde{r}(\phi(x))$. Thus, using relation (15), resp. (16), we obtain

$$\begin{aligned} \frac{\partial \phi_1}{\partial z} &= \frac{\tilde{r}}{r} \cos(\alpha) = \frac{\partial \phi_2}{\partial y}, & \frac{\partial \phi_1}{\partial y} &= \frac{\tilde{r}}{r} \sin(\alpha) = -\frac{\partial \phi_2}{\partial z}, \\ \text{resp. } \frac{\partial \phi_1}{\partial z} &= \frac{\tilde{r}}{r} \cosh(\alpha) = \frac{\partial \phi_2}{\partial y}, & \frac{\partial \phi_1}{\partial y} &= -\frac{\tilde{r}}{r} \sinh(\alpha) = \frac{\partial \phi_2}{\partial z}, \end{aligned}$$

from which we deduce condition (20). Conversely, applying the diffeomorphism ϕ given by (20), together with the feedback $w = \tilde{w} + \alpha$, with α being a solution of

$$\begin{aligned} \cos(\alpha) &= \frac{r}{\tilde{r}} \frac{\partial \phi_1}{\partial z}, & \sin(\alpha) &= -\frac{r}{\tilde{r}} \frac{\partial \phi_1}{\partial y}, \\ \text{resp. } \cosh(\alpha) &= \frac{r}{\tilde{r}} \frac{\partial \phi_1}{\partial z}, & \sinh(\alpha) &= -\frac{r}{\tilde{r}} \frac{\partial \phi_1}{\partial y}, \end{aligned}$$

we transform Ξ_E^c into $\tilde{\Xi}_E^c$, resp. Ξ_H^c into $\tilde{\Xi}_H^c$. □

Remark. In the above proof we used the metric \mathbf{g}_\pm on \mathcal{X} defined by

$$\mathbf{g}_\pm(A, A) = 1, \quad \mathbf{g}_\pm(B, B) = \pm 1, \quad \mathbf{g}_\pm(A, B) = 0. \quad (21)$$

This object will play a special role in the interpretation of the conditions describing the existence of a commutative EH-frame.

The above proposition shows that elliptic and hyperbolic systems Ξ_{EH} are parametrized by three smooth functions of Ξ variables (and not by 6 functions defining the triple (A, B, C)). Now we will pass to the problem of commutative frames and the following proposition gives equivalent algebraic and geometric conditions for the existence of a commutative EH-frame.

Proposition 4.5 (Existence of a commutative EH-frame). *Consider an elliptic/hyperbolic system $\Xi_{EH} = (A, B, C)$ with structure functions (μ_0, μ_1) of the EH-frame (A, B) . The following statements are equivalent locally around x_0 :*

(i) *There exists a commutative EH-frame.*

(ii) *The structure functions (μ_0, μ_1) attached to the EH-frame (A, B) satisfy*

$$-(\mu_0)^2 \mp (\mu_1)^2 \pm L_A(\mu_1) - L_B(\mu_0) = 0. \quad (22)$$

(iii) *The Gaussian curvature κ_{\pm} of the metric \mathbf{g}_{\pm} vanishes.*

Notice that item (i) describes the following normal forms,

$$\Xi'_E : \begin{cases} \dot{z} = \cos(w) + c_0(x) \\ \dot{y} = \sin(w) + c_1(x) \end{cases}, \quad \text{and} \quad \Xi'_H : \begin{cases} \dot{z} = \cosh(w) + c_0(x) \\ \dot{y} = \sinh(w) + c_1(x) \end{cases},$$

whose structure functions are $\mu_0 = \mu_1 = 0$, $\gamma_0 = c_0$, and $\gamma_1 = c_1$. We call Ξ'_E a *flat elliptic system* and Ξ'_H a *flat hyperbolic system*.

Proof. The equivalence between (ii) and (iii) is immediate since the left hand side of (22) is the Gaussian curvature κ_{\pm} of \mathbf{g}_{\pm} (details of the computations are in Appendix D). We show that (i) is equivalent to (ii). If the EH-frame (A, B) is equivalent via $w = \tilde{w} + \alpha(x)$ to a commutative EH-frame (\tilde{A}, \tilde{B}) , then by (19) we immediately have $L_A(\alpha) = \pm\mu_0$ and $L_B(\alpha) = \mu_1$; the integrability condition of this system of first order partial differential equations gives (22). Conversely, consider the system $\Xi_{EH} = (A, B, C)$ and construct α as a solution of the system $L_A(\alpha) = \pm\mu_0$ and $L_B(\alpha) = \mu_1$, whose solvability is guaranteed by the integrability condition given by (22). Then by (19) we see that the resulting EH-frame (\tilde{A}, \tilde{B}) , of the system $\tilde{\Xi}_{EH}$ obtained by the reparametrisation $w = \tilde{w} + \alpha$, is commutative. \square

Notice that when proving Proposition 4.5 we have shown that the Gaussian curvature κ_{\pm} of the metric \mathbf{g}_{\pm} is given by the left hand side of (22). Moreover relation (19) implies that κ_{\pm} is invariant under reparametrisations $w = \tilde{w} + \alpha$ and is therefore an equivariant of the feedback transformations of the system Ξ_{EH} .

In the following proposition, we give first a classification of flat elliptic/hyperbolic systems, second we characterise those without functional parameters, i.e. constant-forms, and third we provide a canonical form for the latter. Recall that $\iota = (\gamma_1, \mp\gamma_0)$ and that for flat elliptic/hyperbolic systems Ξ'_{EH} we have $(\gamma_0, \gamma_1) = (c_0, c_1)$ so all statements of the proposition below are actually expressed in terms of structure functions. From now on, we will consider the group of feedback transformations consisting of $\tilde{x} = \phi(x)$ and $w = \pm\tilde{w} + \alpha(x)$. The additional transformation $w = -\tilde{w} + \alpha$ implies $(\tilde{A}, \tilde{B}) = (A, B)\bar{\bar{R}}_{EH}(\pm\alpha)$, where $\bar{\bar{R}}_E(\alpha) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix}$ and $\bar{\bar{R}}_H(\alpha) = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ -\sinh(\alpha) & -\cosh(\alpha) \end{pmatrix}$, and the corresponding structure functions change by, compare (19),

$$(\tilde{\mu}_0, \tilde{\mu}_1) = -(\mu_0 \mp L_A(\alpha), \mu_1 - L_B(\alpha))\bar{\bar{R}}_{EH}(\alpha) \quad \text{and} \quad \tilde{\gamma} = \gamma\bar{\bar{R}}_{EH}(\alpha). \quad (23)$$

Proposition 4.6 (Characterisation and classification of flat elliptic/hyperbolic systems).

