

# THE $KK$ -THEORY OF AMALGAMATED FREE PRODUCTS

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**ABSTRACT.** In the presence of conditional expectations, we prove a long exact sequence in  $KK$ -theory for both the maximal and the vertex reduced amalgamated free product of unital  $C^*$ -algebras that is valid even for non GNS-faithful conditional expectations. However, in the degenerated case, one has to introduce a new reduced amalgamated free product, that we call vertex-reduced. In the course of the proof we established the  $KK$ -equivalence between the full amalgamated free product and the vertex-reduced amalgamated free product. This results generalize and simplify the results obtained before by Germain and Thomsen. When the conditional expectations are extremely degenerated, i.e. when they are  $*$ -homomorphisms, our vertex-reduced amalgamated free product is isomorphic to the fiber direct sum. Hence our results also generalize a result of Cuntz.

## 1. INTRODUCTION

After the development of  $KK$ -theory by Kasparov [Ka80b, Ka88], J. Cuntz obtained in 1982 a very elegant result about the full free product of unital  $C^*$ -algebras with one-dimensional representations that leads to a conjectural long exact sequence in  $KK$ -theory for amalgamated free products in a general situation [Cu82]. At about the same time M. Pimsner's and D. Voiculescu's computation of the  $KK$ -theory for some group  $C^*$ -algebras (see [PV82]) culminated in the computation of full and reduced crossed products by groups acting on trees [Pi86] (or by the fundamental group of a graph of groups in Serre's terminology). To go beyond the group situation has been difficult and it relied heavily on various generalizations of Voiculescu's absorption theorem (see [Th03] for the most general results in that direction). Note also that G. Kasparov and G. Skandalis had another proof of Pimsner long exact sequence when studying  $KK$ -theory for buildings [KS91].

Section 2 is a preliminary section in which we investigate the notion of reduced amalgamated free products of unital  $C^*$ -algebras  $A_1 *_B A_2$  in the presence of not necessarily GNS-faithful conditional expectations. The usual reduced version, due to D. Voiculescu, which is obtained by looking at the module over  $B$ , is often too small. Indeed, when the conditional expectations onto  $B$  are both  $*$ -homomorphisms, the Voiculescu's reduced amalgamated free product is isomorphic to  $B$  and all the information about  $A_1$  and  $A_2$  is lost. This is why we consider another reduced amalgamated free product, that we call vertex-reduced, which is obtained by looking at the two modules over  $A_1$  and  $A_2$  and is an intermediate quotient between the full amalgamated free product and Voiculescu's reduced amalgamated free product. When the conditional expectations are GNS-faithful, these two reduced amalgamated free products coincide and when the conditional expectations are  $*$ -homomorphisms the vertex reduced amalgamated free product is isomorphic to the fiber sum  $A_1 \oplus_B A_2$ . Hence, even in the extreme degenerated case, the information on  $A_1$

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and  $A_2$  is still contained in the vertex-reduced amalgamated free product. As the vertex-reduced free product is a new construction, we devote some time to show some of its properties.

Before proving our long exact sequence in  $KK$ -theory we start with an auxiliary and easy result in Section 3. This result states that the full free product is always  $K$ -equivalent to the vertex-reduced free product. In particular, when the conditional expectations are morphisms, we get exactly Cuntz result [Cu82]. This result also generalizes and simplifies the previous result obtained by the second author [Ge96]. The proof is very natural, just a rotation trick. While finishing writing this paper, the authors have been made aware that K. Hasegawa [Ha15] just obtained the same result in the particular case of GNS-faithful conditional expectations (see also [Ha19]). By a remark by Ueda ([Ue08]), this result also proves the  $K$ -equivalence between full and (vertex) reduced HNN extensions.

The main part and also the most difficult part of our paper comes in Section 4. Under the presence of conditional expectations, we show that the full amalgamated free product  $A_1 *_B A_2$  is  $K$ -equivalent with the algebra  $D$  of continuous functions  $f$  from  $] - 1, 1[$  to the full free product such that  $f(] - 1, 0]) \subset A_1$ ,  $f([0, 1[) \subset A_2$  and  $f(0) \in B$ . This is done by generalizing a result in a paper by one of the authors ([Ge97]). Therefore the full amalgamated free product  $A_1 *_B A_2$  sits inside a long exact sequence for the computation of its  $KK$ -groups. Of course the vertex reduced free product has the same long exact sequence. Explicitly, if  $C$  is any separable  $C^*$ -algebra, then we have the two 6-terms exact sequences (see Corollary 4.12),

$$\begin{array}{ccccc} KK^0(C, B) & \longrightarrow & KK^0(C, A_1) \oplus KK^0(C, A_2) & \longrightarrow & KK^0(C, A_1 *_B A_2) \\ \uparrow & & & & \downarrow \\ KK^1(C, A_1 *_B A_2) & \longleftarrow & KK^1(C, A_1) \oplus KK^1(C, A_2) & \longleftarrow & KK^1(C, B) \end{array}$$

and

$$\begin{array}{ccccc} KK^0(B, C) & \longleftarrow & KK^0(A_1, C) \oplus KK^0(A_2, C) & \longleftarrow & KK^0(A_1 *_B A_2, C) \\ \downarrow & & & & \uparrow \\ KK^1(A_1 *_B A_2, C) & \longrightarrow & KK^1(A_1, C) \oplus KK^1(A_2, C) & \longrightarrow & KK^1(B, C) \end{array}$$

Again the HNN extension case follows using the isomorphism with an amalgamated free product. Note that this result greatly simplifies and generalizes the results of Thomsen [Th03] about  $KK$ -theory for amalgamated free products which are valid only when the amalgam is finite dimensional.

Let us mention some applications. As a direct corollary, we obtain that the amalgamated free product of discrete quantum groups is  $K$ -amenable if and only if the initial quantum groups are  $K$ -amenable. This generalizes the result of Vergnioux [Ve04] which was valid only for amenable discrete quantum groups and this also implies that a graph product of discrete quantum groups (see [CF14]) is  $K$ -amenable if and only if the initial quantum groups are  $K$ -amenable. Finally, let us mention that our results will be applied in a future paper to deduce a long exact sequence in  $KK$ -theory for fundamental  $C^*$ -algebras of graph of  $C^*$ -algebras, generalizing and simplifying the results of Pimsner [Pi86] and, as an application, the results of Fima-Freslon [FF13] (and those of Julg and Valette [JV84]).

