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QUANTUM CLUSTER ALGEBRAS AND THEIR SPECIALIZATIONS

CHRISTOF GEISS, BERNARD LECLERC, AND JAN SCHRÖER

ABSTRACT. We show that if a cluster algebra coincides with its upper cluster algebra and admits a grading with finite dimensional homogeneous components, the corresponding Berenstein-Zelevinsky quantum cluster algebra can be viewed as a flat deformation of the classical cluster algebra.

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1. INTRODUCTION AND MAIN RESULT

Throughout, let $\mathbb{K}$ be a field of characteristic 0. Let q be a variable, and set $R := \mathbb{K}[q^{\pm 1/2}]$, which is a principal ideal domain. We write $\mathbb{K}_1 := R/pR$, where $p := q^{1/2} - 1$. Of course, as a ring $\mathbb{K}_1 \cong \mathbb{K}$. However, we want to stress the way how $\mathbb{K}$ is considered as an R -module.

We fix some positive integers $m > n$. Let $(\Lambda, \tilde{B})$ be a compatible pair in the sense of [BZ, Section 3]. We can see $\Lambda = (\lambda_{ij}) \in M_m(\mathbb{Z})$ as a skew-symmetric $(m \times m)$ -matrix over the integers, and $\tilde{B} \in M_{m, m-n}(\mathbb{Z})$ is an $(m \times (m-n))$ -matrix over the integers such that the first $m-n$ rows of $\tilde{B}$ form a skew-symmetrizable matrix, which is denoted by B . The matrices Λ and $\tilde{B}$ need to satisfy a compatibility condition.

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Let $\mathcal{T}_q(\Lambda)$ be the R -algebra with generators $X_1, \dots, X_m, X_1^{-1}, \dots, X_m^{-1}$ subject to the relations

$$X_i X_i^{-1} = X_i^{-1} X_i = 1 \quad \text{and} \quad X_i X_j = q^{\lambda_{ij}} X_j X_i$$

for all $1 \leq i, j \leq m$. Then $\mathcal{T}_q(\Lambda)$ is an Ore domain and can be considered as a subring of its skew field of fractions $\mathcal{F}$, compare [BZ]. The algebra $\mathcal{T}_q(\Lambda)$ is called a *based quantum torus*. The quantum cluster algebra $\mathcal{A}_q := \mathcal{A}_q(\Lambda, \tilde{B})$ is then a certain R -subalgebra of the skew field $\mathcal{F}$. (We slightly deviate from the conventions of [BZ] by considering the quantum cluster algebra $\mathcal{A}_q$ as an R -algebra and not as a $\mathbb{Z}[q^{\pm 1/2}]$ -algebra as in [BZ].)

Let $\mathcal{A} := \mathcal{A}(\tilde{B})$ be the (commutative) cluster algebra (which is a $\mathbb{K}$ -algebra) associated with $\tilde{B}$, and let $\mathcal{U} := \mathcal{U}(\tilde{B})$ be its upper cluster algebra. The *specialization of $\mathcal{A}_q$ for $q = 1$* is defined as

$$\mathcal{A}_1 := \mathbb{K}_1 \otimes_R \mathcal{A}_q.$$

It is not hard to see that there is a surjective $\mathbb{K}$ -algebra homomorphism

$$\mathcal{A}_1 \rightarrow \mathcal{A}.$$

It seems that several authors (including us) implicitly assumed that the above is an isomorphism. However, this appears to be far from obvious. Up to our knowledge this question was not discussed in the existing literature. The following theorem is a positive result for a restricted class of quantum cluster algebras. This fixes a gap in the proof of [GLS2, Proposition 12.2]. This proposition is crucial for the proof of the main result [GLS2, Theorem 12.3]. We thank Alastair King [K] for pointing out the gap.

Theorem 1.1. *Let $\mathcal{A}_q$ be a quantum cluster algebra, and let $\mathcal{A}$ be the associated (commutative) cluster algebra. Let $\mathcal{U}$ be the upper cluster algebra of $\mathcal{A}$. We assume the following:*

- (i) $\mathcal{A} = \mathcal{U}$;
- (ii) $\mathcal{A}$ is a $\mathbb{Z}$ -graded cluster algebra with finite-dimensional homogeneous components.