(i) *Two flat elliptic systems Ξ'_E and $\tilde{\Xi}'_E$, resp. two flat hyperbolic systems Ξ'_H and $\tilde{\Xi}'_H$, are locally feedback equivalent around $x_0 = 0 \in \mathbb{R}^2$ if and only if there exists a constant $\alpha \in \mathbb{R}$ satisfying*

$$R_{EH}^{-1}(\pm\alpha)C(x) = \tilde{C}(R_{EH}^{-1}(\pm\alpha)x), \quad (24)$$

where R_{EH} stands for either $\bar{\bar{R}}_{EH}$ or $\bar{\bar{R}}_{EH}$.

(ii) *An elliptic/hyperbolic system Ξ_{EH} is locally feedback equivalent to a constant-form, i.e. Ξ'_{EH} with $(c_0, c_1) \in \mathbb{R}^2$, if and only if one of the equivalent conditions of Proposition 4.5 holds and, additionally,*

$$L_A(\gamma) + \iota\mu_0 = 0 \quad \text{and} \quad L_B(\gamma) \pm \iota\mu_1 = 0. \quad (25)$$

(iii) A constant-form elliptic system is always feedback equivalent, locally around $x_0 = 0 \in \mathbb{R}^2$, to the canonical form

$$\Xi_E^{\Gamma_E} : \begin{cases} \dot{z} = \cos(w) + \sqrt{\Gamma_E} \\ \dot{y} = \sin(w) \end{cases},$$

where $\Gamma_E = (c_0)^2 + (c_1)^2 \in \mathbb{R}$ is a complete invariant of constant-form elliptic systems.

(iv) A constant-form hyperbolic system is always feedback equivalent, locally around $x_0 = 0 \in \mathbb{R}^2$, to one of the following canonical forms

$$\begin{aligned} \Xi_H^{\Gamma_H, \varepsilon} : \begin{cases} \dot{z} = \cosh(w) + \varepsilon\sqrt{\Gamma_H} \\ \dot{y} = \sinh(w) \end{cases}, \quad \text{or} \quad \Xi_H^{-\Gamma_H} : \begin{cases} \dot{z} = \cosh(w) \\ \dot{y} = \sinh(w) + \sqrt{-\Gamma_H} \end{cases}, \\ \text{or} \quad \Xi_H^{0, \varepsilon} : \begin{cases} \dot{z} = \cosh(w) + \varepsilon \\ \dot{y} = \sinh(w) + 1 \end{cases}, \quad \text{or} \quad \Xi_H^{0, 0} : \begin{cases} \dot{z} = \cosh(w) \\ \dot{y} = \sinh(w) \end{cases}, \end{aligned}$$

where $\Gamma_H = (c_0)^2 - (c_1)^2 \in \mathbb{R}$ and satisfies $\Gamma_H > 0$ for the first form, $\Gamma_H < 0$ for the second form, and $\Gamma_H = 0$ for the third and fourth ones, where $\varepsilon = \text{sgn}(c_0) = \pm 1$. Moreover (Γ_H, ε) is a complete invariant of constant-form hyperbolic systems.

Observe that if in items (i), (iii), and (iv) the considered systems are defined globally, then their feedback equivalence is also global and, in particular, the proposed canonical forms are also global.

Remark. In item (iv), notice that there are two orbits of the local action of feedback transformations group for any $\Gamma_H > 0$, corresponding to $\text{sgn}(c_0) = \varepsilon = \pm 1$, one orbit for any $\Gamma_H < 0$, and three orbits for $\Gamma_H = 0$ corresponding, respectively, to $\text{sgn}(c_0) = \varepsilon = \pm 1$ and to $(c_0, c_1) = (0, 0)$. The invariant $\varepsilon = \pm 1$ corresponds to the parametrisation of one of two branches of the hyperbola $(z - \sqrt{\Gamma_H})^2 - y^2 = 1$.

Proof.

(i) Consider, locally around $0 \in \mathbb{R}^2$, two equivalent flat elliptic/hyperbolic systems Ξ'_{EH} and $\tilde{\Xi}'_{EH}$ given by structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1) = (0, 0, c_0, c_1)$ and $(\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\gamma}_0, \tilde{\gamma}_1) = (0, 0, \tilde{c}_0, \tilde{c}_1)$, respectively. Since they both have a commutative EH-frame, by (19) (and (23)) they differ by a reparametrisation $w = \pm\tilde{w} + \alpha$ satisfying $L_A(\alpha) = L_B(\alpha) = 0$ and thus $\alpha \in \mathbb{R}$. Applying this reparametrisation together with a diffeomorphism ϕ satisfying $\phi_* = R_{EH}^{-1}(\pm\alpha)$, that is $\tilde{x} = \phi(x) = R_{EH}^{-1}(\pm\alpha)x$, transforms Ξ'_{EH} into $\tilde{\Xi}'_{EH}$ if and only if

$$\begin{pmatrix} \tilde{c}_0(\tilde{x}) \\ \tilde{c}_1(\tilde{x}) \end{pmatrix} = R_{EH}^{-1}(\pm\alpha) \begin{pmatrix} c_0(x) \\ c_1(x) \end{pmatrix},$$

which is (24).

(ii) Assume that Ξ_{EH} , given by structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$, is equivalent via $\tilde{x} = \phi(x)$ and $w = \pm\tilde{w} + \alpha$ to Ξ'_{EH} with structure functions $(\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\gamma}_0, \tilde{\gamma}_1) = (0, 0, c_0, c_1)$, where $(c_0, c_1) \in \mathbb{R}^2$. Necessity of one (and thus any) of the conditions of Proposition 4.5 is clear, and by (19) and (23) we have first, $L_A(\alpha) = \pm\mu_0$ and $L_B(\alpha) = \mu_1$ and second, $\gamma R_{EH}(\alpha) = \tilde{\gamma} = (c_0, c_1)$; recall that R_{EH} stands for either \bar{R}_{EH} or $\bar{\bar{R}}_{EH}$. By differentiating this last relation along A and B we obtain

$$\begin{aligned} 0 &= L_A(\gamma) R_{EH}(\alpha) + \gamma L_A(R_{EH}(\alpha)) \\ &= L_A(\gamma) R_{EH}(\alpha) + \gamma \begin{pmatrix} 0 & \mp 1 \\ 1 & 0 \end{pmatrix} R_{EH}(\alpha) \\ &= L_A(\gamma) + \lrcorner\mu_0, \quad \text{and} \\ 0 &= L_B(\gamma) R_{EH}(\alpha) + \gamma L_B(R_{EH}(\alpha)) \\ &= L_B(\gamma) R_{EH}(\alpha) + \gamma \begin{pmatrix} 0 & \mp 1 \\ 1 & 0 \end{pmatrix} R_{EH}(\alpha) \\ &= L_B(\gamma) \pm \lrcorner\mu_1, \end{aligned}$$

thus proving that (25) holds. Conversely, assume that (22) and (25) hold for Ξ_{EH} . By Proposition 4.5, Ξ_{EH} is equivalent to Ξ'_{EH} with a commutative EH-frame (A, B) , and applying (25) to the latter we get $L_A(\gamma) = L_B(\gamma) = 0$ and therefore, we have $(c_0, c_1) \in \mathbb{R}^2$.