## 2. PRELIMINARIES

**2.1. Notations and conventions.** All  $C^*$ -algebras and Hilbert modules are supposed to be separable. For a  $C^*$ -algebra  $A$  and a Hilbert  $A$ -module  $H$  we denote by  $\mathcal{L}_A(H)$  the  $C^*$ -algebra of  $A$ -linear adjointable operators from  $H$  to  $H$  and by  $\mathcal{K}_A(H)$  the sub- $C^*$ -algebra of  $\mathcal{L}_A(H)$  consisting of  $A$ -compact operators. For  $a \in A$ , we denote by  $L_A(a) \in \mathcal{L}_A(H)$  the left multiplication operator by  $a$ . We refer the reader to [Bl86] for the basics on KK-theory. In general KK-theory is a bi-functor in the category of  $\mathbb{Z}/2\mathbb{Z}$ -graded  $C^*$ -algebras. When the two  $C^*$ -algebras are trivially graded, we end up with what is called  $KK^0(A, B)$ . It follows from the standard simplifications that any element in  $KK^0(A, B)$  is the homotopy class of a  $A$ - $B$ -Kasparov's module of the form  $(H, \pi, T)$ , with  $H$  a  $\mathbb{Z}/2\mathbb{Z}$ -graded Hilbert  $B$ -module, i.e.  $H = H_0 \oplus H_1$  is a direct sum of Hilbert  $B$ -modules,  $\pi$  a morphism of graded  $C^*$ -algebras ( $\mathcal{L}_B(H)$  is a naturally graded  $C^*$ -algebra). As  $A$  is trivially graded,  $\pi = \pi_0 \oplus \pi_1$ , where  $\pi_k : A \rightarrow \mathcal{L}_B(H_k)$  are  $*$ -homomorphisms. And  $T$  a self-adjoint 1-graded operator in  $\mathcal{L}_B(H)$  with compact commutator with any element of  $\pi(A)$ .

Therefore  $T = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}$  with  $F \in \mathcal{L}_B(H_0, H_1)$  intertwines  $\pi_0$  and  $\pi_1$  up to compact operators.

The operator  $T$  also has the additional property that  $T^2 = 1$  modulo compact operator ( $A$  unital) and hence  $F$  is unitary up to compact operators in the case  $A$  is unital. In part 3 of this article, we refer to such a Kasparov module as  $(H, \pi, F)$  to simplify notation.

In part 4 of this article we must deal with  $KK^1$  elements. Any element in  $KK^1(A, B)$  has a simple description. It is the homotopy class of a triple  $(H, \pi_0, F)$ , where  $H$  is a Hilbert  $B$ -module,  $\pi_0 : A \rightarrow \mathcal{L}_B(H)$  is a  $*$ -homomorphism and  $F \in \mathcal{L}_B(H)$  a selfadjoint operator which is unitary up to compact operators and commutes with  $\pi_0$  up to compact operators. But it actually fits in the general description of Kasparov module but for the couple  $(A, B \otimes \mathbb{C}_1)$  where  $\mathbb{C}_1$  is the first non trivial Clifford algebra (see section 17.5.2 of [Bl86]). As a  $\mathbb{Z}/2\mathbb{Z}$ -graded algebra,  $\mathbb{C}_1 = \mathbb{C} \oplus \mathbb{C}$  where  $(1, 1)$  is 0 graded and  $(1, -1)$  is 1-graded. If  $E$  is a Hilbert  $B$ -module then  $E \otimes \mathbb{C}_1$  naturally becomes a  $\mathbb{Z}/2\mathbb{Z}$ -graded Hilbert module over  $B \otimes \mathbb{C}_1$ . If  $\pi$  is an action of the trivially graded  $C^*$ -algebra  $A$  on this module, then compatibility with the grading as well as  $\mathbb{C}_1$ -linearity imply that  $\pi$  decomposes as  $\pi_0 \oplus \pi_1$  with  $\pi_0$  an action of  $A$  onto  $E$ . Now a self-adjoint 1-graded operator  $T$  in  $\mathcal{L}_{B \otimes \mathbb{C}_1}(E \otimes \mathbb{C}_1)$  must be of the form  $(F, -F)$  where  $F$  is a self-adjoint operator of  $\mathcal{L}_B(E)$ . So the simple description of a  $KK^1(A, B)$  element gives a natural triple  $(H \otimes \mathbb{C}_1, \pi_0 \oplus \pi_1, (F, -F))$  in  $KK(A, B \otimes \mathbb{C}_1)$ . It must be noted, although we don't use it, that by Kasparov stabilisation any element of  $KK(A, B \otimes \mathbb{C}_1)$  is in the same class as an element of this simple form. For the largest part of section 4, we use the first description except for proposition 4.6 where Connes-Skandalis characterization of the Kasparov product between a  $KK^0$  and a  $KK^1$  element forces us to use the general description.

**2.2. Conditional expectations.** Let  $A, B$  be unital  $C^*$ -algebras and  $\varphi : A \rightarrow B$  be a unital completely positive map (ucp). A GNS construction of  $\varphi$  is a triple  $(K, \rho, \eta)$ , where  $K$  is a Hilbert  $B$ -module,  $\eta \in K$  and  $\rho : A \rightarrow \mathcal{L}_B(K)$  is a unital  $*$ -homomorphism such that  $K = \overline{\rho(A)\eta \cdot B}$  and  $\langle \eta, \rho(a)\eta \rangle = \varphi(a)$  for all  $a \in A$ . A GNS construction always exists and is unique, up to a canonical isomorphism. Note that, if  $B \subset A$  and  $E : A \rightarrow B$  is a conditional expectation, then the Hilbert  $B$ -submodule  $\eta \cdot B$  of  $K$ , where  $(K, \rho, \eta)$  is a GNS construction of  $E$ , is complemented. Indeed, we have  $K = \eta \cdot B \oplus K^\circ$ , where  $K^\circ = \overline{\text{Span}\{\rho(a)\eta \cdot b : a \in A^\circ \text{ and } b \in B\}}$  and  $A^\circ = \text{Ker}(E)$ . Since  $E$  is a conditional expectation onto  $B$  we have  $bA^\circ \subset A^\circ$  for all  $b \in B$ . It follows that  $\rho(b)K^\circ \subset K^\circ$

for all  $b \in B$ . Hence, the restriction of  $\rho$  to  $B$  (and to  $K^\circ$ ) gives a unital  $*$ -homomorphism  $\rho : B \rightarrow \mathcal{L}_B(K^\circ)$ .

A conditional expectation is called *GNS-faithful* (or *non-degenerate*) if for a given GNS construction (and hence for all GNS constructions)  $(K, \rho, \eta)$ , the homomorphism  $\rho$  is faithful. In this paper we will consider reduced amalgamated free product with respect to non-necessary GNS-faithful conditional expectations. Actually, the degeneracy of the conditional expectations will naturally produce different types of reduced amalgamated free products. This is why we include the next proposition, which is well known to specialists but helps to understand the extreme degenerated case: when  $E$  is a homomorphism. We include a complete proof for the convenience of the reader.