Then we have

$$\mathbb{K}_1 \otimes_R \mathcal{A}_q \cong \mathcal{A}.$$

The paper is organized as follows: After some preparations in Section 2, we consider specializations of upper quantum cluster algebras and based quantum tori in Section 3. Section 4 deals with gradings for different kinds of (quantum) cluster algebras. The proof of the main result Theorem 1.1 is completed in Section 5.

2. DIVISIBILITY AND INJECTIVITY OF TENSOR PRODUCT MAPS

Let R be a (commutative) principal ideal domain, and let $p \in R$ be a prime element. We abbreviate $Q := R/(p)$.

If X is an R -module, we say that $x \in X$ is *divisible* by p (in X), if there exists some $x' \in X$ with $x = p \cdot x'$. The following result is probably well known. We include the easy proof for the convenience of the reader.

Lemma 2.1. *Let F be a free R -module, and let $U \subseteq F$ be a submodule. Let $j: U \rightarrow F$ be the inclusion map. Then the following are equivalent:*

(i) *The map*

$$Q \otimes_R j: Q \otimes_R U \rightarrow Q \otimes_R F$$

is injective;

(ii) *We have $\text{Tor}_1^R(Q, F/U) = 0$.*

(iii) *For all $u \in U$ we have*

$$u \text{ is divisible by } p \text{ in } U \iff u \text{ is divisible by } p \text{ in } F \quad (1)$$

Proof. We consider the short exact sequence

$$0 \rightarrow U \xrightarrow{j} F \rightarrow F/U \rightarrow 0.$$

Since F is free, and in particular flat as an R -module, $Q \otimes_R j$ is injective if and only if

$$0 = \text{Tor}_1^R(Q, F/U) \equiv \{\bar{f} \in F/U \mid p \cdot \bar{f} = 0\},$$

see for example [W, Section 3.1, Example 3.1.7]. Thus, $\text{Tor}_1^R(Q, F/U) = 0$ if and only if for all $f \in F \setminus U$ we have $p \cdot f \notin U$.

Thus, we have to show that the following are equivalent:

(a) $\{f \in F \setminus U \mid p \cdot f \in U\} = \emptyset$;

(b) Let $u \in U$ with $u = pf$ for some $f \in F$. Then $u = pf'$ for some $f' \in U$.

Suppose (a) holds. Then condition (b) becomes empty and is therefore satisfied. Suppose (b) holds. Thus let $u \in U$ with $u = pf = pf'$ with $f \in F$ and $f' \in U$. This implies $0 = p(f - f')$. Since R is a domain and F is free, this yields $f = f'$. Therefore (a) holds. $\square$

3. QUANTUM CLUSTER ALGEBRAS

As in the introduction, let $R := \mathbb{K}[q^{\pm 1/2}]$, $p := q^{1/2} - 1$, and $\mathbb{K}_1 := R/pR$. Let $(\Lambda, \tilde{B})$ be a compatible pair, and let $\mathcal{T}_q(\Lambda)$, $\mathcal{F}$ and $\mathcal{A}_q := \mathcal{A}_q(\Lambda, \tilde{B})$ be defined as before.

The *initial quantum seed* of $\mathcal{A}_q$ is denoted by $(M, \tilde{B})$, where $M: \mathbb{Z}^m \rightarrow \mathcal{F}$ is defined by $M(a_1, \dots, a_m) := X_1^{a_1} \cdots X_m^{a_m}$ for all $(a_1, \dots, a_m) \in \mathbb{Z}^m$.

Let $(M', \tilde{B}')$ be a quantum seed of $\mathcal{A}_q$ in the sense of [BZ, Definition 4.5]. In particular, M' is a map $\mathbb{Z}^m \rightarrow \mathcal{F}$ such that $M'(\mathbf{e}_1), \dots, M'(\mathbf{e}_m)$ is a free generating set of $\mathcal{F}$ and $M'(a_1, \dots, a_m) = M'(\mathbf{e}_1)^{a_1} \cdots M'(\mathbf{e}_m)^{a_m}$ for all $(a_1, \dots, a_m) \in \mathbb{Z}^m$. The based quantum torus $\mathcal{T}_q M'$ is a free R -module, which has the set $\{M'(c) \mid c \in \mathbb{Z}^m\}$ as an R -basis.

The following lemma is straightfoward.

