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Abstract - The effective wavenumber model of Linton and Martin [1] for the dilute monodisperse case (obstacles of identical sizes) is used to calculate the effective modulus and mass density of a polydisperse assembly of poroelastic spheres. We focus on the Rayleigh limit (low frequency regime) where the wavelengths can be considered very large compared to the size of the obstacles. In order to show the influence of the distribution in size of the obstacles, the special case of Schulz probability density function is considered.

Keywords: - Random media, effective moduli, polydispersity.

1. INTRODUCTION

The properties of the coherent wave propagating in a fluid containing randomly distributed poroelastic spheres are studied. The polydispersity - defined as the variation in size of the obstacles in a representative volume - is taken into account. The starting point is the effective wavenumber model of Linton and Martin (LM) [1] in the monodisperse case (identical sizes). This model based on a closure assumption often called the Quasi-Crystalline Approximation (QCA) of Lax, is an improvement of the models of Foldy [2] and Waterman and Truell [3] initially which were built for the dilute case (low concentrations). The following effective quantities will be derived: wavenumber, modulus and mass density of a polydisperse suspension in the low frequency regime where the wavelength can be assumed very large compared to the size of the obstacles (or large with regard to the representative volume). The monodisperse case (all the obstacles have the same radius) can then be seen as a particular case of polydispersity.

The phenomenon of polydispersity is met in dispersions [4] and colloids [5-6]. It also manifests in the flowing and transport of solid particles [7] and in the distribution of air bubbles in fluids [8-10]. An understanding of the effects induced by the distribution in size of the obstacles on the effective acoustic properties of the dispersions can be of great interest, especially to higher orders in concentration - it should be noted that so far, the most used formulas are those of Foldy which only accounts first order terms in concentration and Waterman and Truell which contains an incorrect second order term. Moreover, considering porous obstacles, previous results on elastic spheres are extended.

2. BACKGROUND

The fluid surrounding the porous obstacles has a mass density ρ_0 and a speed of sound c_0. The fluid saturating the porous space has a mass density ρ_f, a speed of sound c_f and a kinematic viscosity η. It will be assumed for the sake of simplicity that $\rho_f = \rho_0$ and $c_f = c_0$. We denote by ρ_s the mass density of the material constituting the solid grains and by $\rho = (1-\phi)\rho_s + \phi\rho_0$ the density of the fluid saturated porous medium of porosity ϕ. The porous medium obeys Biot's theory and is also characterized by a tortuosity $\tilde{\alpha}$ sometimes referred to as the structure factor [11] (see also Table 1, Ref. [12] for a complete list of the parameters constituting the saturated porous medium).

Let ω be the angular frequency and $k_0 = \omega/c_0$ the wavenumber of the longitudinal incident wave in the absence of the obstacles. When it encounters the obstacles, the incident wave is partially reflected. The part of the wave penetrating the obstacles is converted into three waves according to Biot's theory of poroelastic media : two longitudinal waves of respective wavenumbers $k_1 = \omega/c_1$ (fast) and $k_2 = \omega/c_2$ (slow) and a transverse wave of wavenumber $k_t = \omega/c_t$. Since the three wavenumbers are all complex-valued, the waves attenuate when propagating in the porous space. The scattering of an incident plane wave by a spherical obstacle of radius a is described by the scattering coefficients T_n ($T_{-n} = T_n$) where n is a relative integer (*cf.* Appendix A). The T_ns depend on spherical Bessel and Hankel functions which in tour depend on one of the normalized wavenumbers $k_\alpha a$ ($\alpha = 0,1,2,t$).

The far-field scattering function $f(\theta)$ is a modal series depending on the angle of observation θ and on the coefficients T_n as follows

$$f(\theta) = \frac{1}{ik_0} \sum_{n=0}^{+\infty} (2n+1) T_n P_n(\cos\theta). \quad (1)$$