**Proposition 2.1.** *Let  $B \subset A$  be a unital inclusion of unital  $C^*$ -algebras and  $E : A \rightarrow B$  be a conditional expectation with GNS construction  $(K, \rho, \eta)$ . The following are equivalent.*

- (1)  $E$  is a homomorphism.
- (2)  $K = \eta \cdot B$ .
- (3)  $K^\circ = \{0\}$ .

*Proof.* Since  $K = \eta \cdot B \oplus K^\circ$  the equivalence between (2) and (3) is obvious.

(1)  $\Rightarrow$  (3). If  $E$  is a homomorphism from  $A$  to  $B$  then, since  $E$  is ucp, it is a unital  $*$ -homomorphism and we have for all  $b \in B$  and all  $a \in A^\circ$ ,

$$\langle \rho(a)\eta \cdot b, \rho(a)\eta \cdot b \rangle_K = b^* \langle \eta, \rho(a^*a)\eta \rangle_K b = b^* E(a^*a) b = b^* E(a)^* E(a) b = 0.$$

(3)  $\Rightarrow$  (1). If  $K^\circ = \{0\}$  then, for all  $a \in A^\circ$ , we have  $E(a^*a) = \langle \rho(a)\eta, \rho(a)\eta \rangle_K = 0$ . Hence

$$E((a - E(a))^*(a - E(a))) = 0 = E(a^*a) - E(a^*)E(a) - E(a)^*E(a) + E(a)^*E(a) \quad \text{for all } a \in A.$$

It follows that, for all  $a \in A$ , we have  $E(a^*a) = E(a)^*E(a)$ . Hence, the multiplicative domain of the ucp map  $E$  is equal to  $A$  which implies that  $E$  is a homomorphism.  $\square$

**2.3. The full and reduced amalgamated free products.** Let  $A_1, A_2$  be two unital  $C^*$ -algebras with a common  $C^*$ -subalgebra  $B \subset A_k, k = 1, 2$  and denote by  $A_f$  the full amalgamated free product. To be more precise, we sometimes write  $A_f = A_1 *_B A_2$ . It is well known that the canonical map from  $A_k$  to  $A_f$  is faithful for  $k = 1, 2$ . Hence, we will always view  $A_1$  and  $A_2$  as subalgebras of  $A_f$ .

We will now construct, in the presence of conditional expectations, two different reduced amalgamated free products. One of them, that we call the *edge-reduced amalgamated free product* has been extensively studied and it is called, in the literature, the reduced amalgamated free product. The other one, that we call the *vertex-reduced amalgamated free product*, does not seem to be known, even from specialists. As it will become gradually clear, the vertex-reduced amalgamated free product is actually much more natural than the edge-reduced amalgamated free product. It is an intermediate quotient of the full amalgamated free product and it is isomorphic to the edge-reduced amalgamated free product if the conditional expectations are assumed GNS-faithful. This is the reason why it has not appeared before in the literature since many authors only consider amalgamated free product in the presence of GNS-faithful conditional expectations. Since the vertex-reduced and the edge-reduced amalgamated free product are the foundations of our proofs we will now explain in great detail their constructions.

In the sequel, we always assume that, for  $k = 1, 2$ , there exists a conditional expectation  $E_k : A_k \rightarrow B$ . We write  $A_k^\circ = \{a \in A_k : E_k(a) = 0\}$ , we denote by  $(K_k, \rho_k, \eta_k)$  a GNS construction of  $E_k$  and by  $K_k^\circ$  the canonical orthogonal complement of  $\eta_k \cdot B$  in  $K_k$  as explained in Section 2.2. Recall that the restriction of  $\rho_k$  to  $B$  (and to  $K_k^\circ$ ) gives a unital  $*$ -homomorphism  $\rho_k : B \rightarrow \mathcal{L}_B(K_k^\circ)$ .

We denote by  $I$  the subset of  $\cup_{n \geq 1} \{1, 2\}^n$  defined by

$$I = \{(i_1, \dots, i_n) \in \{1, 2\}^n : n \geq 1 \text{ and } i_k \neq i_{k+1} \text{ for all } 1 \leq k \leq n-1\},$$

Recall that an operator  $x \in A_f$  is called *reduced* if  $x \neq 0$  and  $x$  can be written as  $x = a_1 \dots a_n$  with  $n \geq 1$  and  $a_k \in A_{i_k}^\circ - \{0\}$  such that  $\underline{i} = (i_1, \dots, i_n) \in I$ .

2.3.1. *The vertex-reduced amalgamated free products.* For  $\underline{i} = (i_1, \dots, i_n) \in I$ , we define a  $A_{i_1}$ - $A_{i_n}$ -bimodule  $H_{\underline{i}}$ . As Hilbert  $A_{i_n}$ -module we have:

$$H_{\underline{i}} = \begin{cases} K_{i_1} \otimes_B K_{i_2}^\circ \otimes_B \dots \otimes_B K_{i_{n-1}}^\circ \otimes_B A_{i_n} & \text{if } n \geq 3, \\ K_{i_1} \otimes_B A_{i_2} & \text{if } n = 2, \\ A_{i_1} & \text{if } n = 1. \end{cases}$$

The left action of  $A_{i_1}$  on  $H_{\underline{i}}$  is given by the unital  $*$ -homomorphism defined by

$$\lambda_{\underline{i}} : A_{i_1} \rightarrow \mathcal{L}_{A_{i_n}}(H_{\underline{i}}); \quad \lambda_{\underline{i}} = \begin{cases} \rho_{i_1} \otimes \text{id} & \text{if } n \geq 2, \\ L_{A_{i_1}} & \text{if } n = 1. \end{cases}$$

We consider, for  $k, l \in \{1, 2\}$ , the subset  $I_{k,l} = \{\underline{i} = (i_1, \dots, i_n) \in I : i_1 = k \text{ and } i_n = l\}$  and the  $A_k$ - $A_l$ -bimodule defined by

$$H_{k,l} = \bigoplus_{\underline{i} \in I_{k,l}} H_{\underline{i}} \quad \text{and} \quad \lambda_{k,l} = \bigoplus_{\underline{i} \in I_{k,l}} \lambda_{\underline{i}} : A_k \rightarrow \mathcal{L}_{A_l}(H_{k,l}).$$

For  $k \in \{1, 2\}$  we denote by  $\bar{k}$  the unique element in  $\{1, 2\} \setminus \{k\}$ .