Lemma 3.1. *Let S be a commutative ring, and let X be an S -module. For any ideal I in S there is an isomorphism*

$$\eta_{I,X}: S/I \otimes_S X \rightarrow X/IX$$

defined by $(s + I) \otimes x \mapsto sx + IX$ for all $s \in I$ and $x \in X$.

Lemma 3.2. *For $T := \mathcal{T}_q(\Lambda)$ the following hold:*

- (i) $\eta_{pR,T}: \mathbb{K}_1 \otimes_R T \rightarrow T/pT$ *is an isomorphism;*
- (ii) *The R -basis $\{M(\mathbf{e}) \mid \mathbf{e} \in \mathbb{Z}^m\}$ of T yields a $\mathbb{K}$ -basis $\{\overline{M}(\mathbf{e}) \mid \mathbf{e} \in \mathbb{Z}^m\}$ of T/pT , where $\overline{M}(\mathbf{e}) := \eta_{pR,T}(1 \otimes M(\mathbf{e}))$.*

Proof. Part (i) is just a special case of Lemma 3.1. We observe that $\{1 \otimes M(\mathbf{e}) \mid \mathbf{e} \in \mathbb{Z}^m\}$ is a $\mathbb{K}$ -basis of $\mathbb{K}_1 \otimes_R T$. (Here we use that T is a free R -module and that $\mathbb{K}_1 \otimes_R R \cong \mathbb{K}_1$.) Now (ii) follows from (i). $\square$

Lemma 3.3. *With $T = \mathcal{T}_q(\Lambda)$ as above, the algebra isomorphism*

$$\eta_{pR,T}: \mathbb{K}_1 \otimes_R T \rightarrow T/pT$$

restricts to a surjective algebra homomorphism

$$\eta: \mathbb{K}_1 \otimes_R \mathcal{A}_q(\Lambda, \tilde{B}) \rightarrow \mathcal{A}(\tilde{B}).$$

Proof. Comparing the mutation of quantum seeds of the quantum cluster algebra $\mathcal{A}_q(\Lambda, \tilde{B})$ as described in [BZ, Proposition 4.9] with the mutation of seeds of the cluster algebra $\mathcal{A}(\tilde{B})$, shows that $\eta_{pR,T}$ maps the set

$$\{1 \otimes x_q \mid x_q \text{ is a quantum cluster variable of } \mathcal{A}_q(\Lambda, \tilde{B})\}$$

surjectively onto the set of cluster variables of $\mathcal{A}(\tilde{B})$. $\square$

The surjective algebra homomorphism η in Lemma 3.3 might not be an isomorphism. Here one has to keep in mind that the quantum exchange relations and the exchange relations might not be the defining relations of $\mathcal{A}_q(\Lambda, \tilde{B})$ and $\mathcal{A}(\tilde{B})$, respectively. Our aim is to show that under some assumptions on $\mathcal{A}(\tilde{B})$, η is indeed an isomorphism.

We follow the convention from Section 2 and say that $x \in \mathcal{T}_q(\Lambda)$ is *divisible* by p if $x \in p\mathcal{T}_q(\Lambda)$.

Lemma 3.4. *Let $T = \mathcal{T}_q(\Lambda)$, and let $x, y \in T$ such that xy is divisible by p . Then at least one of x and y is divisible by p .*

Proof. By assumption we have $xy = pz$ for some $z \in T$. Furthermore, we know that

$$T/pT \cong \mathbb{K}_1 \otimes_R T \cong \mathbb{K}[x_1^{\pm 1}, \dots, x_m^{\pm 1}].$$

So T/pT is a domain, and therefore has no zero divisors. Now $xy \in pT$ implies that $x \in pT$ or $y \in pT$. This finishes the proof. $\square$

Let

$$\mathcal{U}_q(\Lambda, \tilde{B}) := \mathcal{T}_q M \cap \mathcal{T}_q M_1 \cap \cdots \cap \mathcal{T}_q M_n = \bigcap_{(M', \tilde{B}') \in \mathcal{S}} \mathcal{T}_q M',$$

be the *upper quantum cluster algebra*, where we abbreviated $(M_k, \tilde{B}_k) := \mu_k(M, \tilde{B})$ and $\mathcal{S}$ denotes the mutation class of $(M, \tilde{B})$, compare [BZ, Section 5].

The following is our key observation.

