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Despite a blossoming of research activity on racks and their homology for over two decades, with a record of diverse applications to central parts of contemporary mathematics, there are still very few examples of racks whose homology has been fully calculated. In this paper, we compute the entire integral homology of all permutation racks. Our method of choice involves homotopical algebra, which was brought to bear on the homology of racks only recently. For our main result, we establish a spectral sequence, which reduces the problem to one in equivariant homology, and for which we show that it always degenerates. The blueprint given in this paper demonstrates the high potential for further exploitation of these techniques.

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Permutations, racks, rack homology, equivariant homology, spectral sequences.

Racks are fundamental algebraic structures boasting applications to knots [FR92, EN15], singularities [Bri88], monodromy [Yet03], branched covers [EVW16, Ran19], Yang–Baxter equations [Leb18, LV19, CJO20], Hopf algebras [AG03], and the integration of Leibniz algebras [Kin07], for instance. Just as group homology is an important invariant of groups, racks come with rack homology [FRS04], and these rack homology groups are what the applications most often require. So far, complete computations of homological invariants of racks were limited to a few isolated cases, such as in [Cla11, Nos13, FGG14]. In this paper, we pursue the approach to rack homology via Quillen's homotopical algebra, initiated in [Szy19]. It allows us to compute the entire integral homology for a whole family of racks: the permutation racks.

Recall that a *rack* $(X, \triangleright)$ is a set X together with a binary operation $\triangleright$ such that the left multiplications $y \mapsto x \triangleright y$ are automorphisms for all x in X. Basic examples are the *permutation racks* (X, φ), where these automorphisms are independent of the left factor: we have $x \triangleright y = \varphi(y)$ for a permutation φ on the set X.

Our main result reads as follows:

Theorem A. *For every permutation rack (X, φ), there is a spectral sequence of homological type whose* E^2 *page is given by*

$$\mathrm{E}^2_{\bullet,q} \cong \overline{\mathrm{H}}_\bullet(X /\!/ \varphi)^{\otimes(q-1)} \otimes \mathrm{H}_\bullet(X /\!/ \varphi),$$

and which abuts to the rack homology $\mathrm{HR}_\bullet(X, \varphi)$ *of* (X, φ). *This spectral sequence always degenerates from its* E^2 *page on.*

The spectral sequence in Theorem A is constructed in Theorem 4.1 of the main text, and the degeneracy is proven in Theorem 6.1. As for the description of the E^2 page, the spaces $X/\!/\varphi$ are the homotopy orbit spaces of the permutation φ acting on X. We review their definition in Section 3. They decompose, up to homotopy, into a disjoint union of contractible lines, one for each infinite orbit of φ, and circles, one for each finite orbit of φ. In particular, we know their homology, the $\mathbb{Z}$–equivariant homology of (X,φ), see Proposition 3.2. This extra information makes the spectral sequence efficient. The rest of this introduction spells out the computational consequences of Theorem A.

For *free* permutation racks, where all orbits are infinite, there is no difference between the homotopy orbit space $X/\!/\varphi$ and the usual orbit space $X/\varphi = S$, which is discrete. This implies that, in this particular case, the spectral sequence has only one column and degeneracy is obvious.

Theorem B. *Let (X,φ) be a free permutation rack with the set of orbits $S = X/\varphi$. There are natural isomorphisms*

$$\mathrm{HR}_n(X,\varphi) \;\cong\; \overline{\mathbb{Z}S}^{\otimes(n-1)} \otimes \mathbb{Z}S,$$

where $\mathbb{Z}S$ is the free abelian group on the set S, and $\overline{\mathbb{Z}S}$ is the free abelian subgroup of linear combinations whose coefficients add up to zero. In particular, the homology is a free abelian group in each degree. If the number $|S| = r$ of orbits is finite, then

$$r(r-1)^{n-1} = (r-1)^n + (r-1)^{n-1}$$

is the n–th Betti number of the free permutation rack.

Actually, we use this result in the proof of Theorem A, and therefore provide an independent proof beforehand, as Theorem 2.3.

In general, the spectral sequence is concentrated in the region of the first quadrant where $p \leqslant q$. This easily implies the vanishing of the differentials in low dimensions and gives:

Theorem C. *For any permuation rack (X,φ) we have*

$$\begin{aligned}
&\mathrm{HR}_0(X,\varphi) \cong \mathbb{Z},\\
&\mathrm{HR}_1(X,\varphi) \;\cong\; \mathbb{Z}S \;\cong\; \mathbb{Z}^r,\\
&\mathrm{HR}_2(X,\varphi) \;\cong\; (\overline{\mathbb{Z}S} \otimes \mathbb{Z}S) \oplus \mathbb{Z}S_{\mathrm{fin}} \;\cong\; \mathbb{Z}^{r(r-1)+r_{\mathrm{fin}}},
\end{aligned}$$

where $S_{\mathrm{fin}} \subseteq S$ is the subset of finite orbits, and $|S_{\mathrm{fin}}| = r_{\mathrm{fin}}$ denotes its size.

These formulas for HR_0 and HR_1 agree with the known formulas for general racks. However, our computations for HR_2, especially important for applications, are new.

While our spectral sequence is built upon our direct computation for the free permutation racks, we can use it to compute the entire integral homology for the opposite extreme as well: for permutation racks that do not contain any free orbit, such as all finite permutation racks.

Theorem D. *Let (X,φ) be a finite permutation rack or, more generally, a permutation rack such that all its orbits are finite. Then*

$$\mathrm{HR}_n(X,\varphi) \;\cong\; \mathbb{Z}S^{\otimes n}$$

is a free abelian group on the set of n–tuples of orbits, where $S = X/\varphi$ as before. In particular, there is no torsion in the homology.

This is proven as Theorem 5.1 and Remark 5.2 in the main text. Briefly, we show that the functional equation satisfied by the Poincaré series of the E^2 page implies that the upper bound on the homology given by the E^2 page coincides with the lower bound known from the rational computations in [EG03]: the n–th Betti number of (X,φ) equals r^n, where $r = |X/\varphi|$ is the number of orbits. In the case of a single finite orbit, the result recovers the computation for cyclic racks from [Leb16, Thm. 6].