(iii) Consider a flat elliptic system Ξ'_E with $(c_0, c_1) \in \mathbb{R}^2$, then relation (24) reads

$$\begin{pmatrix} \tilde{c}_0 \\ \tilde{c}_1 \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \mp \sin(\alpha) & \pm \cos(\alpha) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}. \quad (24')$$

Take α as a solution of $-\sin(\alpha)c_0 + \cos(\alpha)c_1 = 0$, then we have $\tilde{c}_1 = 0$ and $\tilde{c}_0 = \pm\sqrt{\Gamma_E}$, with $\Gamma_E = (c_0)^2 + (c_1)^2$. If necessary, apply $\alpha = \pi$ to send $(\tilde{c}_0, \tilde{c}_1) = (-\sqrt{\Gamma_E}, 0)$ into $(\sqrt{\Gamma_E}, 0)$. The proof that $\Xi_E^{\Gamma_E}$ is equivalent to $\tilde{\Xi}_E^{\Gamma_E}$ if and only if $\Gamma_E = \tilde{\Gamma}_E$ is immediate from (24').

(iv) Consider a flat hyperbolic system Ξ'_H with $(c_0, c_1) \in \mathbb{R}^2$ and denote $\Gamma_H = (c_0)^2 - (c_1)^2$, then relation (24) reads

$$\begin{pmatrix} \tilde{c}_0 \\ \tilde{c}_1 \end{pmatrix} = \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ \mp \sinh(\alpha) & \pm \cosh(\alpha) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}. \quad (24'')$$

We consider four cases. First, assume that $\Gamma_H > 0$, that is $c_0 \neq 0$ and $-1 < \frac{c_1}{c_0} < 1$, and take α as the solution of $\tanh(\alpha) = \frac{c_1}{c_0}$ yielding $\tilde{c}_1 = 0$ and $\tilde{c}_0 = \text{sgn}(c_0)\sqrt{\Gamma_H}$, which gives the canonical form $\Xi_H^{\Gamma_H, \varepsilon}$. Second, assume that $\Gamma_H < 0$, that is $c_1 \neq 0$ and $-1 < \frac{c_0}{c_1} < 1$, and take α as the solution of $\tanh(\alpha) = \frac{c_0}{c_1}$ yielding $\tilde{c}_0 = 0$ and $\tilde{c}_1 = \text{sgn}(c_1)\sqrt{-\Gamma_H}$. If $\text{sgn}(c_1) = -1$, then by applying (24'') with $\alpha = 0$ and the bottom sign in the second row, we can always normalize $\text{sgn}(c_1)$ to $+1$ yielding the canonical form $\Xi_H^{-\Gamma_H}$. Third, assume that $\Gamma_H = 0$ and $c_0 = 0$ thus $c_1 = 0$ and therefore we immediately have the canonical form $\Xi_H^{0,0}$. Fourth, and finally, assume that $\Gamma_H = 0$ and $c_0 \neq 0$, thus $c_1 = \varepsilon c_0$ with $\varepsilon = \pm 1$. If necessary, apply (24'') with $\alpha = 0$ and the bottom sign to obtain $c_1 > 0$. Take $\alpha = \varepsilon \ln c_1$ and apply (24'') with the upper sign to obtain $\tilde{c}_1 = 1$ and $\tilde{c}_0 = \varepsilon$. To show that (Γ_H, ε) is a complete invariant is trivial by applying (24'') to the canonical forms $\Xi_H^{\Gamma_H, \varepsilon}$, $\Xi_H^{-\Gamma_H}$, $\Xi_H^{0, \varepsilon}$, and $\Xi_H^{0,0}$.

□

We summarise the results of this subsection: we started from a general elliptic, resp. hyperbolic, system Ξ_E , resp. Ξ_H , which parametrises an elliptic, resp. a hyperbolic, submanifold \mathcal{S}_E , resp. \mathcal{S}_H . We showed that reparametrisations (pure feedback transformations) acting on the class of elliptic/hyperbolic systems Ξ_{EH} are affine (actually they are of the Brockett type $w = \pm\tilde{w} + \alpha$) with respect to the control and thus depend on one function α only. By transforming into the conformal form, we showed that elliptic/hyperbolic systems Ξ_{EH} are given by three arbitrary smooth functions. Next, we showed that the vanishing of the Gaussian curvature of a (pseudo-)Riemannian metric, associated with the EH-frame of Ξ_{EH} , characterises the flat elliptic/hyperbolic systems Ξ'_{EH} , which depend on two arbitrary smooth functions only. Finally, we gave conditions characterising the constant-form systems, i.e. elliptic/hyperbolic systems without functional parameters. In the elliptic case, equivalent constant-form systems correspond to the circles $\Gamma_E = (c_0)^2 + (c_1)^2 = \text{const.}$, and their canonical forms are parametrized by a closed half-line of real constants. On the other hand, in the hyperbolic case the structure is richer because equivalent systems correspond to connected branches of the hyperbolas $\Gamma_H = (c_0)^2 - (c_1)^2 = \text{const.}$; two connected components for $\Gamma_H > 0$, one for $\Gamma_H < 0$, and three for $\Gamma_H = 0$. Thus canonical forms of hyperbolic systems are parametrized by a real line of constants (the value of Γ_H) and by a discrete invariant $\varepsilon = \pm 1$ (if either $\Gamma_H > 0$ or $\Gamma_H = 0$ and $c_0 \neq 0$). Our characterisation of elliptic/hyperbolic systems gives an equivalent classification for elliptic/hyperbolic submanifolds that is summarised in Lemma 4.1 and explicitly given in [24].

4.2 Classification of parabolic systems

We now turn to the classification of parabolic systems $\Xi_P = (A, B, C)$, of the form

$$\Xi_P : \dot{x} = A(x)w^2 + B(x)w + C(x),$$

which is expected to be different from that of elliptic and hyperbolic systems because the allowed reparametrisations of the control w depend on 2 smooth functions (α, β) , see Proposition 4.2. Our

aim is to get a classification of parabolic submanifolds \mathcal{S}_P , as the one presented in [Table 1](#) of the Introduction, via the properties of the triple (A, B, C) presented in [Table 2](#) at the beginning of [Section 4](#). In particular, the existence of a commutative P-frame (A, B) corresponds to weakly-flat parabolic submanifolds; existence of a commutative P-frame, which additionally satisfies $A \wedge C = 0$, corresponds to strongly-flat submanifolds; finally, existence of a commutative P-frame, which additionally satisfies C constant (in the coordinates, where (A, B) is rectified), corresponds to constant-form parabolic submanifolds; in particular, the case of $C = 0$ is called null-form. As in the elliptic/hyperbolic cases, we introduce the structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$ uniquely defined for any triple (A, B, C) by

$$[A, B] = \mu_0 A + \mu_1 B \quad \text{and} \quad C = \gamma_0 A + \gamma_1 B.$$