Proposition 3.5. *For any $1 \leq k \leq n$ we have*

$$((q^{1/2} - 1)\mathcal{T}_q M) \cap \mathcal{T}_q M_k = \mathcal{T}_q M \cap ((q^{1/2} - 1)\mathcal{T}_q M_k).$$

Proof. Without loss of generality, we assume $k = 1$. For $1 \leq i \leq m$ set

$$X_i := M(\mathbf{e}_i) \quad \text{and} \quad X'_i := M_1(\mathbf{e}_i).$$

Following the proof of [BFZ, Lemma 4.1] (see also [BZ, Lemma 5.5]), every element of $\mathcal{T}_q M$ can be written uniquely as

$$y = \sum_{j=-N}^N X_1^j c_j, \quad \text{with} \quad c_j \in \bigoplus_{\mathbf{r} \in \{0\} \times \mathbb{Z}^{m-1}} RM(r_2 \mathbf{e}_2 + \cdots + r_m \mathbf{e}_m)$$

and N large enough. Then y is divisible by p if and only if each c_j is divisible by p . Note that the direct sum on the right hand side is a subring of $\mathcal{T}_q M$. Now,

$$\begin{aligned} X_1 &= (X'_1)^{-1} \cdot (q^{m_+/2} M(\mathbf{b}_+) + q^{m_-/2} M(\mathbf{b}_-)) \\ &= (q^{-m_+/2} M(\mathbf{b}_+) + q^{-m_-/2} M(\mathbf{b}_-)) \cdot (X'_1)^{-1} \end{aligned}$$

for certain $\mathbf{b}_+, \mathbf{b}_- \in \{0\} \times \mathbb{N}^{m-1}$ and $m_+ = \mathbf{e}_1^t \Lambda_1 \mathbf{b}_+$, $m_- = \mathbf{e}_1^t \Lambda_1 \mathbf{b}_-$. For brevity, for $k \in \mathbb{Z}$ we set

$$Q^{km/2} := q^{km_+/2} M(\mathbf{b}_+) + q^{km_-/2} M(\mathbf{b}_-).$$

Note that

$$\begin{aligned} (X'_1)^{-1} M(\mathbf{b}_+) &= q^{-m_+} M(\mathbf{b}_+) (X'_1)^{-1}, \\ (X'_1)^{-1} M(\mathbf{b}_-) &= q^{-m_-} M(\mathbf{b}_-) (X'_1)^{-1}. \end{aligned}$$

This implies

$$Q^{km/2} (X'_1)^{-1} = (X'_1)^{-1} Q^{(k+2)m/2}$$

for all $k \in \mathbb{Z}$, and therefore

$$\begin{aligned} X_1^l &= Q^{-m/2} Q^{-3m/2} \cdots Q^{(1-2l)m/2} (X'_1)^{-l} \\ &= (X'_1)^{-l} Q^{(2l-1)m/2} Q^{(2l-3)m/2} \cdots Q^{m/2} \end{aligned}$$

and

$$\begin{aligned} (X'_1)^l &= X_1^{-l} Q^{-m/2} Q^{-3m/2} \cdots Q^{(1-2l)m/2} \\ &= Q^{(2l-1)m/2} Q^{(2l-3)m/2} \cdots Q^{m/2} X_1^{-l} \end{aligned}$$

for all $l > 0$.

Now observe that $y \in \mathcal{T}_q M_1$ if and only if there are some

$$d_l \in \bigoplus_{\mathbf{r} \in \{0\} \times \mathbb{Z}^{m-1}} RM(r_2 \mathbf{e}_2 + \cdots r_m \mathbf{e}_m)$$

such that

$$y = \sum_{l=-N}^N (X'_1)^l d_l.$$

This yields conditions on the pairs (c_l, d_{-l}) for each $-N \leq l \leq N$.

Using the formulas above (relating X_1^l to $(X'_1)^{-l}$), we get

$$d_{-l} = Q^{(2l-1)m/2} \cdots Q^{3m/2} Q^{m/2} c_l$$

for all $l > 0$, and $d_0 = c_0$. In particular, we have

$$d_{-l} \in \bigoplus_{\mathbf{r} \in \{0\} \times \mathbb{Z}^{m-1}} RM(r_2 \mathbf{e}_2 + \cdots r_m \mathbf{e}_m)$$

for all $0 \leq l \leq N$.