In the final Section 6, we prove the degeneracy of the spectral sequence in general, using most of our earlier results. As a consequence, in Corollary 6.2, we can compute the entire integral homology of any permutation rack:

Theorem E. *Let (X,φ) be any permutation rack. Then the homology is a free abelian group. In particular, it is torsion free. The Poincaré series is given as*

$$\sum_{n=0}^{\infty} \operatorname{rank} \mathrm{HR}_n(X,\varphi) T^n = \frac{1+\mathrm{T}}{1-(r-1)\mathrm{T}-r_{\mathrm{fin}}\mathrm{T}^2}$$

if the number r of orbits is finite and r_{fin} is the number of finite orbits.

Alternatively, the Betti numbers $\beta_n = \operatorname{rank} \mathrm{HR}_n(X,\varphi)$ can be computed by the following recursive formula:

$$\beta_n = \begin{cases} 1, & \text{if } n = 0, \\ r, & \text{if } n = 1, \\ (r-1)\beta_{n-1} + r_{\mathrm{fin}}\beta_{n-2} & \text{if } n \geqslant 2. \end{cases}$$

Beyond this definite result, we expect that the techniques of proof exposed in this paper will find applications to other computations of rack and related homologies.

1 Preliminaries on the homology of permutation racks

In this section, we present some general tools that help us produce and detect homology classes in permutation racks.

Let X be a set. We will write $\mathbb{Z}X$ for the free abelian group on X. Its elements are the formal linear combinations of the elements of X. The subgroup $\overline{\mathbb{Z}X}$ of $\mathbb{Z}X$ consists of those linear combinations whose coefficients sum up to zero.

Let $\mathrm{T}(\mathbb{Z}X)$ be the tensor algebra on the free abelian group $\mathbb{Z}X$. This a graded abelian group that is free in each degree: in degree n, we have the group $(\mathbb{Z}X)^{\otimes n} = \mathbb{Z}(X^n)$, with basis X^n. Using the algebra structure, we can write the elements as non-commutative monomials $x_1 \dots x_n$ rather than $x_1 \otimes \cdots \otimes x_n$ or $(x_1, \dots, x_n)$, which might have been more precise.

If $(X, \triangleright)$ is a rack, then the tensor algebra supports a differential of degree -1 which makes it a chain complex:

$$\begin{aligned}\mathrm{d}(x_1 \cdots x_n) &= \sum_{k=1}^{n-1} (-1)^{k-1} (x_1 \cdots x_{k-1} x_{k+1} \cdots x_n - x_1 \cdots x_{k-1} (x_k \triangleright x_{k+1}) \cdots (x_k \triangleright x_n)) \\ &= \sum_{k=1}^{n-1} (-1)^{k-1} x_1 \cdots x_{k-1} (x_{k+1} \cdots x_n - (x_k \triangleright x_{k+1}) \cdots (x_k \triangleright x_n)).\end{aligned}$$

This chain complex is denoted by $\mathrm{CR}_\bullet(X, \triangleright)$, and its homology $\mathrm{HR}_\bullet(X, \triangleright)$ is the rack homology of the rack $(X, \triangleright)$. Note that the differential is *not* a derivation with respect to the tensor algebra structure.

Example 1.1. A rack $(S, \triangleright)$ is *trivial* if the rack operation $\triangleright = \mathrm{pr}_2$ is given by the projection, that is $s \triangleright t = t$ for all s and t in the set S. In that case, the differential on the chain complex $\mathrm{CR}_\bullet(S, \mathrm{pr}_2)$ is zero by direct inspection, and therefore the homology of the trivial rack on S is the tensor algebra: $\mathrm{HR}_n(S, \mathrm{pr}_2) = (\mathbb{Z}S)^{\otimes n}$.

Trivial racks are examples of permutation racks $(S, \mathrm{pr}_2) = (S, \mathrm{id})$, where $\mathrm{id} = \mathrm{id}_S$ refers to the identity permutation on S. It turns out that the trivial permutation racks help us understand all other permutation racks by means of functoriality: Let (X, φ) be a permutation rack. The canonical projection $X \to X/\varphi$ is a morphism of racks if the set $X/\varphi = S$ of orbits is endowed with the trivial rack structure as in Example 1.1. This leads to an induced homomorphism

$$\mathrm{HR}_n(X, \varphi) \longrightarrow \mathrm{HR}_n(X/\varphi, \mathrm{id}) = \mathbb{Z}(X/\varphi)^n \tag{1.1}$$

in homology. This homomorphism gives us a means to detect elements in the homology of permutation racks. We remark that the underlying homomorphism on the chain level is obviously surjective, but this does not have to be the case on the level of homology.

If (X, φ) is a permutation rack, the permutation φ on X extends to an automorphism of the tensor algebra, and the formula for the differential becomes

$$\begin{aligned}\mathrm{d}(x_1 \cdots x_n) &= \sum_{k=1}^{n-1} (-1)^{k-1} x_1 \cdots x_{k-1} (x_{k+1} \cdots x_n - \varphi(x_{k+1} \cdots x_n)) \\ &= x_2 \cdots x_n - \varphi(x_2 \cdots x_n) - x_1 \mathrm{d}(x_2 \cdots x_n).\end{aligned}$$

We can rewrite this formula as

$$\mathrm{d}(xw) = w - \varphi(w) - x\mathrm{d}(w) \tag{1.2}$$

for all x in X and w in $\mathrm{CR}_\bullet(X, \varphi)$. Equation (1.2) immediately leads to a few fundamental observations. Here is the first:

Proposition 1.2. *The permutation $\varphi\colon X \to X$, which is a rack automorphism, induces the identity map $\mathrm{HR}_n(X,\varphi) \to \mathrm{HR}_n(X,\varphi)$ in rack homology for all n.*

Proof. Recall that two morphisms f and g of chain complexes are chain homotopic if their difference is of the form $f-g = \mathrm{d}h + h\mathrm{d}$ for an operator h that increases degrees by one. Then f and g induce the same homomorphism in homology. In the present situation, the formula (1.2) for the differential shows that multiplication by any element in the rack X is such an operator for $f = \mathrm{id}$ and $g = \varphi$. □

Here is another consequence of Equation (1.2):

Lemma 1.3. *Let T be a subset of (X,φ) that meets every orbit at least once. Then, modulo boundaries, every n–chain in $\mathrm{CR}_n(X,\varphi)$ can be represented as a linear combination of basis elements $w = x_1x_2\ldots x_n$ that start in T in the sense that $x_1 \in T$.*

This lemma shows that in our case it might (and will!) be fruitful to change the usual order of things and study all chains up to boundaries before restricting ourselves to cycles only.

