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► **To cite this version:**

Miguel Abadi, Nicolas Vergne. Sharp error terms for return time statistics under mixing conditions
*. Journal of Theoretical Probability, 2009. hal-02337198

HAL Id: hal-02337198

<https://normandie-univ.hal.science/hal-02337198>

Submitted on 29 Oct 2019

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Sharp error terms for return time statistics under mixing conditions *

Miguel Abadi [†] Nicolas Vergne [‡]

Abstract

We describe the statistics of repetition times of a string of symbols in a stochastic process. We consider a string A of length n and prove: 1) The time elapsed until the process starting with A repeats A , denoted by τ_A , has a distribution which can be well approximated by a degenerated law at the origin and an exponential law. 2) The number of consecutive repetitions of A , denoted by S_A , has a distribution which is approximately a geometric law. We provide sharp error terms for each of these approximations. The errors we obtain are point-wise and allow to get also approximations for all the moments of τ_A and S_A . Our results hold for processes that verify the ϕ -mixing condition.

Keywords: Mixing, recurrence, rare event, return time, sojourn time.

Running head: Return times under mixing conditions.

1 INTRODUCTION

This paper describes the return time statistics of a string of symbols in a mixing stochastic process with a finite alphabet. Generally speaking, the study of the time elapsed until the first occurrence of a small probability event in dependent processes has a long history, see for instance [10] and the references therein. The typical result is:

$$\lim_{n \rightarrow \infty} \mathbb{P}(\tau_{A_n} > t b_n \mid \mu_0) = e^{-t}, \quad (1.1)$$

where τ_{A_n} is the first time the process hits a given measurable set A_n , $n \in \mathbb{N}$ and such that the measure $\mathbb{P}(A_n)$ go to zero as $n \rightarrow \infty$, $\{b_n\}_{n \in \mathbb{N}}$ is a suitable re-scaling sequence of positive numbers and μ_0 is a given initial condition.

*Work done as part of the activities of the project Projeto Temtico-PRONEX “*Stochastic behavior, critical phenomena and rhythmic patterns identification in natural languages*”. Pros. FAPESP 03/09930-9.

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Recently an exhaustive analysis of these statistics was motivated by applications in different areas as entropy estimation, genome analysis, computer science, linguistic, among others. From the point of view of applications, a fundamental task is to understand the rate of convergence of the limit (1.1). A detailed review of such results appearing in the literature can be found in [3].

It is the purpose of this paper to present the following new results: For *any* string A of length n

- A sharp upper bound for the above rate of convergence in general ϕ -mixing processes that holds when $\mu_0 = A$.
- A sharp upper bound for the difference between the law of the number of *consecutive* visits to A and a geometric law.

When μ_0 is taken as A , we refer to the distribution $\mathbb{P}(\tau_A > t \mid A)$ as the *return time*. In general it can not be well approximated by an exponential law. This was firstly noted by Hirata, when he proved the convergence of the number of visits to a small cylinder around a point to the Poisson law. His result holds for axiom A diffeomorphisms (see [11]). The result holds for *almost* every point. Then, he proved that for periodic points, the asymptotic limit law of the return time differs from the one-level Poisson law, namely e^{-t} .

Our first result concerns the rate of convergence of limit in (1.1) when $\mu_0 = A$ for any string A of length n . We prove that the return time law converges to a convex combination of a Dirac law at the origin and an exponential law. Specifically, we show that for large n

$$\mathbb{P}\left(\tau_A > \frac{t}{\mathbb{P}(A)} \mid A\right) \approx \begin{cases} 1 & t \leq \mathbb{P}(A)\tau(A) \\ \zeta_A e^{-\zeta_A t} & t > \mathbb{P}(A)\tau(A) \end{cases} .$$

$\tau(A)$ is the position of the first overlap of A with a copy of itself (see definition below). ζ_A is a parameter related to the overlap properties of the string A . It is worth noting that the parameter of the exponential law is exactly the weight of the convex combination. So far, the overlap properties of a string appears as a major factor to describe the statistical properties of the return time. For instance, if a string overlaps itself, then it will turn out in the sequel that $\zeta_A \neq 1$ and the return time distribution approximates the above mixture of laws. However, for a word which does not overlap itself, it will turn out that $\zeta_A = 1$ and the return time distribution approximates a purely exponential law. For the role of overlaps an a treatment of the independent case with a good introduction to the previous literature see [5], and for the Markov case with a probability generating functions point of view see [16].

It is worth recalling at this point that when in equation (1.1) the initial condition is the equilibrium measure of the process, τ_A is called the *hitting time* of A . In [12] it is proved a rate of convergence of the return time as function of the distance between the hitting time and return time laws. While this result applies only for cylinders around non-periodic points, our result applies to *all* of them.

The great enhancement of our work is that, contrarily to all the previous works which present bounds depending only on the string A , our error estimate

decays exponentially fast in t for all $t > 0$. As a byproduct we obtain *explicit expressions* for *all the moments* of the return time. This also appears as a generalization of the famous Kac's lemma (see [13]) which states that the *first* moment of the return time to a string A of positive measure is equal to $\mathbb{P}(A)^{-1}$ and the result in [7] which presents conditions for the *existence* of the moments of return times. Further, [12] proves that hitting and return times coincide if and only if the return time converges to the exponential law. We extend this result establishing that the laws of hitting and return times coincide if and only if the weight of the Dirac measure in the convex combination of the return time law is zero, which is equivalent to consider a non-overlapping string.

Our framework is the class of ϕ -mixing processes. For instance, irreducible and aperiodic finite state Markov chains are known to be ψ -mixing (and then ϕ -mixing) with exponential decay. Moreover, Gibbs states which have summable variations are ψ -mixing (see [17]). They have exponential decay if they have Hölder continuous potential (see [6]). However, sometimes the ψ -mixing condition is very restricted hypothesis difficult to test. We establish our result under the more general ϕ -mixing condition. Further examples of ϕ -mixing processes can be found in [14]. The error term is explicitly expressed as a function of the mixing rate ϕ . We refer the reader to [9] for a source of examples and definitions of the several kinds of mixing processes.

The base of our proof is a sharp upper bound on the rate of convergence of the hitting time to an exponential law proved in [2].

The self-repeating phenomena in the distribution of the return time leads us to consider the problem of the sojourn time. Our second result states that the law of the number of consecutive repetitions of the string A , denoted by S_A , converges to a geometric law. Namely

$$\mathbb{P}(S_A = k \mid A) \approx (1 - \rho(A))\rho(A)^k . \quad (1.2)$$

Again here, the parameter $\rho(A)$ depends on the overlap properties of the string. Furthermore we show that under suitable conditions one has $\rho(A) \approx 1 - \zeta_A$. As far as we know, this is the first result on this subject for dependent processes.

As in our previous result, the error bound we obtain decreases geometrically fast in k (see (1.2)). This decay on the error bound allows us to obtain an approximation for *all the moments* of S_A for those of a geometrically distributed random variable.

Our results are applied in a forthcoming paper: In [4] the authors prove large deviations and fluctuations properties of the repetition time function introduced by Wyner and Ziv in [18] and further by Ornstein and Weiss in [15], and get entropy estimators.

This paper is organized as follows. In section 2 we establish our framework. In section 3 we describe the self-repeating properties needed to state the return time result. In section 4 we establish the approximation for the return time

law. This is Theorem 4.1. Finally, in section 5 we state and prove the geometric approximation for the consecutive repetitions of a string. This is Theorem 5.1.