It is used in the LM model discussed later. The Legendre polynomial $P_n(\cos\theta)$ of order n is equal to 1 for $\theta = 0$ (forward scattering) or $\theta = \pi$ (backscattering). At low frequencies the normalized wavenumbers $k_\alpha a$ are such that $|k_\alpha a| << 1$ ($\alpha = 0,1,2,t$). Taylor series expansions for these functions show that

$$T_0(a) = \frac{i}{3}(k_0 a)^3 (B_0 - 1) + O\left((k_0 a)^5\right) \quad (2)$$

$$T_1(a) = \frac{i}{3}(k_0 a)^3 B_1 + O\left((k_0 a)^5\right) \quad (3)$$

$$T_n(a) = O\left((k_0 a)^5\right) \quad \text{if } n \geq 2 \quad (4)$$

The coefficients B_0 and B_1 are given in Appendix B. The expansions Eqs. (2-4) for a single obstacle are usually called "Rayleigh limits". The study for any assembly of obstacles presented here remaining restricted to low frequencies, we keep this denomination below.

3. LINTON AND MARTIN EFFECTIVE WAVENUMBER. POLYDISPERSITY.

In 1967, using calculations based on methods of nuclear physics, Lloyd and Berry [13] extended the model of the effective wavenumber of Waterman and Truell for the case of spherical obstacles. They added to the second order in concentration of this wavenumber an integral formula which accounts for a summation on all the scattering angles. Four decades later, working on a new proof of the results of Lloyd and Berry from a more classical approach, LM [1] proposed in the case of cylindrical obstacles a rewrite of the effective wavenumber equivalent to that of Lloyd and Berry, the integral on the angles being transformed into a double series whose general term is proportional to the product $T_n T_m$ of the scattering coefficients. Although extended to any order in n_0 for cylinders [14] and for spheres [15], we consider here only the initial formula of LM which stops at the second order in n_0.

Let us start from a fluid medium containing n_0 obstacles per unit volume. The distribution of radius a is assumed to be continuous and extends from $a_1 = 0$ to $a_2 = +\infty$. It is characterized by a probability density function η such that $\int_0^{+\infty} \eta(a)\,da = 1$. The effective wavenumber of LM is given by [1]

$$\xi_{LM}^2 = k_0^2 + n_0 \delta_1 + n_0^2 \delta_2 + O\left(n_0^3\right) \quad (5)$$

where

$$\delta_1 = \frac{4\pi}{ik_0} \int_{a_1}^{a_2} \sum_{n=0}^{+\infty} (2n+1) T_n(a) \eta(a)\,da \quad (6)$$

and

$$\delta_2 = -\frac{1}{2}\left(\frac{4\pi}{k_0}\right)^4 \times$$
$$\int_{a_1}^{a_2}\int_{a_1}^{a_2} \sum_{n,m=0}^{+\infty} K_{nm} T_n(a) T_m(b) \eta(a) \eta(b)\,da\,db \quad (7)$$

In Eq. (7)

$$K_{nm} = \left(\frac{1}{4\pi}\right)^{3/2} \sqrt{(2n+1)(2m+1)} \times$$
$$\sum_{q=|n-m|}^{n+m} q\sqrt{2q+1}\, G(n,0\,;m,0\,;q) \quad (8)$$

where G is the Gaunt coefficient. The sum on the index q is in steps of two with $n+m+q$ even. Algorithms are provided for the fast calculation of Gaunt coefficients [16].

When the frequency domain of study is arbitrary, one must evaluate numerically the single and double integrals appearing in Eqs. (6-7). The low-frequency approximations of the scattering coefficients T_n given above make it possible to simplify considerably the computations of the integrals. Considering Eqs. (2-4) and (5-7) we obtain

$$\xi_{LM}^2 \simeq k_0^2 (1 + I_1 + I_2) \quad (9)$$

where

$$I_1 = (B_0 - 1 + 3B_1) n_0 \frac{4\pi}{3}\langle a^3 \rangle \quad (10)$$

and

$$I_2 = 3\left[(B_0 - 1)B_1 + 2B_1^2\right]\left(n_0 \frac{4\pi}{3}\langle a^3 \rangle\right)^2 \quad (11)$$

Above, we have introduced the notation

$$\left\langle a^n \right\rangle = \int_0^{+\infty} a^n \eta(a) da \qquad (12)$$

which represents the moment of order n of the probability density function (if $n=1$ it is the average radius $\langle a \rangle$ of the obstacles). Note that $K_{01} = K_{10} = 3/\left(16\pi^2\right)$ and $K_{11} = 3/\left(4\pi^2\right)$ are also used. Equations (10-11) show that there are only two integrals to compute, $\left\langle a^2 \right\rangle$ and $\left\langle a^3 \right\rangle$. In the following $\mathcal{C}_V = n_0 (4\pi/3)\left\langle a^3 \right\rangle$ will denote the average concentration. It is clear that the concentration does not depend on the physical parameters of the media.