Proof. It suffices to show that every basis element $w = x_1x_2\ldots x_n$ can be represented, modulo boundaries, as a linear combination of elements that start in T.

Let $t \in T$ be any element. Then we have $\mathrm{d}(tw) = w - \varphi(w) - t\mathrm{d}(w)$ by the formula (1.2) for the differential. This says that, modulo boundaries, we can make the difference between w and $\varphi(w)$ start with any chosen $t \in T$. By induction, the same holds for all elements $\varphi^k(w)$ in the same orbit as w. It remains to be noticed that $\varphi^k(w)$ itself starts with an element $t \in T$ at some point: take an integer k such that $\varphi^k(x_1) \in T$. This exists by our assumption on the set T. □

Remark 1.4. The lemma with its proof are valid for any rack R, not just permutation racks, and any generating set T of R.

Recall the suspension $C[1]$ of a chain complex C: we have $C[1]_n = C_{n-1}$ and $\mathrm{d}_{C[1]} = -\mathrm{d}_C$. The name is justified by the equation $\mathrm{H}_n(C[1]) = \mathrm{H}_{n-1}(C)$.

Proposition 1.5. *For all $v \in \overline{\mathbb{Z}X}$, the multiplication with v is a morphism*

$$v\colon\ \mathrm{CR}_\bullet(X,\varphi)[1] \longrightarrow \mathrm{CR}_\bullet(X,\varphi)$$

of chain complexes.

Proof. Let us first assume that $v = x - y$ is the difference of two basis elements. The formula (1.2) for the differential implies

$$\mathrm{d}((x-y)w) = -(x-y)\mathrm{d}(w) \tag{1.3}$$

for all chains w in the complex $\mathrm{CR}_\bullet(X,\varphi)$, when all differentials are computed in $\mathrm{CR}_\bullet(X,\varphi)$. This is the equation saying that multiplication with $v = x - y$ is compatible with the differentials

when we use the differential from $\mathrm{CR}_\bullet(X,\varphi)[1]$ on one side: the suspension accounts for the sign.

In general, every element $v \in \overline{\mathbb{Z}X}$ is a linear combination of elements of the form $x-y$, and multiplication with that element v is the corresponding linear combination of the multiplications with the $x-y$'s. □

Proposition 1.5 makes it easy for us to write down cycles for the homology of any permutation rack (X,φ): All elements x of X are 1–cycles. Therefore, for given elements $x_1,\dots,y_1,\dots$ in X, the element

$$(x_1-y_1)(x_2-y_2)\cdots(x_{n-1}-y_{n-1})\,x_n \tag{1.4}$$

is automatically an n–cycle. Of course, many of these elements will be boundaries if not zero. Fortunately, the homomorphism (1.1) already provides a device that allows us to detect elements in the homology of permutation racks.

In general, not all homology classes can be described by cycles of the form (1.4). Here is another device that produces new cycles from old:

Proposition 1.6. *For all fixed points x in X, the multiplication with x^2 is a morphism*

$$x^2\colon \mathrm{CR}_\bullet(X,\varphi)[2] \longrightarrow \mathrm{CR}_\bullet(X,\varphi)$$

of chain complexes.

Proof. This is a straightforward computation: from (1.2) we have

$$\begin{aligned}\mathrm{d}(x^2w) &= xw-\varphi(xw)-x\mathrm{d}(xw)\\ &= xw-x\varphi(w)-x(w-\varphi(w)-x\mathrm{d}(w))\\ &= x^2\mathrm{d}(w)\end{aligned}$$

for all chains w in the complex $\mathrm{CR}_\bullet(X,\varphi)$. □

Remark 1.7. This cycle construction can be generalized to any element x from a finite d–element orbit of X. One just needs to replace the map x^2 above with

$$\begin{aligned}x_{\mathrm{av}}\colon \mathrm{CR}_\bullet(X,\varphi)[2] &\longrightarrow \mathrm{CR}_\bullet(X,\varphi),\\ w &\longmapsto x\sum_{i=0}^{d-1}\varphi^i(xw).\end{aligned}$$

The computation from the proof of Proposition 1.6 can be adapted as follows:

$$\begin{aligned}\mathrm{d}(x\sum_{i=0}^{d-1}\varphi^i(xw)) &= \sum_{i=0}^{d-1}(\varphi^i(xw)-\varphi^{i+1}(xw))-x\sum_{i=0}^{d-1}(\varphi^i(w)-\varphi^{i+1}(w))+x\sum_{i=0}^{d-1}\varphi^i(x\mathrm{d}(w))\\ &= (xw-\varphi^d(xw))-x(w-\varphi^d(w))+x\sum_{i=0}^{d-1}\varphi^i(x\mathrm{d}(w))\\ &= (xw-x\varphi^d(w))-x(w-\varphi^d(w))+x\sum_{i=0}^{d-1}\varphi^i(x\mathrm{d}(w))\\ &= x\sum_{i=0}^{d-1}\varphi^i(x\mathrm{d}(w)).\end{aligned}$$

2 The homology of free permutation racks

In this section, we explain our computation of the homology of free permutation racks.

A permutation φ on a set X is essentially the same structure as an action of the infinite cyclic group $\mathbb{Z}$ on X. A permutation rack (X,φ) is called *free* if the corresponding action is free, that is, all orbits are infinite. In that case, the permutation rack is isomorphic to a permutation rack of the form $(\mathbb{Z}\times S,\varphi)$, where S is some set, the *basis*, and $\varphi(n,s)=(n+1,s)$ is the permutation acting on the Cartesian product $\mathbb{Z}\times S$. In that example, we will shorten $(0,s)$ to s and identify the set $(\mathbb{Z}\times S)/\varphi$ of orbits with S.