By a direct computation, we obtain that under reparametrisations of the form $w = \beta \tilde{w} + \alpha$ the structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$ of (A, B, C) are transformed into the structure functions $(\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\gamma}_0, \tilde{\gamma}_1)$ of $(\tilde{A}, \tilde{B}, \tilde{C})$ via the following relations:

$$\tilde{\gamma}_0 = \frac{1}{\beta^2} (\gamma_0 - 2\alpha\gamma_1 - \alpha^2), \quad \tilde{\gamma}_1 = \frac{1}{\beta} (\gamma_1 + \alpha), \quad (26)$$

$$\begin{aligned} \tilde{\mu}_0 &= \beta\mu_0 - 2\alpha L_A(\beta) + 2\beta L_A(\alpha) - 2L_B(\beta) - 2\alpha(L_A(\beta) + \beta\mu_1), \\ \tilde{\mu}_1 &= \beta^2\mu_1 + \beta L_A(\beta). \end{aligned} \quad (27)$$

There are two main questions that we will answer. First, when does a commutative P-frame (\tilde{A}, \tilde{B}) exist, i.e. $\tilde{\mu}_0 = \tilde{\mu}_1 = 0$? Second, provided that a commutative P-frame (A, B) has been normalised, how can we additionally simplify C ? Contrary to the elliptic and hyperbolic cases the answer to the first question is always positive without any additional assumption, as ensured by the next result.

Proposition 4.7 (Existence of a commutative P-frame). *(i) For any P-frame (A, B) there exists, locally around x_0 , a reparametrisation (α, β) such that (\tilde{A}, \tilde{B}) is a commutative P-frame.*

(ii) If (A, B) is a commutative P-frame, then (\tilde{A}, \tilde{B}) is also a commutative P-frame if and only if the reparametrisation (α, β) satisfies

$$L_A(\beta) = 0 \quad \text{and} \quad \frac{1}{\beta} L_B(\beta) = L_A(\alpha). \quad (28)$$

Proof.

(i) Consider a P-frame (A, B) whose structure functions are (μ_0, μ_1) . Apply a reparametrisation (α, β) , $\beta \neq 0$, given by a solution of the following system of equations, in which we solve the first equation for β , and plug in β into the second equation to solve it for α

$$\begin{cases} L_A(\beta) &= -\beta\mu_1 \\ L_A(\alpha) + \alpha\mu_1 &= \frac{1}{2\beta} (2L_B(\beta) - \beta\mu_0) \end{cases} .$$

Then formula (27) implies $\tilde{\mu}_0 = \tilde{\mu}_1 = 0$, i.e. $[\tilde{A}, \tilde{B}] = 0$. Notice that to ensure $\beta \neq 0$, we may actually solve $L_A(\ln \beta) = -\mu_1$.

(ii) Using relation (27) with $\mu_i = \tilde{\mu}_i = 0$, for $i = 0, 1$, we see that all reparametrisations (α, β) have to satisfy the relations of (28). Conversely, if (A, B) is a commutative P-frame ($\mu_0 = \mu_1 = 0$) and (α, β) is any solution of (28), with $\beta \neq 0$, then by (27) we obtain that $\tilde{\mu}_0 = \tilde{\mu}_1 = 0$, i.e. (\tilde{A}, \tilde{B}) is also a commutative P-frame. □

Immediately, item *(i)* of [Proposition 4.7](#) gives the following prenormal forms of parabolic systems Ξ_P . Recall that systems and equivalence are considered locally with respect to the state x and globally with respect to the control $w \in \mathbb{R}$.

Corollary 4.8 (Prenormal forms of Ξ_P). *The parabolic system Ξ_P is always feedback equivalent to the following prenormal forms, locally around x_0 :*

$$\Xi'_P : \begin{cases} \dot{z} &= w^2 + c_0(x) \\ \dot{y} &= w + c_1(x) \end{cases}, \quad \Xi''_P : \begin{cases} \dot{z} &= w^2 + b(x)w + \Gamma_P(x) \\ \dot{y} &= w \end{cases},$$

whose structure functions are $(\mu'_0, \mu'_1, \gamma'_0, \gamma'_1) = (0, 0, c_0, c_1)$, and $(\mu''_0, \mu''_1, \gamma''_0, \gamma''_1) = (\frac{\partial b}{\partial z}, 0, \Gamma_P, 0)$, respectively.

A parabolic system of the form Ξ'_P is called weakly-flat, this terminology is justified by generalisations of the previous result to higher dimensions that we will present in a future paper.

Remark. *Since any parabolic system can be brought into Ξ'_P (and into Ξ''_P), it follows that all parabolic systems are locally parametrized by two functions of two variables (c_0 and c_1 , or, equivalently, b and Γ_P). This is in contrast with elliptic/hyperbolic systems Ξ_{EH} parametrized by three functions of two variables (compare [Proposition 4.4](#)).*

Proof. Apply to Ξ_P a reparametrisation (α, β) transforming its P-frame into a commutative P-frame (\tilde{A}, \tilde{B}) and let ϕ be a local diffeomorphism introducing coordinates $x = (z, y)$ such that $\phi_*\tilde{A} = \frac{\partial}{\partial z}$ and $\phi_*\tilde{B} = \frac{\partial}{\partial y}$. In this system of coordinates, Ξ_P takes the form Ξ'_P . Then apply to Ξ'_P the reparametrisation $\tilde{w} = w + c_1(x)$ to obtain the form Ξ''_P with $b = -2c_1$ and $\Gamma_P = c_0 + (c_1)^2$. The computation of the structure functions is straightforward. \square

Notice that the normal forms Ξ'_P and Ξ''_P are related by the reparametrisation $\tilde{w} = w + c_1(x)$. The function Γ_P (appearing in Ξ''_P) will be of special importance in the remaining part of this section, and in any P-frame (A, B) , commutative or not, we define it by setting

$$\Gamma_P = \gamma_0 + (\gamma_1)^2.$$

Recall that (γ_0, γ_1) are defined by $C = \gamma_0 A + \gamma_1 B$. Clearly, diffeomorphisms act on Γ_P by conjugation and reparametrisations (α, β) act by $\beta^2 \tilde{\Gamma}_P = \Gamma_P$ (as it can be computed from formula [\(26\)](#)).

The remaining part of this section shows how to additionally normalize Ξ'_P and Ξ''_P while preserving the commutativity of the P-frame (A, B) . Although for a parabolic system Ξ_P , there always exists a commutative P-frame (A, B) , its explicit construction can be difficult or even impossible, as it requires to solve a system of first order PDEs. For this reason, we will state our results for a general, not necessarily commutative, P-frame (A, B) .