3.1. Concentrations

We will examine the effect of polydispersity in the case of the Schulz distribution, Fig. 1, and then deduce the formula of the concentration. In their work, Leroy et al. [9] considered the log-normal distribution, Mascaro *et al.* [10] the Gaussian distribution which can be seen as a particular case of the Schulz distribution. The Schulz distribution [18-21] is a special case of the Gamma distribution. It probability density function is given by

$$\eta(a) = \frac{(j+1)^{j+1}}{\langle a \rangle \Gamma(j+1)} \left(\frac{a}{\langle a \rangle} \right)^j \exp\left[-(j+1) \frac{a}{\langle a \rangle} \right] \qquad (13)$$

where $\Gamma(.)$ is Euler's gamma function. The number j measures the width of the distribution and is connected to the polydispersity because $p = \sigma/\langle a \rangle = 1/\sqrt{j+1}$ (σ^2 is the variance). The case $j=0$ gives $p=1$ and corresponds to the exponential distribution (where the number of small-radius diffusers is preponderant).

When j is large, p is small; the dispersion is small and the distribution is close to a Gaussian. In the following, j is a positive integer and we use the relation $\Gamma(j+1) = j!$. The moment of order n is then given by

$$\left\langle a^n \right\rangle = \int_0^{+\infty} a^n \eta(a) da = \frac{(j+n)!}{j!} \frac{\langle a \rangle^n}{(j+1)^n} \qquad (14)$$

The average concentration for a distribution of spherical obstacles is

$$\begin{aligned} \mathcal{C}_V &= n_0 \frac{4\pi \langle a \rangle^3}{3} \frac{(j+3)(j+2)}{(j+1)^2} \\ &= n_0 \langle \mathcal{V} \rangle \left(1 + 3p^2 + 2p^4\right) \end{aligned} \qquad (15)$$

where $\langle \mathcal{V} \rangle = 4\pi \langle a \rangle^3 / 3$. The special case of monodispersity is obtained when $j \to +\infty$ or $p \to 0$ (the probability density function then tends towards an increasingly narrow peak of increasing amplitude and centered on the value 1). The result is the following limits $\left\langle a^n \right\rangle \to \langle a \rangle^n = a^n$ ($n \geq 1$), $\sigma \to 0$ and $\mathcal{C}_V \to n_0 V$.

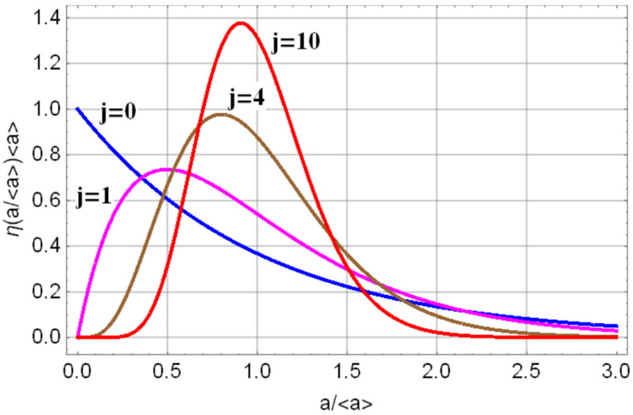

Figure 1. The Schulz distribution

3.2. Effective wavenumber

Making the appropriate substitutions in Eq. (9), one arrives at a formula for the effective wavenumber going up to the second order in concentration of the form

$$\begin{aligned} \left(\frac{\xi_{LM}}{k_0} \right)^2 &= 1 + \mathcal{C}_V \left[B_0 - 1 + 3B_1 \right] + \\ & 3\mathcal{C}_V^2 \left[B_1 (B_0 - 1) + 2B_1^2 \right] \end{aligned} \qquad (16)$$