2 FRAMEWORK AND NOTATION

Let \mathcal{C} be a finite set. Put $\Omega = \mathcal{C}^{\mathbb{Z}}$. For each $x = (x_m)_{m \in \mathbb{Z}} \in \Omega$ and $m \in \mathbb{Z}$, let $X_m : \Omega \rightarrow \mathcal{C}$ be the m -th coordinate projection, that is $X_m(x) = x_m$. We denote by $T : \Omega \rightarrow \Omega$ the one-step-left shift operator, namely $(T(x))_m = x_{m+1}$.

We denote by \mathcal{F} the σ -algebra over Ω generated by strings. Moreover we denote by \mathcal{F}_I the σ -algebra generated by strings with coordinates in I , $I \subseteq \mathbb{Z}$.

For a subset $A \subseteq \Omega$, $A \in \mathcal{C}^n$ if and only if

$$A = \{X_0 = a_0; \dots; X_{n-1} = a_{n-1}\},$$

with $a_i \in \mathcal{C}$, $i = 0, \dots, n-1$.

We consider an invariant probability measure \mathbb{P} over \mathcal{F} . We shall assume without loss of generality that there is no singleton of probability 0.

For two measurable sets V and W , we denote as usual $\mathbb{P}(V|W) = \mathbb{P}_W(V) = \mathbb{P}(V; W) / \mathbb{P}(W)$ the conditional measure of V given W . We write $\mathbb{P}(V; W) = \mathbb{P}(V \cap W)$.

We say that the process $\{X_m\}_{m \in \mathbb{Z}}$ is ϕ -mixing if the sequence

$$\phi(l) = \sup |\mathbb{P}_B(C) - \mathbb{P}(C)|,$$

converges to zero. The supremum is taken over B and C such that $B \in \mathcal{F}_{\{0, \dots, n\}}$, $n \in \mathbb{N}$, $\mathbb{P}(B) > 0$, $C \in \mathcal{F}_{\{m \geq n+l+1\}}$.

We use the measure theoretic notation: $\{X_n^m = x_n^m\} = \{X_n = x_n, \dots, X_m = x_m\}$. For an n -string $A = \{X_0^{n-1} = x_0^{n-1}\}$ and $1 \leq w \leq n$, we write $A^{(w)} = \{X_{n-w}^{n-1} = x_{n-w}^{n-1}\}$ for the w -string belonging to the σ -algebra $\mathcal{F}_{\{n-w, \dots, n-1\}}$ and consisting of the *last* w symbols of A . We write $V^c = \Omega \setminus V$, for the complement of V .

The conditional mean of a r.v. X with respect to any measurable set V will be denoted by $\mathbb{E}_V(X)$ and we put $\mathbb{E}(X)$ when $V = \Omega$. Wherever it is not ambiguous we will write C for different positive constants even in the same sequence of equalities/inequalities. For brevity we put $(a \vee b) = \max\{a, b\}$ and $(a \wedge b) = \min\{a, b\}$.

3 PERIODS

Definition 3.1 *Let $A \in \mathcal{C}^n$. We define the period of A (with respect to T) as the number $\tau(A)$ defined as follows:*

$$\tau(A) = \min \{k \in \{1, \dots, n\} \mid A \cap T^{-k}(A) \neq \emptyset\}.$$

By definition, if $A \in \mathcal{C}^n$, then $A = (a_0, \dots, a_{n-1})$, $a_i \in \mathcal{C}$ for $0 \leq i \leq n-1$. For instance, pick up $A = (aaaabbbaaaabbaaaa) \in \mathcal{C}^{15}$. Then shift a copy of A until there is a fit between them. Namely

$$\begin{array}{l} A = \text{aaaabbbaaaabbaaaa} \\ T^{-6}(A) = \text{aaaabbbaaaaabbaaaa} \end{array} .$$

Let us take $A \in \mathcal{C}^n$, and write $n = q\tau(A) + r$, with $q = [n/\tau(A)]$ and $0 \leq r < \tau(A)$. Thus

$$A = \left\{ X_0^{\tau(A)-1} = X_{\tau(A)}^{2\tau(A)-1} = \dots = X_{(q-1)\tau(A)}^{q\tau(A)-1} = a_0^{\tau(A)-1} ; X_{q\tau(A)}^{n-1} = a_0^{r-1} \right\} .$$

So, we say that A has *period* $\tau(A)$ and *rest* r . We remark that periods can be “read backward” (and for the purpose of section 5 it will be more useful to do it in this way), that is

$$\begin{aligned} A &= \left\{ X_0^{r-1} = a_{n-r}^{n-1}, X_{n-q\tau(A)}^{n-(q-1)\tau(A)-1} = \dots = X_{n-\tau(A)}^{n-\tau(A)-1} = X_{n-\tau(A)}^{n-1} = a_{n-\tau(A)}^{n-1} \right\} \\ &= \bigcap_{j=1}^{(q-1)\tau(A)} T^{j\tau(A)}(A^{(\tau(A))}) \cap T^{q\tau(A)}(A^{(r)}) . \end{aligned}$$

We recall the definition of $A^{(w)}$, $1 \leq w \leq n$, at the end of section 2. For instance, using the previously chosen A ,

$$A = \overbrace{(aaaabb)}^{\text{period}} \overbrace{(aaaabb)}^{\text{period}} \overbrace{(aaa)}^{\text{rest}} = \overbrace{(aaa)}^{\text{rest}} \overbrace{(abbaaa)}^{\text{period}} \overbrace{(abbaaa)}^{\text{period}} . \quad (3.1)$$

$T^{12}A^{(3)}$ $T^6A^{(6)}$ $A^{(6)}$

In the middle of the above equality, periods are read forward while in the right hand side periods are read backward.

Consider the set of overlapping positions of A :

$$\{k \in \{1, \dots, n-1\} \mid A \cap T^{-k}(A) \neq \emptyset\} = \{\tau(A), \dots, [n/\tau(A)]\tau(A)\} \cup \mathcal{R}(A) ,$$

where

$$\mathcal{R}(A) = \{k \in \{[n/\tau(A)]\tau(A) + 1, \dots, n-1\} \mid A \cap T^{-k}(A) \neq \emptyset\} .$$

The set $\{\tau(A), \dots, [n/\tau(A)]\tau(A)\}$ is called the set of principal periods of A while $\mathcal{R}(A)$ is called the set of secondary periods of A . Furthermore, put $r_A = \#\mathcal{R}(A)$. Observe that one has $0 \leq r_A < n/2$.

The notion of period is related to the notion of *return times*.

Definition 3.2 *Given $A \in \mathcal{C}^n$, we define the hitting time $\tau_A : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$ as the following random variable: For any $x \in \Omega$*

$$\tau_A(x) = \inf\{k \geq 1 : T^k(x) \in A\} .$$

The return time is the hitting time restricted to the set A , namely $\tau_A|_A$.

We remark the difference between τ_A and $\tau(A)$: while $\tau_A(x)$ is the first time A appears in x , $\tau(A)$ is the first overlapping position of A .