Theorem 4.9 (Normalisation of parabolic systems). *Let $\Xi_P = (A, B, C)$ be a parabolic system with structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$. Then the following statements hold, locally around x_0 :*

- (i) Ξ_P is strongly-flat, i.e. feedback equivalent to Ξ'_P with $c_1 \equiv 0$, we will denote that form Ξ'''_P , if and only if

$$L_A^2(\gamma_1) + \gamma_1 (L_A(\mu_1) - (\mu_1)^2) = \frac{\mu_0 \mu_1}{2} + \frac{1}{2} L_A(\mu_0) + L_B(\mu_1). \quad (29)$$

- (ii) Ξ_P is a constant-form, i.e. feedback equivalent to Ξ'_P with $(c_0, c_1) \in \mathbb{R}^2$, satisfying $c_0 + (c_1)^2 \in \mathbb{R}^*$, if and only if $\Gamma_P \neq 0$ and

$$\begin{aligned} L_A(\Gamma_P) + 2\mu_1 \Gamma_P &= 0, \\ L_B(\Gamma_P) + 2\Gamma_P L_A(\gamma_1) &= \Gamma_P \mu_0 - 2\Gamma_P \gamma_1 \mu_1. \end{aligned} \quad (30)$$

Moreover, in this case we can always normalize $c_0 = \pm 1$ and $c_1 = 0$ and we will denote that form Ξ_P^\pm .

- (iii) Ξ_P is a null-form, i.e. feedback equivalent to Ξ'_P with $c_0 \equiv c_1 \equiv 0$, which we will denote Ξ_P^0 , if and only if [\(29\)](#) holds and, additionally, $\Gamma_P \equiv 0$.

Notice that items (i), (ii), and (iii) of the above theorem characterise, respectively, the following normal forms around x_0 (recall that we work globally with respect to w)

$$\Xi_P''' : \begin{cases} \dot{z} = w^2 + c_0(x) \\ \dot{y} = w \end{cases}, \quad \Xi_P^\pm : \begin{cases} \dot{z} = w^2 \pm 1 \\ \dot{y} = w \end{cases}, \quad \text{and} \quad \Xi_P^0 : \begin{cases} \dot{z} = w^2 \\ \dot{y} = w \end{cases}.$$

A parabolic system of the form Ξ_P''' is called strongly-flat (the terminology is justified by generalisations of the previous result to higher dimensions). Equivalent statements can be formulated to obtain special structures of the second prenormal form Ξ_P'' , e.g. the normal form Ξ_P''' corresponds to Ξ_P'' with $b \equiv 0$ and thus describes the intersection of the prenormal forms Ξ_P' and Ξ_P'' . The constant-forms Ξ_P^\pm and Ξ_P^0 are clearly special cases of Ξ_P''' , where c_0 is constant. Observe that contrary to the elliptic/hyperbolic case, where canonical forms of constant-form systems are parametrised by real-continuous parameters, in the parabolic case, there are only three distinct canonical forms of constant-form systems. The difference between the normal form Ξ_P^\pm and Ξ_P^0 lies in the existence or not of an equilibrium point: the control $w = 0$ defines an equilibrium of Ξ_P^0 while there are no equilibria for Ξ_P^\pm .

Proof.

(i) *Sufficiency:* Consider a parabolic system $\Xi_P = (A, B, C)$, with structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$, and assume that relation (29) holds. Introduce the reparametrisation $\alpha = -\gamma_1$ and β given as a solution of the following system of equations

$$\begin{cases} \frac{1}{\beta} L_A(\beta) = -\mu_1 \\ \frac{1}{\beta} L_B(\beta) = \frac{1}{2}(\mu_0 - 2\gamma_1\mu_1 - 2L_A(\gamma_1)) \end{cases}. \quad (31)$$

This system, rewritten for $\ln(\beta)$, admits solutions since the integrability condition

$$L_A(L_B(\ln \beta)) - L_B(L_A(\ln \beta)) = L_{[A,B]}(\ln \beta) = \mu_0 L_A(\ln \beta) + \mu_1 L_B(\ln \beta) \quad (31')$$

takes the form

$$\begin{aligned} \frac{1}{2} L_A(\mu_0) - \gamma_1 L_A(\mu_1) - L_A^2(\gamma_1) - \mu_1 L_A(\gamma_1) + L_B(\mu_1) \\ = -\frac{1}{2} \mu_0 \mu_1 - \gamma_1 (\mu_1)^2 - \mu_1 L_A(\gamma_1) \end{aligned} \quad (31'')$$

and is guaranteed by condition (29). Consider the system $\tilde{\Xi}_P = (\tilde{A}, \tilde{B}, \tilde{C})$ obtained by the above defined reparametrisation (α, β) , and using relations (26) and (27) we deduce that the structure functions $(\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\gamma}_0, \tilde{\gamma}_1)$ of $\tilde{\Xi}_P$ satisfy $\tilde{\mu}_0 = \tilde{\mu}_1 = 0$ and $\tilde{\gamma}_1 = 0$. By choosing local coordinates (z, y) such that $(\tilde{A}, \tilde{B}) = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial y} \right)$, we obtain the system Ξ_P' with $c_1 = \tilde{\gamma}_1 = 0$, i.e. a strongly-flat parabolic system Ξ_P''' .

(i) *Necessity:* Assume that $\Xi_P = (A, B, C)$ is feedback equivalent to $\tilde{\Xi}_P$ of the form Ξ_P' with $c_1 \equiv 0$, via ϕ and (α, β) . Then for $\tilde{\Xi}_P$ we have $\tilde{\mu}_0 = \tilde{\mu}_1 = \tilde{\gamma}_1 = 0$. First, by (26) we obtain $\alpha = -\gamma_1$ and by (27) we obtain that β satisfies the relations given by system (31) above. Therefore, by computing the integrability condition (31'), equivalently given by (31''), we conclude relation (29).

(ii) *Sufficiency:* Consider a parabolic system $\Xi_P = (A, B, C)$, with structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$, and assume that $\Gamma_P = \gamma_0 + (\gamma_1)^2 \neq 0$ and (30) holds. We follow the same reasoning as in the proof of sufficiency of item (i). Introduce a reparametrisation (α, β) , where $\alpha = -\gamma_1$ and β is a solution of the system (31). To assure the existence of β , we have to fulfil the integrability condition of (31), which is (31'), equivalently (31''). To this end, we differentiate the second condition of (30) along A and use $L_A(L_B(\cdot)) = L_B(L_A(\cdot)) + \mu_0 L_A(\cdot) + \mu_1 L_B(\cdot)$ to conclude (31''). Consider the system $\tilde{\Xi}_P = (\tilde{A}, \tilde{B}, \tilde{C})$ obtained by the above defined reparametrisation (α, β) and using relations (26) and (27), we deduce that the structure functions $(\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\gamma}_0, \tilde{\gamma}_1)$ of $\tilde{\Xi}_P = (\tilde{A}, \tilde{B}, \tilde{C})$ satisfy $\tilde{\mu}_0 = \tilde{\mu}_1 = \tilde{\gamma}_1 = 0$. Therefore $\tilde{\Gamma}_P = \tilde{\gamma}_0$, for which condition (30) yields