Lemma 2.1. *On a free permutation rack (X,φ) with basis S, let c be any chain in $\mathrm{CR}_\bullet(X,\varphi)$ that differs from its image $\varphi(c)$ under φ only by monomials that start in S. Then the chain c is zero.*

Proof. We can write any chain c as a linear combination of monomials $w=x_1\dots x_n$ with integral coefficients. Fix one monomial w. We shall show that the coefficient in front of w is zero if c satisfies the assumption in the statement.

By freeness, there is exactly one integer m such that the monomial $\varphi^m(w)$ starts in S. Either m is positive and $\varphi^n(w)$ does not start in S for all $n\leqslant 0$, or m is non-positive and $\varphi^n(w)$ does not start in S for all $n\geqslant 1$. We shall show that the coefficient in front of w is zero in both cases.

Assume first that $\varphi^n(w)$ does not start in S for all $n\leqslant 0$. Then w does not start in S and, by assumption, it has to appear in $\varphi(c)$ with the same coefficient. It follows that $\varphi^{-1}(w)$ appears in c with the same coefficient as w. Inductively, we see that all the $\varphi^n(w)$ with $n\leqslant -1$ appear in c with the same coefficient. Since only finitely many coefficients can be non-zero, all the coefficients of these $\varphi^n(w)$ have to be zero, and this holds, in particular, for the one in front of w.

Assume now that $\varphi^n(w)$ does not start in S for all $n\geqslant 1$. Then the monomial $\varphi(w)$ appears in c and $\varphi(c)$ with the same coefficient, which is the coefficient of w in c. Inductively, we see that all $\varphi^n(w)$ with $n\geqslant 0$ have the same coefficient in c. As above, this coefficient has to vanish. □

Lemma 2.2. *On a free permutation rack (X,φ) with basis S, let c be an n–cycle, with $n\geqslant 2$. Then c can be represented in $\overline{\mathbb{Z}S}\otimes\mathrm{CR}_{n-1}(X,\varphi)$ modulo boundaries.*

Proof. We already know from Lemma 1.3 that we can represent any chain c modulo boundaries by an element that lives in $\mathbb{Z}S\otimes\mathrm{CR}_{n-1}(X,\varphi)$. This means that we can assume c to be of the form

$$c=\sum_{s\in S} sc_s$$

for suitable chains c_s. We choose an element t in S and rewrite the expression for c as

$$c=\sum_{s\in S}(s-t)c_s+t\left(\sum_{s\in S}c_s\right).$$

Since $s-t \in \overline{\mathbb{Z}S}$, it is sufficient to show that the rightmost sum vanishes.

We recall that c is assumed to be a cycle, so that we know

$$0 = \mathrm{d}(c) = -\sum_{s\in S}(s-t)\mathrm{d}(c_s) + \left(\sum_{s\in S} c_s\right) - \varphi\left(\sum_{s\in S} c_s\right) - t\mathrm{d}\left(\sum_{s\in S} c_s\right)$$

from (1.3) and (1.2). We see that the sum $\sum_{s\in S} c_s$ satisfies the conditions of Lemma 2.1: it differs from its image under φ only by monomials that start in S. It follows from the lemma that the sum is zero, as claimed. □

We are now ready for the main result of this section.

Theorem 2.3. *For any sets S, let $(\mathbb{Z}\times S, +1)$ be the free permutation rack on S, with the permutation $+1(n,s) = (n+1,s)$. The natural homomorphism*

$$\overline{\mathbb{Z}S}^{\otimes(n-1)} \otimes \mathbb{Z}S \xrightarrow{\cong} \mathrm{HR}_n(\mathbb{Z}\times S, +1),$$

given my multiplication, is an isomorphism.

Taking a one-element set S, we recover the homology of the free monogenic rack, cf. [FRS07, FGG14]:

$$\mathrm{HR}_n(\mathbb{Z}, +1) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0,1, \\ 0 & \text{if } n \neq 0,1. \end{cases}$$

Proof. Let us first note that the map is well-defined: Take a typical element $v_1 \otimes \cdots \otimes v_{n-1} \otimes s$ with $v_j \in \overline{\mathbb{Z}S}$ and $s \in S$. It follows from Proposition 1.5 and the discussion following it that this defines a cycle in $\mathrm{CR}_\bullet(\mathbb{Z}\times S, +1)$, and then it represents a homology class.

It follows from Lemma 2.2, inductively, that the resulting homomorphism is surjective onto the homology. It remains to be seen that it is injective. We do this by composing it with the morphism

$$\mathrm{HR}_n(\mathbb{Z}\times S, +1) \longrightarrow (\mathbb{Z}S)^{\otimes n}$$

from (1.1). Since all maps are the identity on representatives, this composition is the inclusion, hence injective. Then the first map also has to be injective, as claimed. □

Corollary 2.4. *The homology of a free permutation rack (X,φ) is free as an abelian group. In particular, it is torsion-free. If the number of orbits is finite, say r, then the Betti numbers are given by*

$$\operatorname{rank}\mathrm{HR}_n(X,\varphi) = (r-1)^{n-1}r = (r-1)^n + (r-1)^{n-1}$$

for all $n \geqslant 1$.

Remark 2.5. The free permutation rack is the product of racks:

$$(\mathbb{Z}\times S, +1) \cong (\mathbb{Z}, +1) \times (S, \mathrm{id}).$$

Our theorem and Example 1.1 yield the entire homology of the three racks involved, and we will use the occasion to point out that a naive version of the Künneth theorem for rack homology is

false. Indeed, given two racks $(X, \triangleright_X)$ and $(Y, \triangleright_Y)$, and their direct product $(X \times Y, \triangleright_X \times \triangleright_Y)$, it is tempting to conjecture the existence of a short exact sequence

$$0 \to \bigoplus_{p+q=n} \mathrm{HR}_p(X) \otimes \mathrm{HR}_q(Y) \longrightarrow \mathrm{HR}_n(X \times Y) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}^1_{\mathbb{Z}}(\mathrm{HR}_p(X), \mathrm{HR}_q(Y)) \to 0.$$

For a free permutation rack, this sequence would imply

$$\begin{aligned} \mathrm{HR}_n(\mathbb{Z} \times S, +1) &\cong \bigoplus_{p+q=n} \mathrm{HR}_p(\mathbb{Z}, +1) \otimes \mathrm{HR}_q(S, \mathrm{id}) \\ &\cong (\mathbb{Z}S)^{\otimes n} \oplus (\mathbb{Z}S)^{\otimes (n-1)}, \end{aligned}$$

contradicting our theorem.