**Example 2.2.** If, for  $k \in \{1, 2\}$ ,  $E_k$  is a homomorphism from  $A_k$  to  $B$  it follows from Proposition 2.1 that  $K_k^\circ = \{0\}$ . Hence,  $H_{k,k} = A_k \oplus K_k \otimes_B K_k^\circ \otimes_B A_k$  and  $H_{\bar{k},k} = K_{\bar{k}} \otimes_B A_k$ . Note that, since  $K_k \simeq B$ , we have  $H_{k,k} \simeq A_k \oplus K_k^\circ \otimes_B A_k \simeq K_{\bar{k}} \otimes_B A_k = H_{\bar{k},k}$ . Also we have  $H_{k,\bar{k}} = K_k \otimes_B A_{\bar{k}}$  and  $H_{\bar{k},\bar{k}} = A_{\bar{k}}$ . Again,  $H_{k,\bar{k}} \simeq A_{\bar{k}} = H_{\bar{k},\bar{k}}$ . Actually the isomorphism of Hilbert  $A_l$ -modules  $H_{k,l} \simeq H_{\bar{k},l}$  is true in full generality as explained below.

For  $k, l \in \{1, 2\}$  we define a unitary  $u_{k,l} \in \mathcal{L}_{A_l}(H_{k,l}, H_{\bar{k},l})$ , by the following formula. Let  $\underline{i} = (i_1, \dots, i_n) \in I$ , with  $i_1 = k$  and  $i_n = l$ . For  $\xi \in H_{\underline{i}}$  we define  $u_{k,l}\xi \in H_{\bar{k},l}$  in the following way.

- If  $n \geq 2$ , write  $\underline{i} = (k, \underline{i}')$ , where  $\underline{i}' = (i_2, \dots, i_n) \in I_{\bar{k},l}$ . For  $\xi = \rho_k(a)\eta_k \otimes \xi'$ , with  $a \in A_k$  and  $\xi' \in H_{\underline{i}'}$ , we define  $u_{k,l}\xi := \begin{cases} \eta_{\bar{k}} \otimes \xi & \text{if } E_k(a) = 0, \\ \lambda_{\underline{i}'}(a)\xi' & \text{if } a \in B. \end{cases}$
- If  $n = 1$  then  $k = l$ ,  $\underline{i} = (l)$  and  $\xi \in A_l = H_{\underline{i}}$ . We define  $u_{k,l}\xi := \eta_{\bar{k}} \otimes \xi$ .

Since  $\rho_k(b)\eta_k = \eta_k \cdot b$  for all  $b \in B$ , the operators  $u_{k,l}$  are well defined and it is easy to check that, for all  $k, l \in \{1, 2\}$ , the operators  $u_{k,l}$  commute with the right actions of  $A_l$  on  $H_{k,l}$  and

$H_{\bar{k},l}$  and extend to a unitary operators, still denoted  $u_{k,l}$ , in  $\mathcal{L}_{A_l}(H_{k,l}, H_{\bar{k},l})$  such that  $u_{k,l}^* = u_{\bar{k},l}$ . Moreover, the definition of  $u_{k,l}$  implies that,

$$(1) \quad u_{k,l}^* \lambda_{\bar{k},l}(b) u_{k,l} = \lambda_{k,l}(b) \quad \text{for all } b \in B.$$

**Definition 2.3.** Let  $k \in \{1, 2\}$ . The  $k$ -vertex-reduced amalgamated free product is the  $C^*$ -subalgebra  $A_{v,k} \subset \mathcal{L}_{A_k}(H_{k,k})$  generated by  $\lambda_{k,k}(A_k) \cup u_{k,k}^* \lambda_{\bar{k},k}(A_{\bar{k}}) u_{k,k} \subset \mathcal{L}_{A_k}(H_{k,k})$ . To be more precise, we use sometimes the notation  $A_{v,k} = A_1 \underset{B}{*} A_2$ .

For a fixed  $k \in \{1, 2\}$  the relations (1) imply the existence of a unique unital  $*$ -homomorphism  $\pi_k : A_f \rightarrow A_{v,k}$  such that  $\pi_k(a) = \begin{cases} \lambda_{k,k}(a) & \text{if } a \in A_k, \\ u_{k,k}^* \lambda_{\bar{k},k}(a) u_{k,k} & \text{if } a \in A_{\bar{k}}. \end{cases}$

In the sequel we will denote by  $\xi_k$  the vector  $\xi_k := 1_{A_k} \in A_k \subset H_{k,k}$ . We summarize the main properties of  $A_{v,k}$  in the following proposition.

**Proposition 2.4.** Fix  $k \in \{1, 2\}$ . The following facts hold.

- (1) The morphism  $\pi_k$  is faithful on  $A_k$ .
- (2) If  $E_{\bar{k}}$  is GNS-faithful then  $\pi_k$  is faithful on  $A_{\bar{k}}$ .
- (3) There exists a unique ucp map  $\mathbb{E}_k : A_{v,k} \rightarrow A_k$  such that  $\mathbb{E}_k(\pi_k(a)) = a \quad \forall a \in A_k$  and

$\mathbb{E}_k(\pi_k(a_1 \dots a_n)) = 0$  for all  $a = a_1 \dots a_n \in A_f$  reduced with  $n \geq 2$  or  $n = 1$  and  $a = a_1 \in A_{\bar{k}}^\circ$ .

Moreover,  $\mathbb{E}_k$  is GNS-faithful.

- (4) For any unital  $C^*$ -algebra  $C$  with two unital  $*$ -homomorphisms  $\nu_j : A_j \rightarrow C$ ,  $j = 1, 2$ , such that
  - $\nu_1(b) = \nu_2(b)$  for all  $b \in B$ ,
  - $C$  is generated, as a  $C^*$ -algebra, by  $\nu_1(A_1) \cup \nu_2(A_2)$ ,
  - $\nu_k$  is faithful and there exists a GNS-faithful ucp map  $E : C \rightarrow A_k$  such that  $E(\nu_k(a)) = a$  for all  $a \in A_k$  and

$E(\nu_{i_1}(a_1) \dots \nu_{i_n}(a_n)) = 0$  for all  $a = a_1 \dots a_n \in A_f$  reduced with  $n \geq 2$  or  $n = 1$  and  $a = a_1 \in A_{\bar{k}}^\circ$ ,

there exists a unique unital  $*$ -isomorphism  $\nu : A_{v,k} \rightarrow C$  such that  $\nu \circ \pi_k(a) = \nu_k(a)$  for all  $a \in A_1 \cup A_2$ . Moreover,  $\nu$  satisfies  $E \circ \nu = \mathbb{E}_k$ .