$L_{\tilde{A}}(\tilde{\Gamma}_P) = L_{\tilde{B}}(\tilde{\Gamma}_P) = 0$, implying that $\tilde{\Gamma}_P$ is constant (we still have $\tilde{\Gamma}_P \neq 0$ since $\beta^2\tilde{\Gamma}_P = \Gamma_P$). Introduce coordinates (\tilde{z}, \tilde{y}) such that $\tilde{A} = \frac{\partial}{\partial \tilde{z}}$ and $\tilde{B} = \frac{\partial}{\partial \tilde{y}}$, in which the system takes the form (recall that $\tilde{\gamma}_1 = 0$)

$$\begin{cases} \dot{\tilde{z}} &= \tilde{w}^2 + c_0 \\ \dot{\tilde{y}} &= \tilde{w} \end{cases},$$

with $c_0 \in \mathbb{R}^*$. Finally, defining new coordinates (z, y) by $z = \frac{\tilde{z}}{|c_0|}$ and $y = \frac{\tilde{y}}{\sqrt{|c_0|}}$, and reparametrising by $w = \frac{\tilde{w}}{\sqrt{|c_0|}}$, yields the normal form Ξ_P^\pm .

(ii) *Necessity*: Assume that Ξ_P , whose structure functions are $(\mu_0, \mu_1, \gamma_0, \gamma_1)$ and $\Gamma_P = \gamma_0 + \gamma_1^2$, is feedback equivalent, via ϕ and (α, β) , to $\tilde{\Xi}_P$ of the form $\tilde{\Xi}'_P$ with $(c_0, c_1) \in \mathbb{R}^2$ satisfying $c_0 + c_1^2 \neq 0$. For $\tilde{\Xi}_P$ we have $\tilde{\mu}_0 = \tilde{\mu}_1 = 0$ and $\tilde{\gamma}_0 = c_0$, $\tilde{\gamma}_1 = c_1$, hence $\tilde{\Gamma}_P = c_0 + c_1^2 \neq 0$ implying $\Gamma_P \neq 0$ since $\beta^2\tilde{\Gamma}_P = \Gamma_P$. By relation (27), we obtain that β satisfies the relations of system (31). Differentiating $\Gamma_P = \beta^2\tilde{\Gamma}_P$ along A we deduce

$$L_A(\Gamma_P) = L_A(\beta^2\tilde{\Gamma}_P) = 2\tilde{\Gamma}_P\beta L_A(\beta) = -2\tilde{\Gamma}_P\beta^2\mu_1 = -2\mu_1\Gamma_P,$$

giving the first relation of (30). A similar computation, by taking the derivative of $\Gamma_P = \beta^2\tilde{\Gamma}_P$ along B , implies the second relation of (30).

(iii) The proof of that statement is a special case of the proof of item (i) with the additional condition $\Gamma_P \equiv 0$.

Sufficiency: Use the proof of the sufficiency of item (i) to bring the system Ξ_P into Ξ_P''' . For the latter form we have $\Gamma_P = c_0(x)$, hence $c_0(x) \equiv 0$ (due to $\beta^2\tilde{\Gamma}_P = \Gamma_P$ and assumption $\Gamma_P \equiv 0$) and we obtain the normal form Ξ_P^0 .

Necessity: Assume that Ξ_P , whose structure functions are $(\mu_0, \mu_1, \gamma_0, \gamma_1)$ and $\Gamma_P = \gamma_0 + \gamma_1^2$, is feedback equivalent, via ϕ and (α, β) , to $\tilde{\Xi}_P$ of the form $\tilde{\Xi}_P^0$ (which is, actually, $\tilde{\Xi}'_P$ with $c_0 \equiv c_1 \equiv 0$). For that system we have $\tilde{\mu}_0 = \tilde{\mu}_1 = 0$ and $\tilde{\Gamma}_P \equiv 0$ and since Γ_P is transformed under (α, β) by $\beta^2\tilde{\Gamma}_P = \Gamma_P$, we obtain the necessity of $\Gamma_P \equiv 0$. The necessity of (29) is deduced from the necessity part of statement (i).

□

Observe that item (ii) of the above theorem does not explicitly require condition (29), while the normal form Ξ_P^\pm satisfies $c_1 \equiv 0$ and hence that condition has to be hidden in (30). Indeed, this can be observed by differentiating Γ_P along $[A, B]$ and using the constraint (30), which after a short computation gives condition (29).

Remark (Interpretation of the conditions). *We now give a tangible interpretation of our conditions. To this end, consider the system $\tilde{\Xi}'_P$ for which we have $\mu_0 = \mu_1 = 0$, $\gamma_0 = c_0(x)$, $\gamma_1 = c_1(x)$ and thus $\Gamma_P(x) = c_0(x) + (c_1(x))^2$. First, condition (29) implies $\frac{\partial^2 c_1}{\partial z^2} = 0$, that is, c_1 is affine with respect to z , namely, $c_1(x) = c_1^0(y)z + c_1^1(y)$, and thus, $\Gamma_P(x)$ is given by $c_0(x) + (c_1^0(y)z + c_1^1(y))^2$. This means that if a weakly-flat system $\tilde{\Xi}'_P$ is feedback equivalent to a strongly-flat system $\tilde{\Xi}_P'''$, then it is parametrized by 3 smooth functions, two of them being functions of y only, and it has the following form*

$$\begin{cases} \dot{z} &= w^2 + \Gamma_P(x) - (c_1^0(y)z + c_1^1(y))^2 \\ \dot{y} &= w + c_1^0(y)z + c_1^1(y) \end{cases}.$$

By additionally applying the first equation of (30), we obtain $\frac{\partial c_0}{\partial z} + 2c_1^0(y)^2z + 2c_1^0(y)c_1^1(y) = 0$ and thus $c_0(x)$ is a polynomial of degree 2 in z , related to $c_1(x)$ by $c_0(x) = -(c_1(x))^2 + c_2(y)$, for an arbitrary smooth function $c_2(y)$. We now have $\Gamma_P = \Gamma_P(y) = c_2(y)$ and we use the second equation of (30). Thus, we get $\Gamma_P(y) = G \exp(-2 \int c_1^1(y) dy)$, with $G \in \mathbb{R}$. To summarise, any system $\tilde{\Xi}'_P$ satisfying

(29) and (30), i.e. equivalent to a constant-form parabolic system, is parametrized by two arbitrary smooth functions of y and a constant $G \in \mathbb{R}$, and is expressed by the form

$$\begin{cases} \dot{z} &= w^2 + \Gamma_P(y) - (c_1^0(y)z + c_1^1(y))^2 \\ \dot{y} &= w + c_1^0(y)z + c_1^1(y) \end{cases},$$

where $\Gamma_P(y) = G \exp(-2 \int c_1^1(y) dy)$. Finally, Ξ'_P (satisfying (29) and (30)) is feedback equivalent to Ξ_P^0 if and only if $G = 0$, and is feedback equivalent to Ξ_P^+ , respectively to Ξ_P^- , if $G > 0$, resp. $G < 0$. The distinction between the three normal forms comes from the sign of Γ_P , which is thus a discrete invariant (and that sign is dictated by the value of the constant G).