Return times before $\tau(A)$ are not possible, thus, $\mathbb{P}_A(\tau_A < \tau(A)) = 0$. Still, if A does not return at time $\tau(A)$, then it can not return at times $k\tau(A)$, with $2 \leq k \leq \lfloor n/\tau(A) \rfloor$, so one has

$$\mathbb{P}_A(\tau(A) < \tau_A \leq \lfloor n/\tau(A) \rfloor \tau(A)) = 0.$$

The first possible return time after $\tau(A)$ is

$$n_A = \begin{cases} \min \mathcal{R}(A) & \mathcal{R}(A) \neq \emptyset \\ n & \mathcal{R}(A) = \emptyset \end{cases}.$$

Furthermore, by definition of $\mathcal{R}(A)$ one has $A \cap T^{-j}(A) = \emptyset$ for all j such that $\lfloor n/\tau(A) \rfloor \tau(A) < j \leq n-1$ and $j \notin \mathcal{R}(A)$. Thus

$$\mathbb{P}_A(\{\lfloor n/\tau(A) \rfloor \tau(A) + 1 \leq \tau_A \leq n-1\} \cap \{\tau_A \notin \mathcal{R}(A)\}) = 0.$$

We finally remark that

$$T^{-i}A \cap T^{-j}A = \emptyset \quad \forall i, j \in \mathcal{R}(A).$$

Otherwise it would contradict the fact that the first return time to A is $\tau(A)$ since for $i, j \in \mathcal{R}(A)$ one has $|i-j| < \tau(A)$. We conclude that

$$\mathbb{P}_A(T^{-i}A \cap T^{-j}A \mid i, j \in \mathcal{R}(A)) = 0. \quad (3.2)$$

4 RETURN TIMES

For $A \in \mathcal{C}^n$ define

$$\zeta_A \stackrel{def}{=} \mathbb{P}_A(\tau_A \neq \tau(A)) = \mathbb{P}_A(\tau_A > \tau(A)).$$

The equality follows by the comment at the end of the previous section.

It would be useful for the reader to note now that according to the comments of the previous section, one has

$$\tau_A|_A \in \{\tau(A)\} \cup \mathcal{R}(A) \cup \{k \in \mathbb{N} \mid k \geq n\}. \quad (4.1)$$

We now introduce the error terms that appear in the statement of our main result of this section.

Definition 4.1 *Let us define*

$$\epsilon(A) \stackrel{def}{=} \inf_{0 \leq w \leq n_A} \left[(2n + \tau(A)) \mathbb{P}(A^{(w)}) + \phi(n_A - w) \right]. \quad (4.2)$$

Theorem 4.1 Let $\{X_m\}_{m \in \mathbb{Z}}$ be a ϕ -mixing process. Then, for all $A \in \mathcal{C}^n$, $n \in \mathbb{N}$ the following inequality holds for all t :

$$\left| \mathbb{P}_A(\tau_A > t) - \mathbb{1}_{\{t < \tau(A)\}} - \mathbb{1}_{\{t \geq \tau(A)\}} \zeta_A e^{-\zeta_A \mathbb{P}(A)(t - \tau(A))} \right| \leq 54\epsilon(A)f(A, t), \quad (4.3)$$

where $f(A, t) = \mathbb{P}(A)te^{-(\zeta_A - 16\epsilon(A))\mathbb{P}(A)t}$.

We postpone an example showing the sharpness of $\epsilon(A)$ after Lemma 4.2.

Remark 4.1 $A^{(n_A)}$ is the part of the string A which does not overlap itself in $A \cap T^{-n_A}A$. Note that n_A is the position of the first possible return time after $\tau(A)$. Recall that $r_A = \#\mathcal{R}(A)$ and $n_A = n$ if $\mathcal{R}(A) = \emptyset$. Thus $A^{(w)}$ with $1 \leq w \leq n_A$ is the part of the string $A^{(n_A)}$ after taking out its first $n_A - w$ letters (this will be to create a gap of length $n_A - w$ to use the mixing property).

Remark 4.2 When $\mathcal{R}(A) = \emptyset$, namely, A does not have secondary periods, the error $\epsilon(A)$ of Theorem 4.1 becomes $\inf_{0 \leq w \leq n} [n\mathbb{P}(A^{(w)}) + \phi(n - w)]$.

Remark 4.3 In the error term of the theorem, $\epsilon(A)$ provides a bound which shows the convergence uniform in t of the return time law to that mixture of laws as the length of the string grows. The factor $\mathbb{P}(A)t$ provides an extra bound for values of t smaller than $1/\mathbb{P}(A)$. The factor $e^{-(\zeta_A - 16\epsilon(A))\mathbb{P}(A)t}$ provides an extra bound for values of t larger than $1/\mathbb{P}(A)$.

Remark 4.4 On one hand $\mathbb{P}(A) \leq Ce^{-cn}$ (see [1]). On the other hand, by construction $n_A > n/2$. Further $\phi(n) \rightarrow 0$ as $n \rightarrow \infty$. Taking for instance $w = n/4$ in (4.2) we warrant the smallness of $\epsilon(A)$ for large enough n .

Corollary 4.1 Let the process $\{X_m\}_{m \in \mathbb{Z}}$ be ϕ -mixing. Let $\beta > 0$. Then, for all $A \in \mathcal{C}^n$, $n \in \mathbb{N}$, the β -moment of the re-scaled time $\mathbb{P}(A)\tau_A$ approaches, as $n \rightarrow \infty$, to $\Gamma(\beta + 1)/\zeta_A^{\beta-1}$. Moreover

$$\left| \mathbb{P}(A)^\beta \mathbb{E}_A(\tau_A^\beta) - \frac{\Gamma(\beta + 1)}{\zeta_A^{\beta-1}} \right| \leq \epsilon^*(A) \frac{C\beta e^{2\epsilon(A)(\beta+1)/\zeta_A}}{\zeta_A^2} \frac{\Gamma(\beta + 1)}{\zeta_A^{\beta-1}}, \quad (4.4)$$

where $\epsilon^*(A) = (\epsilon(A) \vee (n\mathbb{P}(A))^\beta)$, $C > 0$ is a constant and Γ is the analytic gamma function.

Remark 4.5 In particular, the corollary establishes that all the moments of the return time are finite.

Remark 4.6 In the special case when $\beta = 1$, the above corollary establishes a weak version of Kac's Lemma (see [13]).

Remark 4.7 For each β fixed and n large enough one has $\beta e^{2\epsilon(A)(\beta+1)}/\zeta_A^2$ is close to β/ζ_A^2 . Thus in virtue of inequality (4.4), the corollary reads not just as a difference result but also as a ratio result.

The next corollary extends Theorem 2.1 in [12].

Corollary 4.2 *Let the process $\{X_m\}_{m \in \mathbb{Z}}$ be ϕ -mixing. There exists a constant $C > 0$ such that, for all $A \in \mathcal{C}^n$, $n \in \mathbb{N}$ and all $t > 0$ the following conditions are equivalent:*

- (a) $|\mathbb{P}_A(\tau_A > t) - e^{-\mathbf{P}(A)t}| \leq C \epsilon(A) f(A, t)$,
- (b) $|\mathbb{P}_A(\tau_A > t) - \mathbb{P}(\tau_A > t)| \leq C \epsilon(A) f(A, t)$,
- (c) $|\mathbb{P}(\tau_A > t) - e^{-\mathbf{P}(A)t}| \leq C \epsilon(A) f(A, t)$,
- (d) $|\zeta_A - 1| \leq C \epsilon(A)$.

Moreover, if $\{A_n\}_{n \in \mathbb{N}}$ is a sequence of strings such that $\mathbb{P}(A_n) \rightarrow 0$ as $n \rightarrow \infty$, then the following conditions are equivalent:

- (\tilde{a}) the return time law of A_n converges to a parameter one exponential law,
- (\tilde{b}) the return time law and the hitting time law of A_n converge to the same law,
- (\tilde{c}) the hitting time law of A_n converges to a parameter one exponential law,
- (\tilde{d}) The sequence $(\zeta_{A_n})_{n \in \mathbb{N}}$ converges to one.