*Proof.* Fix  $k \in \{1, 2\}$ . By definition of  $\pi_k$  we have, if  $a \in A_k$ ,  $\langle \xi_k, \pi_k(a) \xi_k \rangle = a$ . It follows directly that  $\pi_k$  is faithful on  $A_k$ . Moreover, the map  $\mathbb{E}_k : A_{v,k} \rightarrow A_k$ ,  $x \mapsto \langle \xi_k, x \xi_k \rangle$  satisfies  $\mathbb{E}_k(\pi_k(a)) = a \quad \forall a \in A_k$ . By definition we have, for all reduced operators  $x = a_1 \dots a_n$  with  $\underline{i} = (i_1, \dots, i_n) \in I$  and  $a_s \in A_{i_s}^\circ$  for all  $s \in \{1, \dots, n\}$ ,

$$(2) \quad \pi_k(a_1 \dots a_n) \xi_k = \begin{cases} \rho_{i_1}(a_1) \eta_{i_1} \otimes \dots \otimes \rho_{i_{n-1}}(a_{n-1}) \eta_{i_{n-1}} \otimes a_n & \text{if } i_1 = k \text{ and } i_n = k, \\ \eta_k \otimes \rho_{i_1}(a_1) \eta_{i_1} \otimes \dots \otimes \rho_{i_{n-1}}(a_{n-1}) \eta_{i_{n-1}} \otimes a_n & \text{if } i_1 \neq k \text{ and } i_n = k, \\ \rho_{i_1}(a_1) \eta_{i_1} \otimes \dots \otimes \rho_{i_n}(a_n) \eta_{i_n} \otimes 1_{A_k} & \text{if } i_1 = k \text{ and } i_n \neq k, \\ \eta_k \otimes \rho_{i_1}(a_1) \eta_{i_1} \otimes \dots \otimes \rho_{i_n}(a_n) \eta_{i_n} \otimes 1_{A_k} & \text{if } i_1 \neq k \text{ and } i_n \neq k. \end{cases}$$

Hence we have  $\mathbb{E}_k(\pi_k(a_1 \dots a_n)) = 0$  for all  $a = a_1 \dots a_n \in A_f$  reduced with  $n \geq 2$  or  $n = 1$  and  $a = a_1 \in A_{\bar{k}}^\circ$ . It also follows easily from the previous set of equations that  $\overline{\pi_k(A_f) \xi_k \cdot A_k} = H_{k,k}$ . Hence the triple  $(H_{k,k}, \text{id}, \xi_k)$  is a GNS construction for  $\mathbb{E}_k$ . This shows that  $\mathbb{E}_k$  is GNS-faithful. Note that the uniqueness statement of the third assertion is obvious since  $A_f$  is the linear span

of  $B$  and the reduced operators. Also, the second statement becomes now obvious since, by the properties of  $\mathbb{E}_k$  we have, for all  $x \in A_{\bar{k}}$ ,  $\mathbb{E}_k(\pi_k(x)) = \mathbb{E}_k(\pi_k(x - E_{\bar{k}}(x))) + \mathbb{E}_k(\pi_k(E_{\bar{k}}(x))) = \pi_k(E_{\bar{k}}(x))$ . It follows easily from this equation that  $\pi_k$  is faithful on  $A_{\bar{k}}$  whenever  $E_{\bar{k}}$  is GNS-faithful. Indeed, let  $x \in A_{\bar{k}}$  such that  $\pi_k(x) = 0$ . Then, for all  $y \in A_{\bar{k}}$  we have  $\pi_k(y^*x^*xy) = 0$ . Hence,  $\pi_k \circ E_{\bar{k}}(y^*x^*xy) = \mathbb{E}_k \circ \pi_k(y^*x^*xy) = 0$  for all  $y \in A_{\bar{k}}$ . Since  $\pi_k$  is faithful on  $A_k$  we find  $E_{\bar{k}}(y^*x^*xy) = 0$ , for all  $y \in A_{\bar{k}}$ . Since  $E_{\bar{k}}$  is GNS-faithful we conclude that  $x = 0$ .

(4). The proof is a routine. We write the argument for the convenience of the reader. Let  $(K, \rho, \eta)$  be the GNS construction of  $E$ . Since  $E$  is GNS-faithful we may and will assume that  $\rho = \text{id}$  and  $C \subset \mathcal{L}_{A_k}(K)$ . By the properties of  $\mathbb{E}_k$  and  $E$ , the map  $U : H_{k,k} \rightarrow K$  defined by, for  $x = a_1 \dots a_n \in A_f$  reduced with  $a_k \in A_{i_k}^\circ$ ,  $U(\pi_k(x)\xi_k) := \nu_{i_1}(a_1) \dots \nu_{i_n}(a_n)\eta$  and, for  $x = b \in B$ ,  $U(\pi_k(b)\xi_k) = \nu_1(b)\eta = \nu_2(b)\eta$ , is well defined and extends to a unitary  $U \in \mathcal{L}_{A_k}(H_{k,k}, K)$ . By construction, the map  $\nu(x) := UxU^*$ , for  $x \in A_{v,k}$ , satisfies the claimed properties. The uniqueness is obvious.  $\square$

*Remark 2.5.* It is known that the canonical homomorphism from  $A_k$  to  $A_f$  is faithful for  $k \in \{1, 2\}$  without assuming the existence of conditional expectations from  $A_k$  to  $B$ . However, assertion (1) of Proposition 2.4 gives a very simple proof of this fact, since it shows that the composition of the canonical homomorphism from  $A_k$  to  $A_f$  with the homomorphism  $\pi_k$  is faithful, which implies that the canonical homomorphism from  $A_k$  to  $A_f$  itself is faithful.

**Example 2.6.** Suppose that, for a given  $k \in \{1, 2\}$ ,  $E_k$  is a homomorphism. Then, as observed in Example 2.2, we have  $H_{\bar{k}, \bar{k}} = A_{\bar{k}}$  (and  $\lambda_{\bar{k}, \bar{k}} = L_{A_{\bar{k}}}$ ). It follows from the definition of  $\pi_{\bar{k}}$  that

$$\pi_{\bar{k}}(a) = \begin{cases} L_{A_{\bar{k}}}(a) & \text{if } a \in A_{\bar{k}}, \\ 0 & \text{if } a \in A_k^\circ. \end{cases}$$

Hence, since  $A_f$  the closed linear span of  $A_{\bar{k}}$  and the reduced operators and  $\pi_{\bar{k}} : A_f \rightarrow A_{v, \bar{k}}$  is surjective, we find that  $A_{v, \bar{k}} = \pi_{\bar{k}}(A_{\bar{k}})$ . Moreover, since  $\pi_{\bar{k}}$  is faithful on  $A_{\bar{k}}$  we conclude that the restriction of  $\pi_{\bar{k}}$  to  $A_{\bar{k}}$  gives an isomorphism  $A_{\bar{k}} \simeq A_{v, \bar{k}}$ .