Remark 2.6. Similar ideas yield a completely algebraic computation of the homology of the free rack FR_n on the n generators $X = \{x_1, \ldots, x_n\}$, which is considerably more concise than the method from [FGG14]. Recall that the free rack FR_n can be modelled as the set $\mathrm{F}_n \times X$, where F_n is the free group on the generators X, with the rack operation given as $(u,x) \triangleright (w,y) = (uxu^{-1}w, y)$. Let us show that

$$\mathrm{HR}_m(\mathrm{FR}_n) \cong \begin{cases} \mathbb{Z} & \text{if } m = 0, \\ \mathbb{Z}^n & \text{if } m = 1, \\ 0 & \text{if } m \neq 0, 1. \end{cases}$$

For $m \leqslant 1$ the result is classical, given that the rack FR_n has n orbits. Now, take an $m > 1$ and an m–chain c. As explained in Remark 1.4, we may assume that c is of the form $c = \sum_{i=1}^n x_i c_i$, where the c_i are $(m-1)$–chains. We will prove that all the c_i have to vanish. An analogue of formula (1.2) yields

$$0 = \mathrm{d}(c) = \sum_i c_i - \sum_i x_i \triangleright c_i - \sum_i x_i \mathrm{d}(c_i). \tag{2.1}$$

Here, the action $\triangleright$ has been extended from X to X^{m-1} diagonally. To analyze the monomial cancellation in the above sum, consider a graph whose vertices are monomials from X^{m-1}, and whose x_i–labelled edges point from a monomial w to $x_i \triangleright w$. Cancellations in (2.1) can come either from a cycle in this graph, or from a path starting with a vertex of the form $x_j u$ and ending with some $x_k v$. These cycles and paths can be assumed not to contain the same edge twice in a row. If such a path or cycle starts with some au, with $a = (w, x_j) \in \mathrm{FR}_n$, then it ends with some $a'v$, with $a' = (pw, x_j)$, where $p \in \mathrm{F}_n$ is the product (written from right to left) of the labels $x_s^{\pm 1}$ encountered along our path, with the choice x_s^{-1} corresponding to an edge travelled in the opposite direction. Since $pw \neq w$ for a non-trivial path, there can be no non-trivial cancellation paths. On the other hand, if there is a non-zero c_i, then cancellations are necessary.

3 Permutations and their equivariant homology

Permutation racks (X, φ) are essentially sets X with an action of the infinite cyclic group $\mathbb{Z}$, or $\mathbb{Z}$–sets for short. The integer n acts as φ^n. We refer to [Szy18, Sec. 3] for a detailed discussion

of the relation between racks and permutations on the categorical level. In this section, we will very briefly review the equivariant homology theory for actions of a fixed group G, and then we specialize it to the case of the infinite cyclic group $\mathbb{Z}$. Proposition 3.2, the computation of the equivariant homology of permutation actions, will be used in the following Section 4.

Let us fix a discrete group G. If the group G is not trivial, then not all G–sets X are free. Therefore, we need to choose free resolutions. These are G–maps $F_\bullet \to X$ that are equivalences, where F is a simplicial G–set, and the G–action on F is free (on each set of n–simplices). For instance, the classifying space $\mathrm{E}G_\bullet$ coming from the bar construction is a contractible space on which the group G acts freely; it provides functorial free simplicial resolutions $\mathrm{E}G_\bullet \times X \to X$ for all G–sets X. The equivariant homology $\mathrm{H}^G_\bullet(X)$ is the homology of the orbit space $F_\bullet/G \simeq \mathrm{E}G_\bullet \times_G X$. Up to homotopy, this space does not depend on the resolution $F_\bullet$ used to compute it, and it is common to denote this *homotopy orbit space* by $X/\!/G$. These constructions work more generally for G–spaces, or rather simplicial G–sets, and they are interesting even for the trivial G–set $X = \star$ consisting of one fixed point only: its equivariant homology is the homology of the group G.

Remark 3.1. Equivariant homology in this form was initiated by Borel [Bor60]. It is obvious from the description above that Borel's equivariant homology theory agrees with Quillen's general homology theory [Qui70] when specialized to the algebraic theory of G–sets for a fixed group G. This fact is well-known and we emphasize that it plays no role in the following. We refer to [Szy19] for Quillen's homology theory when specialized to the algebraic theory of racks.

The situation is particularly transparent for the infinite cyclic group $G = \mathbb{Z}$ of interest to us. The action is determined by the permutation φ corresponding to the generator. We need to know the equivariant homology

$$\mathrm{H}^{\mathbb{Z}}_\bullet(X) = \mathrm{H}_\bullet(X/\!/\varphi)$$

of any $\mathbb{Z}$–set X. In this case, the space $X/\!/\varphi$ is the mapping torus of the self-map $\varphi\colon X \to X$. The homology of the mapping torus sits in a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_\bullet(X) \xrightarrow{\mathrm{id}-\varphi_\bullet} \mathrm{H}_\bullet(X) \longrightarrow \mathrm{H}_\bullet(X/\!/\varphi) \longrightarrow \mathrm{H}_{\bullet-1}(X) \xrightarrow{\mathrm{id}-\varphi_\bullet} \cdots .$$

Proposition 3.2. *Let φ be a permutation on a set X. Let $S = X/\varphi$ be the set of orbits of φ, and let $S_{\mathrm{fin}} \subseteq S$ be the subset of finite orbits. Then the equivariant homology of (X, φ) is given as*

$$\mathrm{H}_n(X/\!/\varphi) \cong \begin{cases} \mathbb{Z}S & \text{if } n = 0, \\ \mathbb{Z}S_{\mathrm{fin}} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let us first assume that X is a single orbit. If X is a free orbit, then there is no need to resolve X, and $X/\!/\varphi \simeq X/\varphi$ is a point. This proves the claim for a single free orbit. If X is a finite orbit, then the mapping torus $X/\!/\varphi$ is a circle. Therefore, or by inspection of the long exact sequence, we see that the claim is true for finite orbits, too. In general, any G-set is the disjoint union of its orbits, and the homology of a disjoint union is the direct sum of the homologies. This proves the claim in general. □

This result is the reason why finite and infinite orbits play fundamentally different roles in the homology of permutation racks.