As long as terms of order $\mathcal{C}_V^3$ and higher can be neglected, Eqs. (16) can be rewritten as [12, 22]

$$\left(\frac{\xi_{LM}}{k_0} \right)^2 \simeq \left[1 + \mathcal{C}_V (B_0 - 1) \right] \left[1 + 3\mathcal{C}_V B_1 + 6\mathcal{C}_V^2 B_1^2 \right] \qquad (17)$$

4. EFFECTIVE MODULUS

4.1. Porous obstacles

In the absence of obstacles, the square of the wavenumber in the fluid medium is given by $k_0^2 = \omega^2 / c_0^2$, where $c_0^2 = K_0/\rho_0$ (K_0 is the bulk modulus). We define, by analogy with this last formula, the square of the effective wavenumber $\xi_{LM}^2 = \omega^2 (\rho_{LM}/M_{LM})$ of the medium containing the obstacles, where ρ_{LM} represents the effective mass density and M_{LM} the effective modulus. The ratio of the squared wavenumbers gives [22]:

$$\left(\frac{\xi_{LM}}{k_0}\right)^2 = \frac{\rho_{LM}}{\rho_0} \times \frac{K_0}{M_{LM}} \quad (18)$$

Comparing with Eq. (17), we deduce that

$$\frac{\rho_{LM}}{\rho_0} = 1 + 3\mathcal{C}_V B_1 + 6\mathcal{C}_V^2 B_1^2 \quad (19)$$

and that

$$\frac{1}{M_{LM}} = \frac{1-\mathcal{C}_V}{K_0} + \frac{\mathcal{C}_V B_0}{K_0} \quad (20)$$

Accounting for the explicit form of $\mathcal{C}_V$, Eq. (15), we have also

$$\frac{\rho_{LM}}{\rho_0} = 1 + 3n_0 \langle \mathcal{V} \rangle \left(1 + 3p^2 + 2p^4\right) B_1 + 6n_0^2 \langle \mathcal{V} \rangle^2 \left(1 + 6p^2 + 13p^4 + 12p^6 + 4p^8\right) B_1^2 \quad (19')$$

and

$$\frac{1}{M_{LM}} = \frac{1 - n_0 \langle \mathcal{V} \rangle \left(1 + 3p^2 + 2p^4\right)}{K_0} + \frac{n_0 \langle \mathcal{V} \rangle \left(1 + 3p^2 + 2p^4\right) B_0}{K_0} \quad (20')$$

from which the monodisperse case can be derived by putting $p = 0$. It can be seen that the ratio ρ_{LM}/ρ_0 only takes into account B_1 while M_{LM} only takes into account B_0. In Appendix B, substituting Eqs. (B4-B7) in Eqs. (B2-B3) we arrive at the static expressions of B_0 and B_1 given in Eqs. (B8-B9). We thus obtain simplified forms of Eqs (19-20) showing the role played by the parameters characterizing the porous medium

$$\frac{\rho_{LM}}{\rho_0} = 1 + 3\mathcal{C}_V \frac{(1-\phi)(\rho_s - \rho_0)}{2(1-\phi)\rho_s + (1+2\phi)\rho_0} + \ldots \quad (21)$$

$$\frac{1}{M_{LM}} = \frac{1-\mathcal{C}_V}{K_0} + \frac{\mathcal{C}_V}{K_0}\frac{A}{\overline{A}} \quad (22)$$

where

$$A = 4 - \left(1 - \frac{H}{C}\right)\left[4 - 3\frac{H}{\mu}\left(1 - \frac{\rho_0}{\rho}\right)\right] - 3\frac{HM - C^2}{C\mu}$$

and

$$\overline{A} = \left(4 - 3\frac{H}{\mu}\right)\frac{HM - C^2}{CK_0} + 4\frac{\rho_0 H}{\rho K_0}.$$

The interpretation of the bulk modulus, Eq. (22), is the following: the trace of the mean deformation is given by $Tr\langle \varepsilon \rangle = -P/M_{LM}$ if P represents the hydrostatic pressure. Equations (31-32) do not depend on the viscosity of the fluid saturating the pores nor on the permeability. However, they still depend on the porosity.