From the above expressions we see with Lemma 3.4 that c_l is divisible by p if and only if d_{-l} is divisible by p . This implies our claim since none of the $Q^{k_i m/2}$ is divisible by p . $\square$

Recall that for each $(M', \tilde{B}') \in \mathcal{S}$, the $\mathbb{K}$ -algebra

$$\mathbb{K}_1 \otimes_R \mathcal{T}_q M'$$

is a Laurent polynomial ring in m variables.

Corollary 3.6. *Let $j: \mathcal{U}_q(\Lambda, \tilde{B}) \rightarrow \mathcal{T}_q M$ be the natural inclusion. Then the following hold:*

(i) *The map*

$$\mathbb{K}_1 \otimes_R j: \mathbb{K}_1 \otimes_R \mathcal{U}_q(\Lambda, \tilde{B}) \rightarrow \mathbb{K}_1 \otimes_R \mathcal{T}_q M$$

is injective.

(ii) *Identifying $\mathcal{A}(\tilde{B}) \subseteq \mathcal{U}(\tilde{B})$ naturally with subalgebras of $\mathbb{K}_1 \otimes_R \mathcal{T}_q M$, we have $\mathcal{A}(\tilde{B}) \subseteq \text{Im}(\mathbb{K}_1 \otimes_R j) \subseteq \mathcal{U}(\tilde{B})$.*

Proof. With the help of Proposition 3.5 we can show inductively that each element of $\mathcal{U}_q(\Lambda, \tilde{B})$ fulfills the condition of Lemma 2.1(iii). This shows (i).

For (ii), we observe first that $\mathcal{U}_q(\Lambda, \tilde{B})$ contains all quantum cluster variables, see [BZ, Corollary 5.2]. By [BZ, Section 4], quantum cluster variables specialize to classical cluster variables. (For a quantum cluster variable x_q let $x := 1 \otimes x_q \in \mathbb{K}_1 \otimes_R \mathcal{T}_q M$ be the *specialization* of x_q . One can see $\mathcal{A}(\tilde{B})$ as a subalgebra of $\mathbb{K}_1 \otimes_R \mathcal{T}_q M$. It follows from the quantum exchange relations and the classical exchange relations that x is a cluster variable in $\mathcal{A}(\tilde{B})$.) This yields the first inclusion. The second inclusion follows directly from the definitions and the identification of $\mathbb{K}_1 \otimes_R \mathcal{T}_q M$ with the Laurent polynomial ring $\mathcal{T}M$ arising from the initial cluster of $\mathcal{A}(\tilde{B})$. $\square$

The following corollary is worth noting, but is not used later on. We stress that here $\mathcal{A}_q(\Lambda, \tilde{B})$ and $\mathcal{U}_q(\Lambda, \tilde{B})$ are defined over the Laurent polynomial ring R , and not (as for example in [GY]) over a field (like the field of fractions of R).

Corollary 3.7. *Suppose that $\mathcal{A}_q(\Lambda, \tilde{B}) = \mathcal{U}_q(\Lambda, \tilde{B})$. Then*

$$\mathbb{K}_1 \otimes_R \mathcal{A}_q(\Lambda, \tilde{B}) \cong \mathcal{A}(\tilde{B}).$$

Proof. Combine Corollary 3.6(i) with Lemma 3.3. □

4. GRADED CLUSTER ALGEBRAS

Let $\mathcal{A}$ be a cluster algebra $\mathcal{A}(\tilde{B})$ or a quantum cluster algebra $\mathcal{A}(\Lambda, \tilde{B})$. Then $\mathcal{A}$ is a $\mathbb{Z}$ -graded cluster algebra or $\mathbb{Z}$ -graded quantum cluster algebra, respectively, if the following hold:

(i) There is a direct sum decomposition

$$\mathcal{A} = \bigoplus_{g \in \mathbb{Z}} \mathcal{A}_g$$

such that $\mathcal{A}_g \cdot \mathcal{A}_h \subseteq \mathcal{A}_{g+h}$ for all $g, h \in \mathbb{Z}$;

(ii) All cluster variables (resp. quantum cluster variables) are homogeneous, i.e. for each cluster variable (resp. quantum cluster variable) $x \in \mathcal{A}$ there is some g with $x \in \mathcal{A}_g$.