4 The spectral sequence: construction and first applications

Finally, we can now address the following question: Given a set X with a permutation φ, how can we compute the cohomology of the associated permutation rack (X,φ)? Our answer follows the blueprint given by Quillen in [Qui70, Sec. 8]. He computes the associative algebra homology of commutative algebras from the knowledge of the associative algebra homology of *free* commutative algebras (polynomial rings). Here we adapt his methods and deduce the homology of permutation racks from the homology of *free* permutation racks via a spectral sequence.

Theorem 4.1. *For every permutation rack (X,φ), there is a spectral sequence of homological type which has its* E^2 *page given as*

$$\mathrm{E}^2_{\bullet,q} \cong \overline{\mathrm{H}}_\bullet(X/\!/\varphi)^{\otimes(q-1)} \otimes \mathrm{H}_\bullet(X/\!/\varphi)$$

and which abuts to the rack homology of (X,φ).

The homology groups on the right hand side are the ones computed in Proposition 3.2. We'll spell out the consequences after the proof.

Proof. We choose a simplicial resolution $X \leftarrow F_\bullet$ by free permutations, as described in Section 3. Then we apply the rack chain complex functor $\mathrm{CR}_\bullet(?)$ to it. This gives a simplicial chain complex $\mathrm{CR}_\bullet(F_\bullet)$. The Moore construction, which turns a simplicial abelian group into a chain complex with differential $\partial = \sum_j (-1)^j \partial_j$, turns this into a double complex

$$\mathrm{E}^0_{p,q} = \mathrm{CR}_q(F_p).$$

This double complex comes with two spectral sequences that converge to the same target: the homology of the totalization. We will inspect these in order.

First, let us consider the spectral sequence that computes the differential in the horizontal p–direction first. For a fixed q, Lemma 2.3 in [Szy19] says that the Moore complex $\mathrm{CR}_q(F_\bullet)$ is a free resolution of the abelian group $\mathrm{CR}_q(X)$. It follows that we get

$$\mathrm{E}^1_{p,q} \cong \begin{cases} \mathrm{CR}_q(X) & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases}$$

The vertical differential is the one coming from the rack complex, by naturality, so that we get

$$\mathrm{E}^2_{p,q} \cong \begin{cases} \mathrm{HR}_q(X) & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases}$$

Since this is concentrated in the 0–th column, the spectral sequence necessarily degenerates from E^2 on. This shows that the target of this (and the other!) spectral sequence is the rack homology of X.

Second, let us consider the spectral sequence that computes the differential in the vertical q–direction first. By now, we already know that its target is the rack homology of X, but we haven't given a useful description of the initial terms, yet. The starting page is

$$\mathrm{E}^0_{p,q} = \mathrm{CR}_q(F_p)$$

as before. If we fix p, and compute the vertical differential, we get

$$\mathrm{E}^1_{p,q} = \mathrm{HR}_q(F_p),$$

the rack homology of the free racks F_p. We already have computed this in Section 2. If we let S_p be the set of orbits of F_p, then Theorem 2.3 gives a natural isomorphism

$$\mathrm{HR}_q(F_p) \cong \overline{\mathbb{Z}S}_p^{\otimes(q-1)} \otimes \mathbb{Z}S_p.$$

Naturality implies that, for a fixed degree q, we have an isomorphism

$$\mathrm{HR}_q(F_\bullet) \cong \overline{\mathbb{Z}(X/\!/\varphi)}^{\otimes(q-1)} \otimes \mathbb{Z}(X/\!/\varphi) \tag{4.1}$$

of simplicial abelian groups, where $X/\!/\varphi$ is the simplicial set of orbits of $F_\bullet$. It follows that the q–th row of the E^1 page of the spectral sequences is the Moore complex of the right hand side of (4.1). Up to equivalence, the Moore complex commutes with tensor products. The Moore complex for $\mathbb{Z}(X/\!/\varphi)$ computes the homology of $X/\!/\varphi$, and the Moore complex for $\overline{\mathbb{Z}(X/\!/\varphi)}$ computes the *reduced* homology of $X/\!/\varphi$. Both can be read off immediately from our Proposition 3.2. Since all of these homologies are free, the Künneth theorem implies that we have an isomorphism

$$\mathrm{E}^2_{\bullet,q} \cong \overline{\mathrm{H}}_\bullet(X/\!/\varphi)^{\otimes(q-1)} \otimes \mathrm{H}_\bullet(X/\!/\varphi),$$

as claimed. □

It follows from Proposition 3.2 that we have $\mathrm{E}^2_{p,q} = 0$ whenever $p > q$. This implies that all the differentials involving the $\mathrm{E}^2_{p,q}$ with $p+q \leqslant 2$ are zero. Since all of these abelian groups are also free, again by Proposition 3.2, there are no extension problems, and we get

$$\begin{aligned}
\mathrm{HR}_0(X,\varphi) &\cong \mathbb{Z} \\
\mathrm{HR}_1(X,\varphi) &\cong \mathrm{H}_0(X/\!/\varphi) \\
&\cong \mathbb{Z}S \\
\mathrm{HR}_2(X,\varphi) &\cong \Big(\overline{\mathrm{H}}_0(X/\!/\varphi) \otimes \mathrm{H}_0(X/\!/\varphi)\Big) \oplus \mathrm{H}_1(X/\!/\varphi) \\
&\cong (\overline{\mathbb{Z}S} \otimes \mathbb{Z}S) \oplus \mathbb{Z}S_{\mathrm{fin}},
\end{aligned}$$

where S is the set of orbits of φ on X, and S_{fin} is the subset of finite orbits. Theorem C follows.

Remark 4.2. Another important invariant of a rack $(X, \triangleright)$ is its *structure group*, given by the presentation

$$G(X, \triangleright) = \langle g_a,\, a \in X \,|\, g_a g_b = g_{a \triangleright b} g_a,\, a, b \in X \rangle.$$

This is the value of the left-adjoint to the forgetful functor from groups to racks which sends a group to its conjugation rack. It is easy to compute the structure group of a permutation rack (X, φ): it is the free abelian group on the set of orbits of φ. Up to isomorphism, this group depends on the number of orbits of φ only. According to our preceding computations of $\mathrm{HR}_2(X, \varphi)$, two permutation racks with the same (finite) number of orbits but different numbers of finite orbits then have isomorphic structure groups but non-isomorphic homology groups.