4.2. Elastic obstacles

When the porosity tends towards zero ($\phi \to 0$), the medium becomes an elastic solid described by the Lamé constants λ_s, μ_s and the bulk modulus $K_s = \lambda_s + 2\mu_s/3$. We have

$$H \to H_s = K_s + \frac{4\mu_s}{3}, \quad \mu \to \mu_s, \; C \to C_s = K_s \quad (23)$$

and then

$$B_0 \to E_0 = \frac{K_0}{\lambda_s + 2\mu_s/3} = \frac{3\rho_0 c_0^2}{\rho_s\left(3c_L^2 - 4c_T^2\right)} \quad (24)$$

$$B_1 \to E_1 = \frac{\rho_s - \rho_0}{2\rho_s + \rho_0} \quad (25)$$

In Eqs. (24), we used the relations $\lambda_s = \rho_s\left(c_L^2 - 2c_T^2\right)$ and $\mu_s = \rho_s c_T^2$, where c_L (c_T respectively) represents the phase velocity of the longitudinal wave (transverse wave respectively) in the elastic medium. Substitutions of B_0 by E_0 and B_1 by E_1 in Eqs. (19-20) give the static expressions

$$\frac{\rho_{LM}}{\rho_0} \xrightarrow[\phi\to 0]{} 1+3\mathcal{C}_V \frac{\rho_s-\rho_0}{2\rho_s+\rho_0}+6\mathcal{C}_V^2\left(\frac{\rho_s-\rho_0}{2\rho_s+\rho_0}\right)^2 \quad (26)$$

and

$$\frac{1}{M_{LM}} \xrightarrow[\phi\to 0]{} \frac{1-\mathcal{C}_V}{K_0}+\frac{\mathcal{C}_V}{\lambda_s+2\mu_s/3} \quad (27)$$

Equation (26) provides an extension to order 2 in concentration of the results given to order 1 by Aristégui and Angel who used the wavenumber of Waterman and Truell and considered a monodisperse distribution of obstacles (see Eqs. (37) and (41) of Ref. [23]). In the dilute case and to the first order in concentration, the results of Kuster and Toksöz [24] agree with those given here. The divergence observed from the second order in concentration between the effective mass densities of LM and of Kuster and Töksoz comes from the fact that the latter do not consider the multiple scattering phenomenon between the spheres.

5. CONCLUSIONS

In this paper, we have used the general formula of the effective wavenumber provided by LM to determine the properties - mass densities and effective moduli - of polydisperse media made up of poroelastic spherical obstacles immersed in a fluid. Schulz's statistical distribution was considered for the size of the obstacles. Formulas are obtained for the effective wavenumber, modulus and mass density at low frequency. From there we derive formulas in the static case for porous obstacles and next for elastic obstacles. All formulas relative to the effective quantities depend on the fundamental parameter of concentration, which in turn depends closely on the statistical distribution considered for the size of the obstacles.

APPENDIX A

For a poro-elastic sphere surrounded and saturated by the same liquid, the application of the open-pore boundary conditions (Eq. (A1) of Ref. [12]) allows to write the scattering coefficients T_n as a ratio of determinants of order 4 d_n and $d_n^{[1]}$:

$$T_n = \frac{d_n^{[1]}}{d_n}. \quad (A1)$$

The elements of d_n are

$d_{11} = x_0 h_n^{(1)\prime}(x_0)$, $d_{12} = -\Gamma_1 x_1 j_n{}'(x_1)$,

$d_{13} = -\Gamma_2 x_2 j_n{}'(x_2)$,

$$d_{14} = -n(n+1)\Gamma_t j_n(x_t), \quad (A2)$$

$d_{21} = -h_n^{(1)}(x_0)$, $d_{22} = \rho_{f10} j_n(x_1)$,

$$d_{23} = \rho_{f20} j_n(x_2), \; d_{24} = 0, \quad (A3)$$

$d_{31} = -\rho_{0t} x_t^2 h_n^{(1)}(x_0)$,

$d_{32} = 4x_1 j_n{}'(x_1) + \left(\rho_{1t}x_t^2 - 2n(n+1)\right) j_n(x_1)$,

$d_{33} = 4x_2 j_n{}'(x_2) + \left(\rho_{2t}x_t^2 - 2n(n+1)\right) j_n(x_2)$

$$d_{34} = -2n(n+1)\left[x_t j_n{}'(x_t) - j_n(x_t)\right] \quad (A4)$$

$d_{41} = 0$, $d_{42} = x_1 j_n{}'(x_1) - j_n(x_1)$,

$d_{43} = x_2 j_n{}'(x_2) - j_n(x_2)$,

$$d_{44} = -x_t j_n{}'(x_t) + \left(1 - n(n+1) + x_t^2/2\right) j_n(x_t) \quad (A5)$$