We have $\tilde{B} = (b_{ij}) \in M_{m, m-n}(\mathbb{Z})$. Assume that there is some $d = (d_1, \dots, d_m) \in \mathbb{Z}^n$ such that for each $1 \leq k \leq m-n$ we have

$$\sum_{b_{ik} > 0} d_i b_{ik} = \sum_{b_{ik} < 0} d_i b_{ik}. \quad (2)$$

Let $X = (X_1, \dots, X_m)$ be the initial cluster of $\mathcal{A}$. Then

$$\deg_d(X_i) := d_i$$

extends to all cluster monomials and turns $\mathcal{A}$ into a $\mathbb{Z}$ -graded cluster algebra with

$$\mathcal{A}_g := \text{Span}_R(x \mid x \text{ is a product of cluster variables with } \deg_d(x) = g).$$

For the proof one uses the inductive construction of cluster variables starting with the initial seed $(M, \tilde{B})$. (The initial cluster is then $X = (X_1, \dots, X_m)$ with $X_i = M(\mathbf{e}_i)$ for all $1 \leq i \leq m$.) Equation (2) ensures that all exchange relations are homogeneous. We refer to [GL] for more details.

Lemma 4.1. *Suppose $\mathcal{A}(\tilde{B})$ is a $\mathbb{Z}$ -graded cluster algebra. Then the grading on $\mathcal{A}(\tilde{B})$ induces gradings on the quantum cluster algebra $\mathcal{A}_q(\Lambda, \tilde{B})$, the upper cluster algebra $\mathcal{U}(\tilde{B})$, and the upper quantum cluster algebra $\mathcal{U}_q(\Lambda, \tilde{B})$. Moreover, the homogeneous components of $\mathcal{U}_q(\Lambda, \tilde{B})$ are free R -modules.*

Proof. For a $\mathbb{Z}$ -graded cluster algebra $\mathcal{A}(\tilde{B})$, a corresponding $\mathbb{Z}$ -grading of $\mathcal{A}_q(\Lambda, \tilde{B})$ is constructed analogously, see [GL]. As before, let (Λ, M) be the initial seed of $\mathcal{A}_q(\Lambda, \tilde{B})$ with initial cluster $X = (X_1, \dots, X_m)$ where $X_i := M(\mathbf{e}_i)$ for $1 \leq i \leq m$.

By definition, we have

$$\mathcal{U}_q(\Lambda, \tilde{B}) = \bigcap_{(\Lambda', M') \in \mathcal{S}} \mathcal{T}_q M'.$$

The $\mathbb{Z}$ -grading on $\mathcal{A}_q(\Lambda, \tilde{B})$ induces a $\mathbb{Z}$ -grading on each quantum torus $\mathcal{T}_q M'$. Let

$$\mathcal{T}_q M' = \bigoplus_{g \in \mathbb{Z}} (\mathcal{T}_q M')_g$$

where the $(\mathcal{T}_q M')_g$ are the homogeneous components. Clearly, all $(\mathcal{T}_q M')_g$ are free R -modules (of infinite rank).

Set

$$U_q(\Lambda, \tilde{B})_g := (\mathcal{T}_q M)_g \cap U_q(\Lambda, \tilde{B}).$$

Note that as a submodule of a free R -module, $U_q(\Lambda, \tilde{B})_g$ is also a free R -module.

We claim that

$$\mathcal{U}_q(\Lambda, \tilde{B}) = \bigoplus_{g \in \mathbb{Z}} U_q(\Lambda, \tilde{B})_g.$$

Let $x \in U_q(\Lambda, \tilde{B})$. For each quantum seed $(\Lambda', M') \in \mathcal{S}$, it follows that

$$x = \sum_{g \in \mathbb{Z}} t_{M', g}$$

for uniquely determined $t_{M', g} \in (\mathcal{T}_q M')_g$. Keeping in mind that exchange relations are homogeneous, we get that $t_{M', g} = t_{M, g}$ for all $(\Lambda', M') \in \mathcal{S}$. This proves our claim. $\square$

5. PROOF OF THE MAIN RESULT

Theorem 5.1. *Let $(\Lambda, \tilde{B})$ be a compatible pair such that the following hold:*

- (i) $\mathcal{A}(\tilde{B}) = \mathcal{U}(\tilde{B})$;
- (ii) $\mathcal{A}(\tilde{B})$ is a $\mathbb{Z}$ -graded cluster algebra with finite-dimensional homogeneous components.