Example 4.3. If X is a free permutation rack, then $\mathrm{H}_\bullet(X /\!/ \varphi)$ is concentrated in degree 0, and so the spectral sequence is concentrated in the 0–th column from E^2 on. Then it degenerates, and we recover the result of Theorem 2.3. Of course, we have used that result already when setting up the spectral sequence, and this example only serves as a consistency check.

5 The spectral sequence degenerates for finite racks

We now turn to a more substantial example and prove Theorem D.

Theorem 5.1. *Let (X, φ) be a finite permutation rack. Then the spectral sequence degenerates from its E^2 page on, and the rack homology $\mathrm{HR}_n(X, \varphi)$ is a free abelian group of rank r^n, where $r = |X/\varphi|$ is the number of orbits of φ acting on X.*

Proof. We inspect the E^2 page of the spectral sequence in Theorem 4.1. We have a free abelian group in every bidegree (p, q), hence in every total degree n. We will now show that the rank of the E^2 page in total degree n is r^n.

Let $f(\mathrm{T})$ be the Poincaré series of $\overline{\mathrm{H}}_\bullet(X /\!/ \varphi)$. If r is the number of orbits, then we have

$$f(\mathrm{T}) = (r-1) + r\mathrm{T} \tag{5.1}$$

by Proposition 3.2. This satisfies the functional equation

$$1 - f(\mathrm{T})\mathrm{T} = (1+\mathrm{T})(1-r\mathrm{T}), \tag{5.2}$$

as is straightforward to verify from (5.1). Note that we used that all orbits are finite; otherwise the Poincaré series $f(\mathrm{T})$ is a bit more complicated. From the description of the q–th row of the E^2 page of the spectral sequence, we get that its Poincaré series is 1 for $q = 0$

and $f(\mathrm{T})^q + f(\mathrm{T})^{q-1}$ for all $q \geqslant 1$, regardless of the orbit structure. Therefore, we have

$$\begin{aligned}
\sum_{n\geqslant 0}\Big(\sum_{p+q=n}\operatorname{rank}(\mathrm{E}^2_{p,q})\Big)\mathrm{T}^n &= \sum_{p,q\geqslant 0}\operatorname{rank}(\mathrm{E}^2_{p,q})\mathrm{T}^{p+q} \\
&= \sum_{q\geqslant 0}\Big(\sum_{p\geqslant 0}\operatorname{rank}(\mathrm{E}^2_{p,q})\mathrm{T}^p\Big)\mathrm{T}^q \\
&= 1+\sum_{q\geqslant 1}\Big(f(\mathrm{T})^q + f(\mathrm{T})^{q-1}\Big)\mathrm{T}^q \\
&= \sum_{q\geqslant 0} f(\mathrm{T})^q\mathrm{T}^q + \sum_{q\geqslant 0} f(\mathrm{T})^q\mathrm{T}^{q+1} \\
&= (1+\mathrm{T})\sum_{q\geqslant 0} f(\mathrm{T})^q\mathrm{T}^q \\
&= \frac{1+\mathrm{T}}{1-f(\mathrm{T})\mathrm{T}}.
\end{aligned}$$

In our specific situation, we can use the functional equation (5.2) to see that this equals

$$\frac{1}{1-r\mathrm{T}} = \sum_{n\geqslant 0} r^n\mathrm{T}^n,$$

as claimed.

It is known from the work [EG03, Cor. 4.3] of Etingof and Graña that the n–th Betti number of a finite permutation rack with r orbits is r^n. So r^n is the rank of the E^∞ page in total degree n as well. Given that all the groups on the E^2 page are free abelian, if the spectral sequence would not degenerate from its E^2 page on, the rank of the E^∞ page would be strictly less than that of the E^2 page, contradicting what we have shown above. Therefore the spectral sequence does degenerate from its E^2 page on. Since $\mathrm{E}^\infty \cong \mathrm{E}^2$ is free abelian, there are no extension problems, and we find that $\mathrm{HR}_n(X,\varphi)$ is the totalization of it, which we have just shown to be of rank r^n in total degree n. □

Remark 5.2. It is easy to generalize Theorem 5.1 slightly to cover all permutation racks (X,φ) for permutations φ without free orbits. These are the unions of their finite subracks, and homology is compatible with unions. It follows that the rack homology $\mathrm{HR}_\bullet(X,\varphi)$ of these racks is always free abelian in each degree.

6 The spectral sequence always degenerates

This final section completes the proof of Theorem A by proving the following result.

Theorem 6.1. *For any permutation rack (X,φ), the spectral sequence in Theorem 4.1 degenerates from its E^2 page on.*

Proof. As in Remark 5.2, it is clear that we can assume that the permutation φ on X has only *finitely many orbits*: any differential involves only finitely many of them.

Under this assumption, the E^2 page consists of finitely generated free abelian groups in each bidegree. Therefore, as in our proof of Theorem 5.1, it is sufficient to demonstrate the degeneracy with *rational coefficients*: this is enough to detect non-zero differentials between finitely generated free abelian groups. For the rest of the proof, we can and will assume that (X, φ) is a permutation rack with finitely many orbits, and all homology will be with rational coefficients.

We construct a new permutation rack $(\overline{X}, \overline{\varphi})$ from (X, φ) as a quotient, by collapsing each finite orbit onto a single fixed point, leaving the free orbits as they were. The new permutation rack has the same number of orbits and of finite orbits, but it is *semi-free*: the complement of the fixed point set is free. The quotient map $(X, \varphi) \to (\overline{X}, \overline{\varphi})$ is a morphism of (permutation) racks.

We claim that this morphism induces an isomorphism of spectral sequences from the E^2 pages on. To justify this claim, note that the E^2 pages are given in terms of the equivariant homology of the homotopy orbit spaces $X/\!/\varphi$ and $\overline{X}/\!/\overline{\varphi}$, respectively. The induced map $X/\!/\varphi \to \overline{X}/\!/\overline{\varphi}$ between the homotopy orbit spaces is as follows: The components coming from free orbits are contractible, so any map is a homotopy equivalence. The components coming from the finite orbits are equivalent to circles, and the collapse of an orbit of length $n \geqslant 1$ to a single fixed point induces an n–fold covering. Both maps yield isomorphisms on the level of rational homology. Together with naturality, this proves our claim: we have a morphism of spectral sequences that is an isomorphism from their E^2 pages on.