The determinant $d_n^{[1]}$ is derived from d_n by replacing the elements of the first column of d_n with the following

$\alpha_{11} = -x_0 j_n{}'(x_0)$, $\alpha_{21} = j_n(x_0)$,

$$\alpha_{31} = \rho_{0t} x_t^2 j_n(x_0); \quad \alpha_{41} = 0. \quad (A6)$$

Above, $\Gamma_\alpha = 1 + \gamma_\alpha$ ($\alpha = 1, 2, t$) are compatibility coefficients [25], $x_\alpha = k_\alpha a = \omega a / c_\alpha$ ($\alpha = 0, 1, 2, t$) are dimensionless wavenumbers and j_n, $h_n^{(1)}$ are the spherical Bessel and Hankel functions of order n [26]. Further,

$$\gamma_\alpha = \frac{Hk_\alpha^2 - \rho\omega^2}{\rho_f\omega^2 - Ck_\alpha^2} \quad (\alpha = 1, 2)$$

$$\gamma_t = \frac{\mu k_t^2 - \rho\omega^2}{\rho_f\omega^2} \quad (A7)$$

denote dimensionless coefficients. The elastic moduli H, μ and C (to which a fourth module M must be associated) characterize the porous medium [25]. The coefficients $\rho_{\alpha t} = \rho_\alpha / \rho_t$ ($\alpha = 0, 1, 2$) and $\rho_{f\alpha 0} = \rho_{f\alpha}/\rho_0$, $\rho_{f\alpha 0} = \rho_{f\alpha}/\rho_0$ ($\alpha = 1, 2$) represent mass density ratios [27] where

$$\rho_\alpha = \rho_0\gamma_\alpha + \rho \quad (\alpha = 1, 2, t),$$

$$\rho_{f\alpha} = \rho_0\left(1 + \gamma_\alpha \frac{\tilde{\alpha}}{\phi}\right) \; (\alpha = 1, 2). \quad (A8)$$

It is recalled that $\tilde{\alpha}$ is the dynamic tortuosity [11], $\rho = (1-\phi)\rho_s + \phi\rho_0$ the mass density of the porous medium and ϕ the porosity.

APPENDIX B

The coefficients B_0 and B_1 of Eqs. (1-2) are obtained from Taylor's expansions of the scattering coefficients T_n for $|x_\alpha|$ ($\alpha = 0, 1, 2, t$) small compared to one, Appendix A. We have

$$B_0 = \frac{B_{0n1}\Gamma_2 - B_{0n2}\Gamma_1}{B_{0d1} - B_{0d2}} \tag{B1}$$

where

$$B_{0n1} = \left(4 - 3\frac{\rho_1 c_1^2}{\rho_t c_t^2} + 3\frac{\rho_{f1} c_1^2}{\rho_t c_t^2}\right)$$

$$B_{0n2} = \left(4 - 3\frac{\rho_2 c_2^2}{\rho_t c_t^2} + 3\frac{\rho_{f2} c_2^2}{\rho_t c_t^2}\right)$$

$$B_{0d1} = \frac{\rho_{f2} c_2^2}{\rho_0 c_0^2}\left(4 - 3\frac{\rho_1 c_1^2}{\rho_t c_t^2}\right)$$

$$B_{0d2} = \frac{\rho_{f1} c_1^2}{\rho_0 c_0^2}\left(4 - 3\frac{\rho_2 c_2^2}{\rho_t c_t^2}\right)$$

and

$$B_1 = -\frac{B_{1n1} + B_{1n2} + B_{1n3}}{B_{1d1} + B_{1d2} + B_{1d3}} \tag{B2}$$

$$B_{1n1} = 4\frac{c_2^2}{c_1^2}\left(-\Gamma_2 + \rho_{f20} + \Gamma_t \Delta\rho_2\right)$$