For the initial quantum seed $(M, \tilde{B})$ of $\mathcal{A}_q(\Lambda, \tilde{B})$ let

$$i: \mathcal{A}_q(\Lambda, \tilde{B}) \rightarrow \mathcal{T}_q M$$

be the inclusion map. Then $\mathbb{K}_1 \otimes_R i$ induces an isomorphism

$$\mathbb{K}_1 \otimes_R \mathcal{A}_q(\Lambda, \tilde{B}) \cong \mathcal{A}(\tilde{B}).$$

Proof. By Lemma 3.3 we have $\text{Im}(\mathbb{K}_1 \otimes_R i) = \mathcal{A}(\tilde{B})$. Here we identify $\mathcal{A}(\tilde{B})$ again with a subalgebra of

$$\mathbb{K}_1 \otimes_R \mathcal{T}_q M \cong \mathcal{T}M = \mathbb{K}[X_1^\pm, \dots, X_m^\pm].$$

By assumption (ii) and Lemma 4.1, the upper quantum cluster algebra $\mathcal{U}_q(\Lambda, \tilde{B})$ is a $\mathbb{Z}$ -graded algebra with all homogenous components being free R -modules.

By the hypothesis (i) and Corollary 3.6, the inclusion

$$j: \mathcal{U}_q(\Lambda, \tilde{B}) \rightarrow \mathcal{T}_q M$$

induces an isomorphism

$$\mathbb{K}_1 \otimes_R \mathcal{U}_q(\Lambda, \tilde{B}) \cong \mathcal{A}(\tilde{B}) = \mathcal{U}(\tilde{B}).$$

In particular, we have

$$\text{rank}_R(\mathcal{U}(\Lambda, \tilde{B})_g) = \dim_{\mathbb{K}}(\mathcal{A}(\tilde{B})_g) < \infty$$

for all $g \in \mathbb{Z}$.

Now, consider the inclusion

$$i': \mathcal{A}_q(\Lambda, \tilde{B}) \rightarrow \mathcal{U}_q(\Lambda, \tilde{B}).$$

Since $i = j \circ i'$, by the above remark and Lemma 3.3 we have that $\mathbb{K}_1 \otimes_R i'$ is surjective. Thus if we consider the R -module

$$M := \mathcal{U}_q(\Lambda, \tilde{B}) / \mathcal{A}_q(\Lambda, \tilde{B}),$$

we must have $\mathbb{K}_1 \otimes_R M = 0$, or in other words $M = pM$, compare Lemma 3.1.

For each $g \in \mathbb{Z}$ there exists a short exact sequence

$$0 \rightarrow \mathcal{A}_q(\Lambda, \tilde{B})_g \xrightarrow{i'} \mathcal{U}_q(\Lambda, \tilde{B})_g \rightarrow M_g \rightarrow 0$$

of finitely generated R -modules. (We know that $\mathcal{A}_q(\Lambda, \tilde{B})_g$ and $\mathcal{U}_q(\Lambda, \tilde{B})_g$ are free R -modules of finite rank, in particular both are finitely generated. As a factor module of a finitely generated module, M_g is finitely generated as well.) Since $M = pM$ and therefore also $M_g = pM_g$, it follows that M_g does not have any direct summand isomorphic to R or to $R/(p)$. This implies that M_g is an R -module of finite length. Therefore the surjective R -module homomorphism $M_g \rightarrow M_g$ defined by $x \mapsto px$ has to be injective as well. We conclude that

$$\text{Tor}_1^R(\mathbb{K}_1, M) = \{x \in M \mid px = 0\} = 0.$$

Now Lemma 2.1 implies that $\mathbb{K}_1 \otimes_R i'$ is also injective and therefore an isomorphism. $\square$

Note that Theorem 5.1 implies immediately Theorem 1.1.

Let $\text{Frac}(R) := \mathbb{K}(q^{1/2})$ be the field of fractions of $R = \mathbb{K}[q^{1/2}]$.

Corollary 5.2. *With the assumptions of Theorem 5.1 and the notation used in its proof,*

$$\mathrm{Frac}(R) \otimes i': \mathrm{Frac}(R) \otimes_R \mathcal{A}_q(\Lambda, \tilde{B}) \rightarrow \mathrm{Frac}(R) \otimes_R \mathcal{U}_q(\Lambda, \tilde{B})$$

is an isomorphism.