**Definition 2.7.** The *vertex-reduced amalgamated free product* is the  $C^*$ -algebra obtained by separation and completion of  $A_f$  with respect to the  $C^*$ -semi-norm  $\|\cdot\|_v$  on  $A_f$  defined by

$$\|x\|_v := \text{Max}\{\|\pi_1(x)\|, \|\pi_2(x)\|\} \quad \text{for all } x \in A_f.$$

By *separation and completion* we mean the completion of the pre- $C^*$ -algebra obtained by considering the quotient by the null ideal of the  $C^*$  semi-norm.

We will note it  $A_1 \overset{v}{*}_B A_2$  or  $A_v$  for simplicity in the rest of this section and let  $\pi : A_f \rightarrow A_v$  be the canonical surjective unital  $*$ -homomorphism. Note that, by construction of  $A_v$ , for all  $k \in \{1, 2\}$ , there exists a unique unital (surjective)  $*$ -homomorphism  $\pi_{v,k} : A_v \rightarrow A_{v,k}$  such that  $\pi_{v,k} \circ \pi = \pi_k$ . We describe the fundamental properties of the vertex-reduced amalgamated free product in the following proposition. We call a family of ucp maps  $\{\varphi_i\}_{i \in I}$ ,  $\varphi_i : A \rightarrow B_i$  GNS-faithful if  $\bigcap_{i \in I} \text{Ker}(\pi_i) = \{0\}$ , where  $(H_i, \pi_i, \xi_i)$  is a GNS-construction for  $\varphi_i$ . From Proposition 2.4 and the definition of  $A_v$  we deduce the following result.

**Proposition 2.8.** *The following facts hold.*

- (1)  $\pi$  is faithful on  $A_k$  for all  $k \in \{1, 2\}$ .

(2) For all  $k \in \{1, 2\}$ , there is a unique ucp map  $\mathbb{E}_{A_k} : A_v \rightarrow A_k$  such that  $\mathbb{E}_{A_k} \circ \pi(a) = a$  for all  $a \in A_k$  and all  $k \in \{1, 2\}$  and,

$\mathbb{E}_{A_k}(\pi(a_1 \dots a_n)) = 0$  for all  $a = a_1 \dots a_n \in A_f$  reduced with  $n \geq 2$  or  $n = 1$  and  $a = a_1 \in A_k^\circ$ .

Moreover, the family  $\{\mathbb{E}_{A_1}, \mathbb{E}_{A_2}\}$  is GNS-faithful.

(3) Suppose that  $C$  is a unital  $C^*$ -algebra with  $*$ -homomorphisms  $\nu_k : A_k \rightarrow C$  such that

- $\nu_1(b) = \nu_2(b)$  for all  $b \in B$ ,
- $C$  is generated, as a  $C^*$ -algebra, by  $\nu_1(A_1) \cup \nu_2(A_2)$ ,
- $\nu_1$  and  $\nu_2$  are faithful and, for all  $k \in \{1, 2\}$ , there exists a ucp map  $E_{A_k} : C \rightarrow A_k$  such that  $E_{A_k} \circ \nu_k(a) = a$  for all  $a \in A_k$  and all  $k \in \{1, 2\}$  and,

$E_{A_k}(\nu_{i_1}(a_1) \dots \nu_{i_n}(a_n)) = 0$  for all  $a = a_1 \dots a_n \in A_f$  reduced with  $n \geq 2$  or  $n = 1$  and  $a = a_1 \in A_k^\circ$ ,

and the family  $\{E_{A_1}, E_{A_2}\}$  is GNS-faithful.

Then, there exists a unique unital  $*$ -isomorphism  $\nu : A_v \rightarrow C$  such that  $\nu \circ \pi(a) = \nu_k(a)$  for all  $a \in A_k$  and all  $k \in \{1, 2\}$ . Moreover,  $\nu$  satisfies  $E_{A_k} \circ \nu = \mathbb{E}_{A_k}$ ,  $k \in \{1, 2\}$ .

*Proof.* (1). It is obvious since, by Proposition 2.4,  $\pi_k$  is faithful on  $A_k$  for  $k = 1, 2$ .

(2). By Proposition 2.4, the maps  $\mathbb{E}_{A_k} = \mathbb{E}_k \circ \pi_{v,k}$  satisfy the desired properties and it suffices to check that the family  $\{\mathbb{E}_{A_1}, \mathbb{E}_{A_2}\}$  is GNS-faithful. Let  $x_0 \in A_f$  be such that  $x = \pi(x_0) \in A_v$  satisfies  $\mathbb{E}_{A_k}(y^* x^* x y) = 0$  for all  $y \in A_v$  and all  $k \in \{1, 2\}$ . Then, for all  $k \in \{1, 2\}$  we have  $\mathbb{E}_k(y^* \pi_{v,k}(x^* x) y) = 0$  for all  $y \in A_{v,k}$ . Since  $\mathbb{E}_k$  is GNS-faithful, this implies that  $\pi_{v,k}(x) = \pi_k(x_0) = 0$  for all  $k \in \{1, 2\}$ . Hence,  $\|x\|_{A_v} = \text{Max}(\|\pi_1(x_0)\|, \|\pi_2(x_0)\|) = 0$ .

(3). The proof is a routine. We include it for the convenience of the reader. Let  $(L_k, m_k, f_k)$  be the GNS construction of  $E_{A_k}$ . By the universal property of  $A_{v,k}$ , the  $C^*$ -algebra  $m_k(C) \subset \mathcal{L}_{A_k}(L_k)$  is canonically isomorphic to  $A_{v,k}$ . Hence, in the remainder of the proof we suppose that  $m_k(C) = A_{v,k}$ . By the universal property of  $A_f$ , we have a unital surjective  $*$ -homomorphism  $\nu_f : A_f \rightarrow C$  such that  $\nu_f|_{A_k} = \nu_k$ . Note that, by the identification we made,  $m_k \circ \nu_f = \pi_k$ . Hence, by construction of  $A_v$ , there exists a unique unital (surjective)  $*$ -homomorphism  $\nu_0 : C \rightarrow A_v$  such that  $\pi_{v,k} \circ \nu_0 = m_k$  for all  $k \in \{1, 2\}$ . Note that  $\nu_0$  is faithful since the identity  $\pi_{v,k} \circ \nu_0 = m_k$ ,  $k = 1, 2$ , implies that  $\text{Ker}(\nu_0) \subset \text{Ker}(m_1) \cap \text{Ker}(m_2) = \{0\}$  (because the pair  $(E_{A_1}, E_{A_2})$  is GNS-faithful). Hence  $\nu_0$  is a unital  $*$ -isomorphism and  $\nu := \nu_0^{-1}$  satisfies the required properties.  $\square$