4.1 Preparatory results

Here we collect a number of results that will be useful for the proof of Theorem 4.1. In what follows and for shorthand notation we put $f_A = 1/(2\mathbb{P}(A))$ (factor 2 is rather technical). The next lemma is a useful way to use the ϕ -mixing property.

Lemma 4.1 *Let $\{X_m\}_{m \in \mathbb{Z}}$ be a ϕ -mixing process. Suppose that $A \supseteq B \in \mathcal{F}_{\{0, \dots, b\}}$, $C \in \mathcal{F}_{\{x \in \mathbb{N} \mid x \geq b+n\}}$ with $b, g \in \mathbb{N}$. The following inequality holds:*

$$\mathbb{P}_A(B; C) \leq \mathbb{P}_A(B) (\mathbb{P}(C) + \phi(n)) .$$

Proof Since $B \subseteq A$, obviously $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(B \cap C)$. By the ϕ -mixing property $\mathbb{P}(B; C) \leq \mathbb{P}(B) (\mathbb{P}(C) + \phi(n))$. Dividing the above inequality by $\mathbb{P}(A)$ the lemma follows. \square

The following lemma says that return times over $\mathcal{R}(A)$ have small probability.

Lemma 4.2 *Let $\{X_m\}_{m \in \mathbb{Z}}$ be a ϕ -mixing process. For all $A \in \mathcal{C}^n$, the following inequality holds:*

$$\mathbb{P}_A(\tau_A \in \mathcal{R}(A)) \leq \epsilon(A) . \tag{4.5}$$

Proof For any w such that $1 \leq w \leq n_A$

$$\begin{aligned}
\mathbb{P}_A(\tau_A \in \mathcal{R}(A)) &\leq \mathbb{P}_A\left(\bigcup_{j \in \mathcal{R}(A)} T^{-j}A\right) \\
&\leq \mathbb{P}_A\left(\bigcup_{j \in \mathcal{R}(A)} T^{-j}A^{(w)}\right) \\
&\leq r_A \mathbb{P}\left(A^{(w)}\right) + \phi(n_A - w). \tag{4.6}
\end{aligned}$$

The first inequality follows by (3.2). Since $T^{-j}A \subset T^{-j}A^{(w)}$, second one follows. Third one follows by the above lemma with $B = A$ and $C = \bigcup_{j \in \mathcal{R}(A)} T^{-j}A^{(w)}$. This ends the proof since w is arbitrary. \square

Example 4.1 Consider a process $\{X_m\}_{m \in \mathbb{Z}}$ defined on the alphabet $\mathcal{C} = \{a, b\}$. Consider the string introduced in (3.1):

$$A = \{(X_0 \dots X_{14}) = (\text{aaaabbaaaabbaaa})\}.$$

Then, $n = 15$, $\tau(A) = 6$, $\mathcal{R}(A) = \{13, 14\}$, $r_A = 2$ and $n_A = 13$. Thus

$$A^{(13)} = \{(X_2 \dots X_{14}) = (\text{abbbaaaabbaaa})\}.$$

The ϕ -mixing property factorizes the probability

$$\mathbb{P}_A\left(\bigcup_{j=13}^{14} T^{-j}A\right) = \mathbb{P}_A\left(\bigcup_{j=13}^{14} T^{-j}A^{(13)}\right) \leq \mathbb{P}_A\left(\bigcup_{j=13}^{14} T^{-j}A^{(w)}\right).$$

In such case, a gap at $t = 15$ of length w with $0 \leq w \leq 13$ is the best we can do to apply the ϕ -mixing property.

The next lemma will be used to get the non-uniform factor $f(A, t)$ in the error term of Theorem 4.1.

Lemma 4.3 Let $\{X_m\}_{m \in \mathbb{Z}}$ be a ϕ -mixing process. Let $A \in \mathcal{C}^n$ and let $B \in \mathcal{F}_{\{x \in N \mid x \geq kf_A\}}$, with $k \in \mathbb{N}$. Then the following inequality holds:

$$\mathbb{P}_A(\tau_A > kf_A ; B) \leq [\mathbb{P}(\tau_A > f_A - 2n) + \phi(n)]^{k-1} [\mathbb{P}(B) + \phi(n)].$$

Proof First introduce a gap of length $2n$ between $\{\tau_A > kf_A\}$ and B . Then use Lemma 4.1 to get the inequalities

$$\begin{aligned}
\mathbb{P}_A(\tau_A > kf_A ; B) &\leq \mathbb{P}_A(\tau_A > kf_A - 2n ; B) \\
&\leq \mathbb{P}_A(\tau_A > kf_A - 2n) [\mathbb{P}(B) + \phi(n)]. \tag{4.7}
\end{aligned}$$

Apply this procedure to $\{\tau_A > (k-1)f_A\}$ and $B = \{\tau_A \circ T^{(k-1)f_A} > f_A - 2n\}$ to bound $\mathbb{P}_A(\tau_A > kf_A - 2n)$ by

$$\mathbb{P}_A(\tau_A > (k-1)f_A - 2n) [\mathbb{P}(\tau_A > f_A - 2n) + \phi(n)] .$$

Iterate this procedure to bound $\mathbb{P}_A(\tau_A > kf_A - 2n)$ by

$$\mathbb{P}_A(\tau_A > f_A - 2n) [\mathbb{P}(\tau_A > f_A - 2n) + \phi(n)]^{k-1} .$$

This ends the proof of the Lemma. \square

The next proposition establishes a relationship between *hitting* and *return* times with an error *uniform* in t . In particular, (b) says that they are close (up to $2\epsilon(A)$) if and only if ζ_A is close to 1.

Proposition 4.1 *Let $\{X_m\}_{m \in \mathbb{Z}}$ be a ϕ -mixing processes. Let $A \in \mathcal{C}^n$ and k a positive integer. Then the following holds:*

(a) For all $0 \leq r \leq f_A$,

$$\begin{aligned} & |\mathbb{P}_A(\tau_A > kf_A + r) - \mathbb{P}_A(\tau_A > kf_A)\mathbb{P}(\tau_A > r)| \\ & \leq 2\epsilon(A) \mathbb{P}_A(\tau_A > kf_A - 2n) . \end{aligned}$$

(b) For all $i \geq \tau(A) \in \mathbb{N}$,

$$|\mathbb{P}_A(\tau_A > i) - \zeta_A \mathbb{P}(\tau_A > i)| \leq 2\epsilon(A) . \quad (4.8)$$

Proof To simplify notation, for $t \in \mathbb{Z}$ we write $\tau_A^{[t]}$ to mean $\tau_A \circ T^t$. Assume $r \geq 2n$. We introduce a gap of length $2n$ after coordinate t to construct the following triangular inequality

$$\begin{aligned} & |\mathbb{P}_A(\tau_A > kf_A + r) - \mathbb{P}_A(\tau_A > kf_A)\mathbb{P}(\tau_A > r)| \\ & \leq |\mathbb{P}_A(\tau_A > kf_A + r) - \mathbb{P}_A(\tau_A > kf_A; \tau_A^{[kf_A+2n]} > r - 2n)| \quad (4.9) \end{aligned}$$

$$\begin{aligned} & + |\mathbb{P}_A(\tau_A > kf_A; \tau_A^{[kf_A+2n]} > r - 2n) - \mathbb{P}_A(\tau_A > kf_A)\mathbb{P}(\tau_A > r - 2n)| \\ & \quad (4.10) \end{aligned}$$

$$\begin{aligned} & + |\mathbb{P}_A(\tau_A > kf_A)\mathbb{P}(\tau_A > r - 2n) - \mathbb{P}(\tau_A > r)| . \quad (4.11) \end{aligned}$$