Proof. In the proof of Theorem 5.1 we looked at the short exact sequence

$$0 \rightarrow \mathcal{A}_q(\Lambda, \tilde{B})_g \xrightarrow{i'} \mathcal{U}_q(\Lambda, \tilde{B})_g \rightarrow M_g \rightarrow 0$$

of R -modules for each $g \in \mathbb{Z}$ and saw that M_g is of finite length. It follows that the free R -modules $\mathcal{A}_q(\Lambda, \tilde{B})_g$ and $\mathcal{U}_q(\Lambda, \tilde{B})_g$ have the same rank, i.e.

$$\mathrm{rank}_R(\mathcal{A}_q(\Lambda, \tilde{B})_g) = \mathrm{rank}_R(\mathcal{U}_q(\Lambda, \tilde{B})_g).$$

Since $\mathrm{Frac}(R)$ is a flat R -module, our claim is equivalent to $\mathrm{Frac}(R) \otimes_R M_g = 0$ for all g . But this holds, since M_g is of finite length. $\square$

Question 5.3. *Is it true that $\mathcal{A}(\tilde{B}) = \mathcal{U}(\tilde{B})$ if and only if $\mathcal{A}_q(\Lambda, \tilde{B}) = \mathcal{U}_q(\Lambda, \tilde{B})$? (None of the two implications seems to be obvious.)*

Conjecture 5.4. *Hypothesis (ii) in Theorem 5.1 is not needed.*

6. EXAMPLES

Let $\mathfrak{g} = \mathfrak{g}(C)$ be a symmetric Kac-Moody Lie algebra associated with a symmetric generalized Cartan matrix C . For an element w in the Weyl group of $\mathfrak{g}$, let $\mathfrak{n}(w)$ be the associated nilpotent Lie algebra, and let $N(w)$ be the associated unipotent group, compare for example [GLS1, Section 4.3]. The coordinate ring $\mathbb{C}[N(w)]$ is naturally isomorphic to a cluster algebra $\mathcal{A}(w)$, whose initial exchange matrix $\tilde{B}_{\mathbf{i}}$ is defined via some reduced expression $\mathbf{i} = (i_1, \dots, i_r)$ of w .

To the same data, one can associate a 2-Calabi-Yau Frobenius category $\mathcal{C}_w$, which is by definition a subcategory of the category of finite-dimensional nilpotent modules over a preprojective algebra Λ associated with C . For each Λ -module $X \in \mathcal{C}_w$ there is an evaluation function δ_X , see [GLS1, Section 2.2].

In [GLS1] we proved that $\mathcal{A}(w)$ is equal to its upper cluster algebra and that $\mathcal{A}(w)$ is isomorphic to a polynomial ring $\mathbb{C}[\delta_{M_1}, \dots, \delta_{M_r}]$, where the δ_{M_k} are evaluation functions arising from a Λ -module $M_{\mathbf{i}} = M_1 \oplus \dots \oplus M_r$ in $\mathcal{C}_w$ associated with $\mathbf{i}$. This follows from [GLS1, Theorem 3.1] and [GLS1, Theorem 3.2]. The map $M_k \mapsto \dim(M_k)$ turns $\mathcal{A}(w)$ into a $\mathbb{Z}$ -graded cluster algebra with finite-dimensional homogeneous components. Here we also used that the exchange relations for $\mathcal{A}(w)$ arise from short exact sequences of Λ -modules in $\mathcal{C}_w$, compare [GLS1, Sections 2.7 and 2.8]. This turns the exchange relations into homogeneous relations.

Thus $\mathcal{A}(w)$ satisfies all assumptions of Theorem 1.1. We conclude that the specialized quantum cluster algebra $\mathbb{C}_1 \otimes_R \mathcal{A}_q(w)$ is isomorphic to $\mathcal{A}(w)$. This fixes a gap in the proof of [GLS2, Proposition 12.2], which is essential for the proof of the main result of [GLS2].

Goodearl and Yakimov [GY] construct a large class of quantum cluster algebras, called *quantum nilpotent algebras*, generalizing the quantum cluster algebras $\mathcal{A}_q(w)$ mentioned above, at least if we are in the Dynkin case. They show that these quantum cluster algebras are equal to their upper quantum cluster algebras. But note that Goodearl and Yakimov [GY] work over a field and not (as in our case) over $R = \mathbb{K}[q^{\pm 1/2}]$. An upcoming manuscript of the same authors will contain an integral version of their results.

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