**Corollary 2.9.** *If both  $E_1$  and  $E_2$  are homomorphisms then there is a canonical isomorphism  $A_v \simeq A_1 \oplus_B A_2$ , where  $A_1 \oplus_B A_2 := \{(a_1, a_2) \in A_1 \oplus A_2 : E_1(a_1) = E_2(a_2)\}$ .*

*Proof.* We use the universal property of  $A_v$  described in Proposition 2.8. Define  $\nu_k : A_k \rightarrow A_1 \oplus_B A_2$  by  $\nu_1(x) = (x, E_1(x))$  and  $\nu_2(y) = (E_2(y), y)$ . It is clear that  $\nu_1$  and  $\nu_2$  are both faithful unital  $*$ -homomorphisms such that  $\nu_1(b) = \nu_2(b)$  for all  $b \in B$ . Define  $E_{A_k} : A_1 \oplus_B A_2 \rightarrow A_k$  by  $E_{A_1}(a_1, a_2) = a_1$  and  $E_{A_2}(a_1, a_2) = a_2$ . Then, for all  $k \in \{1, 2\}$ ,  $E_k$  is a unital  $*$ -homomorphism such that  $E_{A_k} \circ \nu_k(a) = a$  for all  $a \in A_k$ . In particular, both  $E_1$  and  $E_2$  are conditional expectations and, since  $\text{Ker}(E_{A_1}) \cap \text{Ker}(E_{A_2}) = \{0\}$ , the family  $\{E_{A_1}, E_{A_2}\}$  is GNS-faithful. Hence, it suffices to check the condition on the reduced operators. Since  $\nu_1(A_1^\circ) = \{(x, 0) : x \in A_1^\circ\}$  and  $\nu_2(A_2^\circ) = \{(0, y) : y \in A_2^\circ\}$ , we have  $\nu_1(A_1^\circ)\nu_2(A_2^\circ) = \nu_2(A_2^\circ)\nu_1(A_1^\circ) = \{0\}$ . Hence, it suffices to check the condition on elements  $(a_1, a_2) \in \nu_1(A_1^\circ) \cup \nu_2(A_2^\circ)$  which is obvious.  $\square$



2.3.2. *The edge-reduced amalgamated free product.* In this section we show how the construction of the edge-reduced (or, in the literature, the reduced) amalgamated free product in full generality is related to the vertex-reduced free product we just defined.

For  $\underline{i} \in I$ , we consider the  $B$ - $B$ -module  $K_{\underline{i}}^{\circ} = K_{i_1}^{\circ} \otimes_B \dots \otimes_B K_{i_n}^{\circ}$  as Hilbert  $B$ -module with the left action of  $B$  given by the unital  $*$ -homomorphism  $\rho_{\underline{i}} : B \rightarrow \mathcal{L}_B(K_{\underline{i}}^{\circ})$ ,  $\rho_{\underline{i}}(b) = \rho_{i_1}(b) \otimes_B \text{id}$  for all  $b \in B$  and we define the Hilbert  $B$ -bimodule  $K = B \oplus \left( \bigoplus_{\underline{i} \in I} K_{\underline{i}}^{\circ} \right)$ .

**Example 2.10.** If, for some  $k \in \{1, 2\}$ ,  $E_k$  is a homomorphism then  $K = B \oplus K_k^{\circ} \simeq K_k^{\circ}$ . Hence, if both  $E_1$  and  $E_2$  are homomorphisms then  $K = B$ .

For  $l \in \{1, 2\}$  define  $K(l) = B \oplus \left( \bigoplus_{\underline{i} \in I, i_1 \neq l} K_{\underline{i}}^{\circ} \right)$  and note that we have a unital  $*$ -homomorphism  $\rho_l : B \rightarrow \mathcal{L}_B(K(l))$  defined by  $\rho_l = L_B \oplus \bigoplus_{\underline{i} \in I, i_1 \neq l} \rho_{\underline{i}}$ . Let  $U_l \in \mathcal{L}_B(K_l \otimes_{\rho_l} K(l), K)$  be the unitary operator defined by

$$\begin{aligned} U_l : \quad K_l \otimes_{\rho_l} K(l) &\longrightarrow K \\ \eta_l \otimes B &\xrightarrow{\simeq} B \\ K_l^{\circ} \otimes_{\rho_l} B &\xrightarrow{\simeq} K_l^{\circ} \\ \eta_l \otimes H_{\underline{i}} &\xrightarrow{\simeq} K_{\underline{i}}^{\circ} \\ K_l^{\circ} \otimes_{\rho_l} H_{\underline{i}} &\xrightarrow{\simeq} K_{(l, \underline{i})}^{\circ} \end{aligned}$$

where  $(l, \underline{i}) = (l, i_1, \dots, i_n) \in I$  if  $\underline{i} = (i_1, \dots, i_n) \in I$  with  $i_1 \neq l$ . We define the unital  $*$ -homomorphisms  $\lambda_l : \mathcal{L}_B(K_l) \rightarrow \mathcal{L}_B(K)$  by  $\lambda_l(x) = U_l(x \otimes 1)U_l^*$ . By definition we have  $\lambda_1(\rho_1(b)) = \lambda_2(\rho_2(b))$  for all  $b \in B$ . It follows that there exists a unique unital  $*$ -homomorphism  $\rho : A_f \rightarrow \mathcal{L}_B(K)$  such that  $\rho(a) = \lambda_k(\rho_k(a))$  for  $a \in A_k$ , for all  $k \in \{1, 2\}$ .