(4.9) is bounded by a direct computation by $\mathbb{P}_A(\tau_A > kf_A; \tau_A^{[kf_A]} \leq 2n)$. This last quantity is bounded using (4.7) by

$$\mathbb{P}_A(\tau_A > kf_A - 2n) [2n\mathbb{P}(A) + \phi(n)] .$$

Term (4.10) is bounded using the ϕ -mixing property by

$$\mathbb{P}_A(\tau_A > kf_A)\phi(n) .$$

The modulus in (4.11) is bounded using stationarity by

$$\mathbb{P}(\tau_A \leq 2n) \leq 2n\mathbb{P}(A) .$$

If $r < 2n$, just change $r - 2n$ by zero and the same proof holds. This ends the proof of (a).

The proof of (b) is very similar to that previous one. We do it briefly. Write the following triangle inequality

$$\begin{aligned} & |\mathbb{P}_A(\tau_A > i) - \zeta_A \mathbb{P}(\tau_A > i)| \\ \leq & |\mathbb{P}_A(\tau_A > i) - \mathbb{P}_A(\tau_A > \tau(A); \tau_A^{[\tau(A)+2n]} > i - \tau(A) - 2n)| \\ + & |\mathbb{P}_A(\tau_A > \tau(A); \tau_A^{[\tau(A)+2n]} > i - \tau(A) - 2n) - \zeta_A \mathbb{P}(\tau_A > i - \tau(A) - 2n)| \\ + & \zeta_A |\mathbb{P}(\tau_A > i - \tau(A) - 2n) - \mathbb{P}(\tau_A > i)| . \end{aligned}$$

The moduli on the right hand side of the above inequality are bounded as follows. The first one by $\mathbb{P}_A(\tau_A > \tau(A); \tau_A^{[\tau(A)]} \leq \tau(A) + 2n - 1)$ which is bounded by $\mathbb{P}_A(\tau_A \in \mathcal{R}(A) \cup \{n, \dots, \tau(A) + 2n - 1\})$. The cardinal of $\mathcal{R}(A) \cup \{n, \dots, \tau(A) + 2n - 1\}$ is less or equal than $n + \tau(A) + \mathcal{R}(A)$. Therefore, the last expression is bounded following the proof of Lemma 4.2 by $(2n + \tau(A))\mathbb{P}(A^{(w)}) + \phi(n_A - w)$.

The second one is bounded using the ϕ -mixing property by $\zeta_A \phi(n)$.

The third one is bounded using stationarity by

$$\mathbb{P}(\tau_A \leq \tau(A) + 2n) \leq (\tau(A) + 2n)\mathbb{P}(A).$$

This ends the proof of (b). \square

The following proposition is the key of the proof of Theorem 4.1.

Proposition 4.2 *Let $\{X_m\}_{m \in \mathbb{Z}}$ be a ϕ -mixing process. Let $A \in \mathcal{C}^n, n \in \mathbb{N}$ and let k be any integer $k \geq 1$. Then the following inequality holds:*

$$\begin{aligned} & |\mathbb{P}_A(\tau_A > kf_A) - \mathbb{P}_A(\tau_A > f_A)\mathbb{P}(\tau_A > f_A)^{k-1}| \\ \leq & 2\epsilon(A)(k-1)\mathbb{P}_A(\tau_A > f_A - 2n)[\mathbb{P}(\tau_A > f_A - 2n) + \phi(n)]^{k-2} . \end{aligned}$$

Proof For $k = 1$ there is nothing to prove. Take $k \geq 2$. The left hand side of the above inequality is bounded by

$$\sum_{j=2}^k |\mathbb{P}_A(\tau_A > jf_A) - \mathbb{P}_A(\tau_A > (j-1)f_A)\mathbb{P}(\tau_A > f_A)|\mathbb{P}(\tau_A > f_A)^{k-j}.$$

The modulus in the above sum is bounded by

$$2\epsilon(A)\mathbb{P}_A(\tau_A > (j-1)f_A - 2n) ,$$

due to Proposition 4.1 (a). The right-most factor is bounded using Lemma 4.3 by $[\mathbb{P}(\tau_A > f_A - 2n) + \phi(n)]^{j-2}$. The conclusion follows. \square

4.2 Proofs of Theorem 4.1 and corollaries

Proof of Theorem 4.1 We divide the proof according to the different values of t : (i) $t < \tau(A)$, (ii) $\tau(A) \leq t \leq f_A$ and (iii) $t > f_A$.

Consider first $t < \tau(A)$. (4.1) says that the left hand side of (4.3) is zero.

Consider now $\tau(A) \leq t \leq f_A$. First write

$$\mathbb{P}_A(\tau_A > t) = \frac{\mathbb{P}_A(\tau_A > t)}{\mathbb{P}(\tau_A > t)} \mathbb{P}(\tau_A > t) = p_{t+1} \mathbb{P}(\tau_A > t), \quad (4.12)$$

and

$$\begin{aligned} \mathbb{P}(\tau_A > t) &= \prod_{i=\tau(A)+1}^t \mathbb{P}(\tau_A > i | \tau_A > i-1) \\ &= \prod_{i=\tau(A)+1}^t (1 - \mathbb{P}(T^{-i}A | \tau_A > i-1)) \\ &= \prod_{i=\tau(A)+1}^t (1 - p_i \mathbb{P}(A)), \end{aligned} \quad (4.13)$$

where

$$p_i \stackrel{\text{def}}{=} \frac{\mathbb{P}_A(\tau_A > i-1)}{\mathbb{P}(\tau_A > i-1)}.$$

Further

$$\left| 1 - p_i \mathbb{P}(A) - e^{-\zeta_A \mathbb{P}(A)} \right| \leq |p_i - \zeta_A| \mathbb{P}(A) + \left| 1 - \zeta_A \mathbb{P}(A) - e^{-\zeta_A \mathbb{P}(A)} \right|. \quad (4.14)$$

Firstly, by Proposition 4.1 (b) and the fact that $\mathbb{P}(\tau_A > i) \geq 1/2$ since $i \leq f_A = 1/(2\mathbb{P}(A))$ we have

$$|p_i - \zeta_A| \leq \frac{2\epsilon(A)}{\mathbb{P}(\tau_A > i)} \leq 4\epsilon(A). \quad (4.15)$$

Secondly, note that $|1 - x - e^{-x}| \leq x^2/2$ for all $0 \leq x \leq 1$. Apply it with $x = \zeta_A \mathbb{P}(A)$ to bound the most right term of (4.14) by $(\zeta_A \mathbb{P}(A))^2/2$. Collecting the last two bounds we get

$$|1 - p_i \mathbb{P}(A) - e^{-\zeta_A \mathbb{P}(A)}| \leq \frac{9}{2} \epsilon(A) \mathbb{P}(A), \quad \forall i = \tau(A) + 1, \dots, f_A.$$

Furthermore, since

$$\left| \prod a_i - \prod b_i \right| \leq \max |a_i - b_i| (\#i) \max\{a_i, b_i\}^{\#i-1} \quad \forall 0 \leq a_i, b_i \leq 1, \quad (4.16)$$

we conclude from (4.13) and (4.12) that

$$|\mathbb{P}(\tau_A > t) - e^{-\zeta_A \mathbb{P}^{(A)}(t-\tau(A))}| \leq \frac{9}{2} \epsilon(A) \mathbb{P}^{(A)} t, \quad (4.17)$$

and

$$|\mathbb{P}_A(\tau_A > t) - \zeta_A e^{-\zeta_A \mathbb{P}^{(A)}(t-\tau(A))}| \leq \frac{9}{2} \epsilon(A) \mathbb{P}^{(A)} t, \quad (4.18)$$

for all $\tau_A \leq t \leq f_A$. This concludes this case.