5 Conclusions

In this paper, we studied the characterisation and the classification problem of 3-dimensional submanifolds of the tangent bundle of a smooth surface. We showed that the equivalence of submanifolds is reflected in the equivalence, under feedback transformations, of their first and second extensions treated as control-nonlinear and control-affine systems, respectively. We gave a complete characterisation of non-degenerate quadratic submanifolds and proposed a classification of the regular ones, namely the classes of elliptic, hyperbolic, and parabolic, submanifolds. To achieve our characterisation results, we introduced the novel class of quadratic control-affine systems and we gave a characterisation of that class in terms of relations between structure functions. Using our characterisation, we identified the subclasses of elliptic, hyperbolic, and parabolic control-affine systems (second extensions of elliptic, hyperbolic, and parabolic submanifolds). Moreover, we constructed a normal-form for all quadratisable systems and thus gave a normal form of quadratic submanifolds, that may smoothly pass from the elliptic to the hyperbolic classes. Finally, working within the class of control-nonlinear systems subject to a regular quadratic nonholonomic constraint, we exhibited several normal forms of elliptic, hyperbolic, and parabolic systems. In particular we highlighted a connection between the Gaussian curvature of a well-defined metric and the existence of a commutative frame for elliptic and hyperbolic systems. Normal forms include systems without functional parameters. As a consequence of our classification of quadratic systems, we obtained a classification of elliptic, hyperbolic, and parabolic submanifolds.

The purpose of future works is two folds. First, we want to study the properties (such as controllability, stability, optimal trajectories...) of the class of control systems that we identified. Second, we plan to extend our results to higher dimensional quadratic nonholonomic constraints, in particular, we will generalise our results for parabolic systems to the case of control-nonlinear systems (with the state-space being an $n \geq 3$ dimensional manifold) subject to paraboloid nonholonomic constraints. Another interesting problem is to characterise control-nonlinear systems subject to any algebraic nonholonomic constraint. For instance, in order to generalise our results for parabolic systems, one can study polynomial systems, that is, systems subject to the nonholonomic constraint $\dot{z} - \sum_{i=0}^d a_i(x)y^i = 0$, $(z, y) \in \mathbb{R}^2$.

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A Resolution of equation (3)

Consider ζ a smooth function of (z, y, w) , we will abbreviate (z, y) as x to shorten the notation. We show how to represent all smooth solutions around (x_0, w_0) of the equation

$$\frac{\partial^3 \zeta}{\partial w^3} = \tau(x) \frac{\partial \zeta}{\partial w}.$$

First, we will find an expression for $\frac{\partial \zeta}{\partial w}$ that will be then integrated to obtain the desired form. Consider the following linear system of PDEs:

$$\begin{pmatrix} \frac{\partial \zeta_1}{\partial w} \\ \frac{\partial \zeta_2}{\partial w} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \tau(x) & 0 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$$

given for the functions $\zeta_1 = \frac{\partial \zeta}{\partial w}$ and $\zeta_2 = \frac{\partial^2 \zeta}{\partial w^2}$. Solutions of this system (interpreted as a system of ODEs with x being a parameter) are expressed by the exponential of the matrix $\begin{pmatrix} 0 & w \\ \tau(x)w & 0 \end{pmatrix}$ given by the formula

$$\exp \begin{pmatrix} 0 & w \\ \tau(x)w & 0 \end{pmatrix} = \sum_{k=0}^{+\infty} \frac{w^{2k+1} \tau^k(x)}{(2k+1)!} \begin{pmatrix} 0 & 1 \\ \tau(x) & 0 \end{pmatrix} + \sum_{k=0}^{+\infty} \frac{w^{2k} \tau^k(x)}{(2k)!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

obtained by expressing the power series of the exponential and by regrouping the terms of odd and even degrees. Denote $b(x) = \zeta_1(x, w_0)$ and $a(x) = \zeta_2(x, w_0)$, thus we obtain

$$\frac{\partial \zeta}{\partial w} = \zeta_1(x, w) = a(x) \sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k+1}}{(2k+1)!} \tau^k(x) + b(x) \sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k}}{(2k)!} \tau^k(x).$$

Integration of this expression yields

$$\zeta(x, w) = a(x) \sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k+2}}{(2k+2)!} \tau^k(x) + b(x) \sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k+1}}{(2k+1)!} \tau^k(x) + c(x).$$

B Detailed computation of the proof of Theorem 3.5

We detail the computation needed in the proof of Theorem 3.5.

B.1 Resolution of equation (12)

We show how to, locally around $0 \in \mathbb{R}^3$, solve the equation $\rho'' - 2\rho\rho' + \frac{4}{9}\rho^3 = 0$, where $\rho = \rho(x, w)$ and the derivatives are taken with respect to w . Introduce the new unknown function $R(x, w) = \exp(-\frac{2}{3} \int_0^w \rho(x, t) dt)$ which satisfies $R(x, 0) \neq 0$ and

$$R' = -\frac{2}{3}\rho R, \quad R'' = -\frac{2}{3}R \left(\rho' - \frac{2}{3}\rho^2 \right), \quad R''' = -\frac{2}{3}R \left(\rho'' - 2\rho\rho' + \frac{4}{9}\rho^3 \right) = 0.$$

Thus, $R(x, w) = a(x)w^2 + b(x)w + c(x)$ yielding $\rho = -\frac{3R'}{R} = -\frac{3}{2} \frac{2aw+b}{aw^2+bw+c}$. Since $R(x, 0) = c(x) \neq 0$, thus taking $d(x) = \frac{a}{c}$ and $e(x) = \frac{b}{c}$ we obtain

$$\rho(x, w) = -\frac{3}{2} \frac{2d(x)w + e(x)}{d(x)w^2 + e(x)w + 1}.$$

B.2 Smooth form of h'

We integrate $h''(x, w) = a(x)(d(x)w^2 + e(x)w + 1)^{-3/2}$ to obtain a smooth expression of $h'(x, w)$ around $0 \in \mathbb{R}^3$. Recall that we denote $p = p(x, w) = dw^2 + ew + 1$ and $\Delta = \Delta(x) = e^2(x) - 4d(x)$. We have

$$h'(x, w) = a(x) \int \frac{1}{p(x, w)^{3/2}} dw = \frac{-2a(2dw + e)}{\Delta\sqrt{p}} + \bar{b}(x).$$