After all this dévissage, we are left to deal with a semi-free permutation rack (X, φ) with finitely many orbits. We can compute the Poincaré series of the E^2 page as in the proof of Theorem 5.1:

$$\sum_{n \geqslant 0} \Big(\sum_{p+q=n} \operatorname{rank}(E^2_{p,q}) \Big) T^n = \frac{1+T}{1-f(T)T}.$$

In contrast to Theorem 5.1, we now have

$$f(T) = (r-1) + r_{\mathrm{fin}} T,$$

where r is the number of orbits, and r_{fin} is the number of finite orbits. Therefore, the Poincaré series of the E^2 page equals

$$\frac{1+T}{1-(r-1)T-r_{\mathrm{fin}}T^2}.$$

Equivalently, the Betti numbers $\beta_n = \sum_{p+q=n} \operatorname{rank}(E^2_{p,q})$ can be computed from the following recursion formula:

$$\beta_0 = 1, \qquad \beta_1 = r, \tag{6.1}$$

$$\beta_{n+2} = (r-1)\beta_{n+1} + r_{\mathrm{fin}}\beta_n, \qquad n \geqslant 0. \tag{6.2}$$

We thus obtain an upper bound on the Poincaré series of the E^∞ page, which agrees with the Poincaré series of the homology $\mathrm{HR}_\bullet(X, \varphi)$. We claim that this bound is sharp: they are equal. Once we have established that claim, the result follows because any non-zero differential would contradict the equality of the Poincaré series.

To prove the claim, we will use Propositions 1.5 and 1.6 to produce a sufficient supply of homology classes in $\mathrm{HR}_\bullet(X, \varphi)$. We choose a set $Q \subseteq X$ of orbit representatives, with a preferred one q_*. Let $T \subseteq Q$ denote the set of fixed points. We define subsets $B_n \subseteq \mathrm{HR}_\bullet(X, \varphi)$

inductively by

$$B_0 = \{\,1\,\}, \qquad B_1 = Q,$$
$$B_n = \{\,(q-q_*)\cdot b_1 \,|\, q \in Q\setminus\{q_*\}, b_1 \in B_{n-1}\,\} \cup \{\,t^2\cdot b_2 \,|\, t\in T, b_2 \in B_{n-2}\,\}.$$

In this description, to exhibit a homology class, we abusively give its representative only. By construction, the cardinalities $|B_n|$ satisfy the same recursive formula as the Betti numbers β_n above, and thus coincide.

It remains to be shown that the homology classes in each B_n are linearly independent. We will actually prove slightly more: the images of these homology classes in $\mathrm{HR}_n(S,\mathrm{id}) = \mathbb{Z}S^n$ under the map (1.1) are linearly independent. Identifying the orbit set $S = X/\varphi$ with the set of orbit representatives Q, we can abusively consider B_n as a subset of $\mathbb{Z}S^n$. We will need to slightly modify some elements b of B_n. By construction, each b is a product of some terms of the form $(q-q_*)$, and some terms of the form t^2. Read b from left to right. Whenever we see a repetition $b = a(q-q_*)(q-q_*)c$, with $q\in T$, we replace it with

$$a(q-q_*)(q-q_*)c - aqqc = a(-qq_* - q_*q + q_*q_*)c.$$

We repeat this transformation for c if necessary, and so on. Denote the element so obtained by b'. At each step, the element we subtract has more terms of the form t^2 than the original element; therefore the modified elements are linearly independent if and only if the original elements of B_n were. Now, we order the orbit set S in any way which makes q_* the minimal element. We extend this order lexicographically to S^n. The maximal monomial of b', with respect to this order, is obtained from b by replacing each $(q-q_*)$ with q, except for neighboring factors $(q-q_*)(q-q_*)$, with $q\in T$, which become qq_*. This latter replacement is again done consecutively from left to right, and one can reconstruct b uniquely from this maximal term. We remark that this is no longer true if one considers the maximal term of b itself, since the two elements $(q-q_*)(q-q_*)p$ and qqp, with $q\in T$, have the same maximal term, and that is why the rather technical deformation procedure from b to b' was necessary. As a consequence of the fact that we can reconstruct b uniquely from the maximal term, the set $B_n \subseteq \mathbb{Z}S^n$ is linearly independent, as desired. □

Corollary 6.2. *Let (X,φ) be a permutation rack with r_{fin} of the r orbits finite. Then all its homology groups $\mathrm{HR}_n(X,\varphi)$ are free abelian, and the Poincaré series of the rack homology is*

$$\sum_{n=0}^{\infty} \operatorname{rank}\mathrm{HR}_n(X,\varphi)T^n = \frac{1+\mathrm{T}}{1-(r-1)\mathrm{T}-r_{\mathrm{fin}}\mathrm{T}^2}.$$

The reader can easily check that this formula specializes to the ones given earlier in the cases when (X,φ) is free ($r_{\mathrm{fin}}=0$) or all orbits are finite ($r_{\mathrm{fin}}=r$).

Remark 6.3. Standard manipulations transform the above formula into an equivalent one, which can be more suitable in practice:

$$\sum_{n=0}^{\infty} \operatorname{rank}\mathrm{HR}_n(X,\varphi)\mathrm{T}^n = \sum_{n=0}^{\infty}(r-1)^n\left(\frac{1+\mathrm{T}}{\mathrm{T}}\right)\left(\frac{\mathrm{T}}{1-r_{\mathrm{fin}}\mathrm{T}^2}\right)^{n+1}.$$

The recursive formulas (6.1) and (6.2) are probably even more convenient for computations.

Remark 6.4. In the proof of the theorem, one could avoid collapsing all finite orbits to fixed points, at the price of replacing in the definition of B_n the elements $t^2 \cdot b_2$ with the more involved construction $t_{\mathrm{av}}(b_2)$ from Remark 1.7. We preferred not to do so, keeping the main line of thought as non-computational as possible.

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