Consider now $t > f_A$. Write it as $t = kf_A + r$ with k a positive integer and $0 \leq r < f_A$. We make the following triangle inequality

$$\begin{aligned} & |\mathbb{P}_A(\tau_A > t) - \zeta_A e^{-\zeta_A \mathbb{P}^{(A)}(t-\tau(A))}| \\ & \leq |\mathbb{P}_A(\tau_A > kf_A + r) - \mathbb{P}_A(\tau_A > kf_A) \mathbb{P}(\tau_A > r)| \end{aligned} \quad (4.19)$$

$$+ |\mathbb{P}_A(\tau_A > kf_A) - \mathbb{P}_A(\tau_A > f_A) \mathbb{P}(\tau_A > f_A)^{k-1} \mathbb{P}(\tau_A > r)| \quad (4.20)$$

$$+ |\mathbb{P}_A(\tau_A > f_A) \mathbb{P}(\tau_A > f_A)^{k-1} - \zeta_A e^{-\zeta_A k/2} \mathbb{P}(\tau_A > r)| \quad (4.21)$$

$$+ \zeta_A e^{-\zeta_A k/2} |\mathbb{P}(\tau_A > r) - e^{-\zeta_A \mathbb{P}^{(A)}(r-\tau(A))}| \quad (4.22)$$

By Proposition 4.1 (a), the modulus in (4.19) is bounded by

$$2\epsilon(A) \mathbb{P}_A(\tau_A > kf_A),$$

and by Lemma 4.3

$$2\epsilon(A) \mathbb{P}_A(\tau_A > kf_A - 2n) \leq 2\epsilon(A) (\mathbb{P}(\tau_A > f_A - 2n) + \phi(n))^{k-1}.$$

The modulus in (4.20) is bounded using Proposition 4.2 by

$$2\epsilon(A)(k-1)(\mathbb{P}(\tau_A > f_A - 2n) + \phi(n))^{k-2}.$$

Thus, the sum of (4.19) and (4.20) is bounded by

$$2\epsilon(A)(\mathbb{P}(\tau_A > f_A - 2n) + \phi(n))^{k-2} [k + \phi(n)]. \quad (4.23)$$

On one hand $k + \phi(n) \leq k + 1 \leq 2k$. On the other hand, applying (4.17) with $t = f_A - 2n$ we get

$$|\mathbb{P}(\tau_A > f_A - 2n) - e^{-\zeta_A/2 + (2n+\tau(A))\mathbb{P}^{(A)}}| \leq \frac{9}{4} \epsilon(A).$$

Furthermore, by the Mean Value Theorem (MVT) we get

$$|e^{-\zeta_A/2 + (2n+\tau(A))\mathbb{P}^{(A)}} - e^{-\zeta_A/2}| \leq (2n + \tau(A)) \mathbb{P}^{(A)} e^{(2n+\tau(A))\mathbb{P}^{(A)}}.$$

We conclude that for large enough n

$$|\mathbb{P}(\tau_A > f_A - 2n) + \phi(n) - e^{-\zeta_A/2}| \leq 4\epsilon(A)$$

And therefore (4.23) is bounded by

$$4\epsilon(A)k(e^{-\zeta_A/2} + 4\epsilon(A))^{k-2} . \quad (4.24)$$

A direct computation using Taylor's expansion gives

$$e^{-\zeta_A/2} \leq e^{-\zeta_A/2} + 4\epsilon(A) \leq e^{-(\zeta_A/2 - 8\epsilon(A))} .$$

Since $t = (k/2\mathbb{P}(A)) + r$ we get

$$e^{-(\zeta_A/2)(k-2)} = e^{-\zeta_A \mathbb{P}(A)t + \zeta_A (\mathbb{P}(A)r + 1)} ,$$

which is bounded by

$$e^{-\zeta_A \mathbb{P}(A)t + 3/2} .$$

Similarly

$$e^{-(\zeta_A/2 - 8\epsilon(A))(k-2)} = e^{-(\zeta_A - 16\epsilon(A)) \mathbb{P}(A)t + (\zeta_A - 16\epsilon(A)) (\mathbb{P}(A)r + 1)} ,$$

which for large enough n is bounded by

$$e^{-(\zeta_A - 16\epsilon(A)) \mathbb{P}(A)t + 3/2} .$$

Thus (4.24) is bounded by

$$36\epsilon(A) \mathbb{P}(A) t e^{-(\zeta_A - 16\epsilon(A)) \mathbb{P}(A)t}$$

To bound (4.21) we proceed as follows. From (4.17) and (4.18) with $t = f_A$ we get that

$$\begin{aligned} & |\mathbb{P}(\tau_A > f_A) - e^{-\zeta_A/2}| \\ & \leq |\mathbb{P}(\tau_A > f_A) - e^{-\zeta_A \mathbb{P}(A)(f_A - \tau(A))}| + e^{-\zeta_A/2} |e^{\zeta_A \mathbb{P}(A)\tau(A)} - 1| \\ & \leq \frac{9}{4}\epsilon(A) + n\mathbb{P}(A) \\ & \leq 3\epsilon(A) , \end{aligned}$$

and similarly

$$|\mathbb{P}_A(\tau_A > f_A) - \zeta_A e^{-\zeta_A/2}| \leq 3\epsilon(A) .$$

Applying the last two inequalities together with (4.16), we get that the modulus in (4.21) is bounded by

$$\begin{aligned} & 3\epsilon(A) k \max\{\mathbb{P}_A(\tau_A > f_A); \mathbb{P}(\tau_A > f_A); e^{-\zeta_A/2}\}^{k-1} \\ & \leq 3\epsilon(A) k \left(e^{-\zeta_A/2} + 3\epsilon(A) \right)^{k-1} . \end{aligned}$$

An argument similar to that used to bound (4.24) let us conclude that the last expression is bounded by

$$10\epsilon(A) \mathbb{P}(A) t e^{-(\zeta_A - 12\epsilon(A)) \mathbb{P}(A)t} .$$

The modulus in (4.22) is bounded using again (4.17) when $r \geq \tau(A)$ by $(9/2)\epsilon(A)$. If $r < \tau(A)$ then it can be rewritten as

$$e^{-\zeta_A \mathbf{P}(A)(r-\tau(A))} - 1 + \mathbf{P}(\tau_A \leq r) ,$$

which is bounded by $2n\mathbf{P}(A)$. We conclude that (4.22) is bounded by

$$(9/2)\epsilon(A)e^{-\zeta_A k/2} = (9/2)\epsilon(A)e^{-\zeta_A \mathbf{P}(A)(t-r)} \leq 8\epsilon(A)\mathbf{P}(A)te^{-\zeta_A \mathbf{P}(A)t} .$$