Since $h'(x, 0) = b(x)$ is smooth, we have $\frac{-2ae}{\Delta} + \bar{b} = b(x)$, where $b(x)$ is a smooth function around 0. We can then derive a smooth closed form expression of $h'(x, w)$:

$$\begin{aligned} h'(x, w) &= \frac{-2a(2dw + e)}{\Delta\sqrt{p}} + \frac{2ae}{\Delta} + b = \frac{-2a}{\Delta\sqrt{p}} (2dw + e - e\sqrt{p}) + b \\ &= \frac{-2a}{\Delta\sqrt{p}(ew + 2 + 2\sqrt{p})} (2dw + e - e\sqrt{p}) (ew + 2 + 2\sqrt{p}) + b \\ &= \frac{-2a}{\Delta\sqrt{p}(ew + 2 + 2\sqrt{p})} (2dew^2 + 4dw + 4dw\sqrt{p} + e^2w + 2e + 2e\sqrt{p} - e^2w\sqrt{p} - 2e\sqrt{p} - 2ep) + b \\ &= \frac{-2a}{\Delta\sqrt{p}(ew + 2 + 2\sqrt{p})} (w\sqrt{p}(4d - e^2) + 4dw + 2dew^2 + 2e + e^2w - 2edw^2 - 2e^2w - 2e) + b \\ &= \frac{-2a}{\Delta\sqrt{p}(ew + 2 + 2\sqrt{p})} (w\sqrt{p}(4d - e^2) + w(4d - e^2)) + b \\ &= \frac{2aw}{\sqrt{p}(ew + 2 + 2\sqrt{p})} (\sqrt{p} + 1) + b. \end{aligned}$$

C Details of the computations of Lemma 4.3

Denoting $c_E(x) = \cos(x)$, $c_H(w) = \cosh(w)$, $s_E(w) = \sin(w)$, and $s_H(w) = \sinh(w)$ and starting from the system

$$\dot{x} = (A, B) \begin{pmatrix} c_{EH}(w) \\ s_{EH}(w) \end{pmatrix} + (A, B) \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix},$$

we apply a reparametrisation $w = \tilde{w} + \alpha(x)$:

$$\begin{aligned}\dot{x} &= (A, B) \bar{R}_{EH}(\pm\alpha) \begin{pmatrix} c_{EH}(\tilde{w}) \\ s_{EH}(\tilde{w}) \end{pmatrix} + (A, B) \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix}, \\ &= (\tilde{A}, \tilde{B}) \begin{pmatrix} c_{EH}(\tilde{w}) \\ s_{EH}(\tilde{w}) \end{pmatrix} + (\tilde{A}, \tilde{B}) \bar{R}_{EH}^{-1}(\pm\alpha) \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix}.\end{aligned}$$

This yields $(\tilde{\gamma}_0, \tilde{\gamma}_1) = (\gamma_0, \gamma_1) \bar{R}_{EH}^{-T}(\pm\alpha)$, and by the definition of $\bar{R}_{EH}(\alpha)$ we have,

$$\bar{R}_E^{-T}(\alpha) = \bar{R}_E^T(-\alpha) = \bar{R}_E(\alpha), \quad \text{and} \quad \bar{R}_H^{-T}(-\alpha) = \bar{R}_H^{-1}(-\alpha) = \bar{R}_H(\alpha).$$

Next, computing separately in the elliptic and hyperbolic cases, we have

$$\begin{aligned}[\tilde{A}, \tilde{B}] &= (\mu_0 \mp L_A(\alpha))A + (\mu_1 - L_B(\alpha))B, \\ (A, B) \bar{R}_E(\pm\alpha) \begin{pmatrix} \tilde{\mu}_0 \\ \tilde{\mu}_1 \end{pmatrix} &= (A, B) \begin{pmatrix} \mu_0 \mp L_A(\alpha) \\ \mu_1 - L_B(\alpha) \end{pmatrix}.\end{aligned}$$

Thus,

$$(\tilde{\mu}_0, \tilde{\mu}_1) = (\mu_0 \mp L_A(\alpha), \mu_1 - L_B(\alpha)) \bar{R}_E^{-T}(\pm\alpha)$$

and relation (19) follows.

D Gaussian curvature for metric given in terms of vector fields

Consider a 2-dimensional manifold \mathcal{X} and two smooth vector fields A and B satisfying $A \wedge B \neq 0$. Construct the (pseudo)-Riemannian metric \mathbf{g}_\pm defined by

$$\mathbf{g}_\pm(A, A) = 1, \quad \mathbf{g}_\pm(B, B) = \pm 1, \quad \text{and} \quad \mathbf{g}_\pm(A, B) = 0.$$

We will give a formula for the Gaussian curvature of \mathbf{g}_\pm in terms of the structure functions (μ_0, μ_1) uniquely defined by $[A, B] = \mu_0 A + \mu_1 B$. We will use the following formula for the covariant derivative

$$\nabla_{E_i} E_j = \frac{1}{2} \sum_k \left(\mathbf{g}_\pm([E_i, E_j], E_k) - \mathbf{g}_\pm([E_i, E_k], E_j) - \mathbf{g}_\pm([E_j, E_k], E_i) \right) E_k$$

for $E_i, E_j, E_k \in \{A, B\}$, and the following formula for the Gaussian curvature of a 2-dimensional manifold, see [17, Proposition 1.11.3],

$$\kappa_\pm = \frac{\mathbf{g}_\pm((\nabla_B \nabla_A - \nabla_A \nabla_B + \nabla_{[A, B]})A, B)}{\det(\mathbf{g}_\pm)},$$

where $\det(\mathbf{g}_\pm) = \mathbf{g}_\pm(A, A)\mathbf{g}_\pm(B, B) - \mathbf{g}_\pm(A, B)^2 = \pm 1$. Computing, we have

$$\nabla_A A = -\mu_0 B, \quad \nabla_A B = \mu_0 A, \quad \nabla_B A = \mp \mu_1 B, \quad \nabla_B B = \pm \mu_1 A.$$

Then we can deduce

$$\begin{aligned}\nabla_B \nabla_A A &= -L_B(\mu_0) B \mp \mu_0 \mu_1 A, \quad \nabla_A \nabla_B A = \mp L_A(\mu_1) B \mp \mu_0 \mu_1 A, \\ \nabla_{[A, B]} A &= \mu_0 \nabla_A A + \mu_1 \nabla_B A = -(\mu_0)^2 B \mp (\mu_1)^2 B.\end{aligned}$$

Thus

$$\begin{aligned}\kappa_\pm &= \pm \mathbf{g}_\pm \left((-L_B(\mu_0) \pm L_A(\mu_1) - (\mu_0)^2 \mp (\mu_1)^2) B, B \right), \\ &= -L_B(\mu_0) \pm L_A(\mu_1) - (\mu_0)^2 \mp (\mu_1)^2.\end{aligned}$$

Therefore, we have obtained the expression of relation (22).