$$B_{1n2} = 4\left(\Gamma_1 - \rho_{f10} - \Gamma_t \Delta\rho_1\right)$$

$$B_{1n3} = 3\frac{c_2^2}{c_t^2}\left(\Gamma_2 \Delta\rho_1 - \Gamma_1 \Delta\rho_2 + \Delta\rho_{12}\right)$$

$$B_{1d1} = 4\frac{c_2^2}{c_1^2}\left(-\Gamma_2 - 2\rho_{f20} + \Gamma_t \Delta\rho_2\right)$$

$$B_{1d2} = -4\left(-\Gamma_1 - 2\rho_{f10} + \Gamma_t \Delta\rho_1\right)$$

$$B_{1d3} = 3\frac{c_2^2}{c_t^2}\left(\Gamma_2 \Delta\rho_1 - \Gamma_1 \Delta\rho_2 - 2\Delta\rho_{12}\right)$$

where

$$\begin{aligned} \Delta\rho_2 &= \rho_{2t} - \rho_{f2t} \\ \Delta\rho_1 &= \rho_{1t} - \rho_{f1t} \\ \Delta\rho_{12} &= \rho_{2t}\rho_{f10} - \rho_{1t}\rho_{f20} \end{aligned}, \tag{B3}$$

with $\rho_{f\alpha t} = \rho_{f\alpha} / \rho_t$ ($\alpha = 1, 2$).

The above parameters as well as fast, slow and shear wave speeds all depend on the frequency. The same is true of the coefficients B_0 and B_1. At low frequencies the following approximations can be used (the porosity of the medium being fixed and different from zero)

$$c_2^2 \simeq -i\frac{\omega}{\omega_c}\frac{HM - C^2}{\rho_0 H}, \quad c_1^2 \simeq \frac{H}{\rho}, \quad c_t^2 \simeq \frac{\mu}{\rho} \tag{B4}$$

$$\gamma_1 \simeq 0, \ \gamma_2 \simeq -\frac{H}{C}, \quad \gamma_t \simeq 0 \tag{B5}$$

$$\rho_1 \simeq \rho, \ \rho_2 \simeq \rho - \frac{\rho_0 H}{C}, \ \rho_t \simeq \rho, \tag{B6}$$

$$\rho_{f1} \simeq \rho_0, \qquad \rho_{f2} \simeq -\frac{i\omega_c}{\omega}\frac{H\rho_0}{C}, \tag{B7}$$

where $\omega_c = \eta / (\rho_0 \kappa)$ is a characteristic frequency built with the permeability κ of the porous medium, the mass density ρ_0 and the kinematic viscosity η of the saturating fluid. We obtain

$$B_0 \simeq \frac{B_{0n1} + B_{0n2} + 4}{B_{0d1} + B_{0d2}} \tag{B8}$$

$$B_{0n1} = \frac{3\left(C^2 - HM\right)}{C\mu}$$

$$B_{0n2} = -\left[3\frac{\rho_0 H}{\rho\mu} + 4 - 3\frac{H}{\mu}\right]\left(1 - \frac{H}{C}\right)$$

$$B_{0d1} = \left(4 - 3\frac{H}{\mu}\right)\left(\frac{HM - C^2}{CK_0}\right)$$

$$B_{0d2} = 4\frac{\rho_0 H}{\rho K_0}$$

and

$$B_1 \simeq \frac{\rho - \rho_0}{2\rho + \rho_0}. \tag{B9}$$

and see that coefficients B_0 and B_1 no longer depend on the frequency. We are in the static limit where the only physical parameters playing a role are those of the fluid (ρ_0, $K_0 = \rho_0 c_0^2$) and the porous medium (H, M, C, μ). We have

$$H = \frac{\left(K_s - K_b\right)^2}{D_b - K_b} + K_b + \frac{4\mu}{3} \tag{B10}$$

$$C = \frac{K_s\left(K_s - K_b\right)}{D_b - K_b} \tag{B11}$$

$$M = \frac{K_s^2}{D_b - K_b} \quad M = \frac{K_s^2}{D_b - K_b} \qquad \text{(B12)}$$

$$D_b = K_s \left[1 + \phi \left(\frac{K_s}{K_0} - 1 \right) \right] \qquad \text{(B13)}$$

where K_s (μ, respectively) is the bulk modulus (shear modulus, respectively) of the frame material and K_b the bulk modulus of the skeletal frame.

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