This ends the proof of the theorem. \square

Proof of Corollary 4.1 Let Y be the r.v. with distribution given by

$$P(Y > t) = \begin{cases} 1 & \mathbf{P}(A) < t \leq \mathbf{P}(A)\tau(A) \\ \zeta_A e^{-\zeta_A(t-\mathbf{P}(A)\tau(A))} & t < \mathbf{P}(A)\tau(A) \end{cases} .$$

Then we can rewrite (4.3) as

$$|\mathbf{P}_A(\mathbf{P}(A)\tau_A > t) - \mathbf{P}(\mathbf{P}(A)\tau_A > t)| \leq C_1\epsilon(A)f(A, t/\mathbf{P}(A)) . \quad (4.25)$$

Integrating (4.25) we get

$$\begin{aligned} & \left| \mathbf{E}_A((\mathbf{P}(A)\tau_A)^\beta) - \mathbf{E}(Y^\beta) \right| \\ &= \left| \int_{\mathbf{P}(A)}^{\infty} \beta t^{\beta-1} (\mathbf{P}(\mathbf{P}(A)\tau_A > t) - \mathbf{P}(Y > t)) dt \right| \\ &\leq \int_{\mathbf{P}(A)}^{\infty} \beta t^{\beta-1} |\mathbf{P}(\mathbf{P}(A)\tau_A > t) - \mathbf{P}(Y > t)| dt \\ &\leq C_1\epsilon(A) \int_{\mathbf{P}(A)}^{\infty} \beta t^{\beta-1} f(A, t/\mathbf{P}(A)) dt . \end{aligned}$$

Now we compute $\mathbf{E}(Y^\beta) = \int_{\mathbf{P}(A)}^{\infty} \beta t^{\beta-1} \mathbf{P}(Y > t) dt$. We do it in each interval $[\mathbf{P}(A), \mathbf{P}(A)\tau(A)]$ and $[\mathbf{P}(A)\tau(A), \infty)$.

The first one is $(\mathbf{P}(A)\tau(A))^\beta - \mathbf{P}(A)^\beta$. The second one can be re-written as

$$\zeta_A e^{\zeta_A \mathbf{P}(A)\tau_A} \left(\int_0^{\infty} - \int_0^{\mathbf{P}(A)\tau_A} \right) \beta t^{\beta-1} e^{-\zeta_A t} dt . \quad (4.26)$$

Consider the exponent of the second factor in (4.26). By definition we have $\zeta_A \mathbf{P}(A)\tau_A \leq \mathbf{P}(A)n$. Moreover, $\mathbf{P}(A)$ decays exponentially fast on n . Then for the second factor we have $|e^{\zeta_A \mathbf{P}(A)\tau_A} - 1| \leq C\mathbf{P}(A)n$. Further, the first integral is $\Gamma(\beta+1)/\zeta_A^\beta$. The second one is bounded by $(\mathbf{P}(A)\tau_A)^\beta$. We recall that the first factor in (4.26) is ζ_A . We conclude that

$$\left| \mathbf{E}(Y^\beta) - \frac{\Gamma(\beta+1)}{\zeta_A^{\beta-1}} \right| \leq Cn\mathbf{P}(A) + 2(n\mathbf{P}(A))^\beta \leq C(n\mathbf{P}(A))^{(\beta \wedge 1)} .$$

Similar computations give

$$\begin{aligned} \int_{\mathcal{P}(A)}^{\infty} \beta t^{\beta-1} f(A, t / \mathbb{P}(A)) dt &\leq \frac{\beta}{\beta+1} \frac{\Gamma(\beta+2)}{(\zeta_A - \epsilon(A))^{\beta+1}} \\ &\leq \frac{\beta e^{2\epsilon(A)(\beta+1)/\zeta_A}}{\zeta_A^2} \frac{\Gamma(\beta+1)}{\zeta_A^{\beta-1}}. \end{aligned}$$

In the last inequality we used $x \leq 2(1 - e^{-x})$ for small enough $x > 0$. This ends the proof of the corollary. \square

Proof of Corollary 4.2. (a) \Leftrightarrow (d). It follows directly from Theorem 4.1.

(b) \Rightarrow (a), (c). It follows by Theorem 4.1 and Theorem 1 in [2]

(a) \Rightarrow (b) and (c) \Rightarrow (b). They follow by Theorem 4.1, Theorem 1 in [2] and (4.15). The corollary is proved. \square

5 SOJOURN TIME

In this section we consider the number of consecutive visits to a fixed string A and prove that the distribution law of this number can be well approximated by a geometric law.

Definition 5.1 Let $A \in \mathcal{C}^n$. We define the sojourn time on the set A as the r.v. $S_A : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$

$$S_A(x) = \sup \left\{ k \in \mathbb{N} \mid x \in A \cap T^{-j\tau(A)} A ; \forall j = 1, \dots, k \right\},$$

and $S_A(x) = 0$ if the supremum is taken over the empty set.

Before to state our main result we have to introduce the following definition about certain continuity property of the probability \mathbb{P} conditioned to i consecutive occurrences of the string A .

Definition 5.2 For each fixed $A \in \mathcal{C}^n$, we define the sequence of probabilities $(\rho_i(A))_{i \in \mathbb{N}}$ as follows:

$$\rho_i(A) \stackrel{def}{=} \mathbb{P} \left(A \mid \bigcap_{j=1}^i T^{j\tau(A)} A \right).$$

If the limit $\lim_{n \rightarrow \infty} \rho_i(A)$ exists then we denote it by $\rho(A)$.

Remark 5.1 By stationarity $\rho_1(A) = 1 - \zeta_A$.

In the following 2 examples, the sequence $(\rho_i(A))_{i \in \mathbb{N}}$ not just converges but even is constant.

Example 5.1 For a i.i.d. Bernoulli process with parameter $0 < \theta = \mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0)$, and for the n -string $A = \{X_0^{n-1} = 1\}$, we have that $\rho_i(A) = 1 - \zeta_A = \theta$ for all $i \in \mathbb{N}$.

Example 5.2 Let $\{X_m\}_{m \in \mathbb{Z}}$ be a irreducible and aperiodic finite state Markov chain. For $A = \{X_0^{n-1} = a_0^{n-1}\} \in \mathcal{C}^n$, the sequence $(\rho_i(A))_{i \in \mathbb{N}}$ is constant. More precisely, by the Markovian property and for all $i \in \mathbb{N}$

$$\begin{aligned} \rho_i(A) &= \mathbb{P}\left(X_{n-\tau(A)}^{n-1} = a_{n-\tau(A)}^{n-1} \mid X_{n-\tau(A)-1} = a_{n-\tau(A)-1}\right) \\ &= \prod_{j=n-\tau(A)}^{n-1} \mathbb{P}(X_j = a_j \mid X_{j-1} = a_{j-1}) . \end{aligned}$$

The next is an example of a process with infinity memory and converging $(\rho_i(A))_{n \in \mathbb{N}}$.

Example 5.3 The following is a family of processes of the renewal type. Define $(X_n)_{n \in \mathbb{N}}$ as the order one Markov chain over \mathbb{N} with transitions probabilities given by

$$Q(n, n+1) = q_n \quad Q(n, 0) = 1 - q_n \quad \forall n \geq 0$$

Define the process

$$Y_n = \begin{cases} 0 & X_n = 0 \\ 1 & X_n \neq 0 \end{cases}$$

The process $(X_n)_{n \in \mathbb{N}}$ is positive recurrent (and then $(Y_n)_{n \in \mathbb{N}}$) if and only if $\sum_{k=0}^{\infty} \prod_{j=0}^k q_j < \infty$. Direct computations show that

$$P(Y_0^{n-1} = 1) = \sum_{k=0}^n \prod_{j=0}^k q_j \quad \forall n \in \mathbb{N} .$$

Now chose q_j such that $P(Y_0^{n-1} = 1) = e^{-n+\delta(n)}$ with $\delta(n)$ any converging sequence (to any real number) and such that $|\delta(i+1) - \delta(i)| < 1$ for all $i \in \mathbb{N}$. Take $A = \{Y_0^{n-1} = 1\}$. Thus $\tau(A) = 1$. Then

$$\rho_i(A) = e^{-1+\delta(n+i+1)-\delta(n+i)} \quad \text{and} \quad \lim_{i \rightarrow \infty} \rho_i(A) = e^{-1} \in (0, 1) .$$

In the following theorem we assume that $(\rho_i(A))_{i \in \mathbb{N}}$ converges with velocity $d_i(A)$. Namely, there is a real number $\rho(A) \in [0, 1)$ such that

$$|\rho_i(A) - \rho(A)| \leq d_i(A) \quad \text{for all } i \in \mathbb{N}, \quad (5.1)$$

where d_i is a sequence converging to zero. For simplicity we put $\bar{d}(A) = \sup\{d_i(A) \mid i \in \mathbb{N}\}$.

Theorem 5.1 Let $\{X_m\}_{m \in \mathbf{Z}}$ be a stationary process. Let $A \in \mathcal{C}^n$. Assume that (5.1) holds. Then, there is $c(A) \in [0, 1)$, such that the following inequalities hold for all $k \in \mathbf{N}$:

$$\left| \mathbb{P}_A(S_A = k) - (1 - \rho(A)) \rho(A)^k \right| \leq c(A)^k \sum_{i=1}^{k+1} d_i(A) \leq c(A)^k (k+1) \bar{d}(A) .$$

We deduce immediately that the β -moments of S_A can be approximated by $\mathbb{E}(Y^\beta)$ where Y is a geometric random variable with parameter $\rho(A)$.

Corollary 5.1 Let Y be a r.v. with geometric distribution with parameter $\rho(A)$. Let $\beta > 0$. Then

$$\left| \mathbb{E}_A(S_A^\beta) - \mathbb{E}(Y^\beta) \right| \leq 2\bar{d}(A) \sum_{k=1}^{\infty} k^{\beta+1} c(A)^k .$$

Remark 5.2 The sum $\sum_{k=1}^{\infty} k^{\beta+1} c(A)^k$ can be approximated using the Gamma function by $\Gamma(\beta + 2)/(-\ln c(A))^{\beta+2}$. When the supremum of the distances $|\rho_i(A) - \rho(A)|$ is small, the approximations given by Theorem 5.1 and Corollary 5.1 are good. The smaller is $c(A)$, the better they are. We compute these quantities for the examples of this section.

Example 5.1 (continuation) It follows straight-forward from definitions that $\rho_i(A) = \rho(A) = \mathbb{P}(A^{(\tau(A))})$ for all i and for any $A \in \mathcal{C}^n, n \in \mathbf{N}$. Thus $c(A) = \mathbb{P}(A^{(\tau(A))})$ and $\bar{d}(A) = 0$.

Example 5.2 (continuation) We already compute that $\rho_i(A) = \rho(A)$ for all i and for any $A \in \mathcal{C}^n, n \in \mathbf{N}$. Thus $c(A) = \rho(A)$ and $\bar{d}(A) = 0$.

Example 5.3 (continuation) For the same n -string there considered, we have

$$d_i(A) = e^{-1} |e^{\delta(n+i+1) - \delta(n+i)} - 1| \leq |\delta(n+i+1) - \delta(n+i)| ,$$

and

$$\bar{d}(A) \leq \sup\{|\delta(n+i+1) - \delta(n+i)|, i \in \mathbf{N}\} .$$

So, for large enough n , $\bar{d}(A)$ is small. Finally,

$$c(A) = \sup\{e^{-1}; e^{-1 + \delta(n+i+1) - \delta(n+i)}, n \in \mathbf{N}\} \in (0, 1) .$$

In the proof of Theorem 5.1 we will use the following lemma.

Lemma 5.1 Let $(l_i)_{i \in \mathbf{N}}$ be a sequence of real numbers such that $0 \leq l_i < 1$, for all $i \in \mathbf{N}$. Let $0 \leq l < 1$ be such that $|l_i - l| \leq d_i$ for all $i \in \mathbf{N}$ with $d_i \rightarrow 0$. Then, there is a constant $c \in [0, 1)$, such that the following inequalities hold for all $k \in \mathbf{N}$:

$$\left| \prod_{i=1}^k l_i - l^k \right| \leq c^{k-1} \sum_{i=1}^k d_i \leq k c^{k-1} \bar{d} .$$

where $\bar{d} = \sup\{d_i, i \in \mathbf{N}\}$.

Proof

$$\begin{aligned}
\left| \prod_{i=1}^k l_i - l^k \right| &= \left| \prod_{i=1}^k l_i - \prod_{i=1}^{k-1} l_i l + \prod_{i=1}^{k-1} l_i l - \prod_{i=1}^{k-2} l_i l^2 + \prod_{i=1}^{k-2} l_i l^2 - \dots - l^k \right| \\
&\leq \sum_{i=1}^k \left(\prod_{j=1}^{k-i} l_j \right) |l_{k-i+1} - l| l^{i-1} \leq c^{k-1} \sum_{i=1}^k d_i \\
&\leq k c^{k-1} \bar{d},
\end{aligned}$$

where $c = \sup \{l; l_i, i \in \mathbb{N}\}$. \square

Proof of Theorem 5.1 For $k = 0$, we just note that $\mathbb{P}_A(S_A = 0) = 1 - \rho_1(A)$ and $|1 - \rho_1(A) - (1 - \rho(A))| \leq d_1(A)$. Suppose $k \geq 1$. Therefore

$$\begin{aligned}
&\mathbb{P}_A(S_A = k) \\
&= \mathbb{P}_A \left(\bigcap_{j=0}^k T^{-j\tau(A)} A ; T^{-(k+1)\tau(A)} A^c \right) \\
&= \mathbb{P} \left(T^{-(k+1)\tau(A)} A^c \mid \bigcap_{j=0}^k T^{-j\tau(A)} A \right) \prod_{i=1}^k \mathbb{P} \left(T^{-i\tau(A)} A \mid \bigcap_{j=0}^{i-1} T^{-j\tau(A)} A \right) \\
&= (1 - \rho_{k+1}(A)) \prod_{i=1}^k \rho_i(A).
\end{aligned}$$

Third equality follows by stationarity. Lemma 5.1 ends the proof of the theorem. \square

Proof of Corollary 5.1 We use the inequality

$$|\mathbb{E}(X^\beta) - \mathbb{E}(Y^\beta)| \leq \sum_{k \geq 0} k^\beta |\mathbb{P}(X = k) - \mathbb{P}(Y = k)|,$$

which holds for any pair of positive r.v. X, Y . We apply the above inequality with $X = S_A$ and Y geometrically distributed with parameter $\rho(A)$.

The exponential decay of the error term in Theorem 5.1 ends the proof of the corollary. \square

Acknowledgments The authors are beneficiaries of a Capes-Cofecub grant. We thank P. Ferrari and A. Galves for useful discussions. We kindly thank also two anonymous referees for their useful comments and suggestions to improve a previous version of this article.

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