**Proposition 2.11.** *There are canonical unitaries  $V_k \in \mathcal{L}_B(H_{k,k} \otimes_{E_k} B, K)$  for  $k = 1, 2$  satisfying  $V_k(\pi_k(a) \otimes 1)V_k^* = \rho(a)$  for all  $a \in A_k$  and all  $k \in \{1, 2\}$ .*

*Proof.* Note that, for  $\underline{i} = (i_1, \dots, i_n) \in I$  with  $i_1 = i_n = k$  (hence  $n$  is odd) we have, if  $n = 1$ ,  $H_{\underline{i}} \otimes_{E_k} B = A_k \otimes B \simeq K_k \simeq K_k^{\circ} \oplus B$ , and, if  $n \geq 3$ ,  $H_{\underline{i}} \otimes_{E_k} B = K_k \otimes_B \left( K_k^{\circ} \otimes_B \dots \otimes_B K_k^{\circ} \right) \otimes_B K_k \simeq K_k^{\circ} \oplus K_{\underline{i}'}^{\circ} \oplus K_{\underline{i}''}^{\circ} \oplus K_{\underline{i}'''}^{\circ}$ , where  $\underline{i}' = (i_2, \dots, i_n)$ ,  $\underline{i}'' = (i_1, \dots, i_{n-1})$  and  $\underline{i}''' = (i_2, \dots, i_{n-1})$ . Hence the existence of  $V_k : H_{k,k} \otimes_{E_k} B \rightarrow K$ . It is easy to check that  $V_k$  satisfies  $V_k(\pi_k(a) \otimes 1)V_k^* = \rho(a)$  for all  $a \in A_k$  and all  $k \in \{1, 2\}$ .  $\square$

**Definition 2.12.** The *edge-reduced* amalgamated free product is the  $C^*$ -subalgebra  $A_e \subset \mathcal{L}_B(K)$  generated by  $\lambda_1(A_1) \cup \lambda_2(A_2) \subset \mathcal{L}_B(K)$ . To be more precise, we use sometimes the notation  $A_e = A_1 \overset{e}{*}_B A_2$ .

The edge-reduced amalgamated free product has been constructed by Voiculescu in [Vo83] and is known in the literature as the Voiculescu's reduced amalgamated free product.

**Example 2.13.** If, for some  $k \in \{1, 2\}$ ,  $E_k$  is a homomorphism then  $A_e$  is the  $C^*$ -algebra  $\rho_{\bar{k}}(A_{\bar{k}}) \subset \mathcal{L}_B(K_{\bar{k}})$ . If both  $E_1$  and  $E_2$  are homomorphisms then  $A_e \simeq B$ .

The preceding example shows that the edge reduced amalgamated free product may forget everything about the initial  $C^*$ -algebras  $A_1$  and  $A_2$  in the extreme degenerated case: it only remembers  $B$ . This shows that, in general, one should consider instead the vertex-reduced amalgamated free product. Indeed, even in the extreme degenerated case, the vertex reduced amalgamated free product correctly remembers the  $C^*$ -algebras  $A_1$  and  $A_2$ , as shown in Corollary 2.9.

In the following proposition we recall the properties of  $A_e$ . The results below are well known when  $E_1$  and  $E_2$  are GNS-faithful. The proof is similar to the proof of Proposition 2.4 and we leave it to the reader.

**Proposition 2.14.** *The following facts hold.*

- (1)  $\rho$  is faithful on  $B$ .
- (2) For any  $k \in \{1, 2\}$ , if  $E_k$  is GNS-faithful then  $\rho$  is faithful on  $A_k$ .
- (3) There exists a unique ucp map  $\mathbb{E} : A_e \rightarrow B$  such that  $\mathbb{E} \circ \rho(b) = b$  for all  $b \in B$  and,

$$\mathbb{E}(\rho(a_1, \dots, a_n)) = 0 \text{ for all } a = a_1 \dots a_n \in A_f \text{ reduced.}$$

Moreover,  $\mathbb{E}$  is GNS-faithful.

- (4) For any unital  $C^*$ -algebra  $C$  with two unital  $*$ -homomorphisms  $\nu_k : A_k \rightarrow C$ ,  $k = 1, 2$ , such that

- $\nu_1(b) = \nu_2(b)$  for all  $b \in B$ ,
- $C$  is generated, as a  $C^*$ -algebra, by  $\nu_1(A_1) \cup \nu_2(A_2)$ ,
- $\nu_1|_B = \nu_2|_B$  is faithful and there exists a GNS-faithful ucp map  $E : C \rightarrow B$  such that  $E \circ \nu_k(b) = b$  for all  $b \in B$ ,  $k = 1, 2$ , and,

$$E(\nu_{i_1}(a_1) \dots \nu_{i_n}(a_n)) = 0 \text{ for all } a = a_1 \dots a_n \in A_f \text{ reduced,}$$

there exists a unique unital  $*$ -isomorphism  $\nu : A_e \rightarrow C$  such that  $\nu \circ \rho(a) = \nu_k(a)$  for all  $a \in A_k$ ,  $k \in \{1, 2\}$ . Moreover,  $\nu$  satisfies  $E \circ \nu = \mathbb{E}$ .

**Proposition 2.15.** *For all  $k \in \{1, 2\}$  there exists a unique unital  $*$ -homomorphism*

$$\lambda_{v,k} : A_{v,k} \rightarrow A_e \quad \text{such that} \quad \lambda_{v,k} \circ \pi_k = \rho.$$

Moreover,  $\lambda_{v,k}$  is faithful on  $\pi_k(A_{\bar{k}})$  and, if  $E_k$  is GNS-faithful,  $\lambda_{v,k}$  is an isomorphism.

*Proof.* The formulae  $\lambda_{v,k}(x) = V_k(x \otimes 1)V_k^*$  defines a unital  $*$ -homomorphism  $\lambda_{v,k} : A_{v,k} \rightarrow A_e$  satisfying  $\lambda_{v,k} \circ \pi_k = \rho$ . The uniqueness of  $\lambda_{v,k}$  is obvious. Let us check that  $\lambda_{v,k}$  is faithful on  $\pi_k(A_{\bar{k}})$ . Suppose that  $x \in A_{\bar{k}}$  and  $\lambda_{v,k}(\pi_k(x)) = 0$ . Then, for all  $y \in A_{\bar{k}}$ , we have  $\rho(y^*x^*xy) = \lambda_{v,k}(\pi_k(y^*x^*xy)) = 0$ . Hence,  $0 = \mathbb{E} \circ \rho(y^*x^*xy) = \mathbb{E} \circ \rho(E_{\bar{k}}(y^*x^*xy)) = E_{\bar{k}}(y^*x^*xy)$ . It follows that  $x \in \text{Ker}(\rho_{\bar{k}})$  hence,  $\lambda_{\bar{k},k}(x) = \bigoplus_{i \in I_{\bar{k},k}} \rho_{\bar{k}}(x) \otimes 1 = 0$  which implies that  $\pi_k(x) = u_{k,k}^* \lambda_{\bar{k},k}(x) u_{k,k} = 0$ . The last statement follows from the universal property of  $A_e$  since the ucp map  $E_k \circ \mathbb{E}_k : A_{v,k} \rightarrow B$  is GNS-faithful whenever  $E_k$  is GNS-faithful.  $\square$

In the next proposition, we study some associativity properties between the edge-reduced and the vertex-reduced amalgamated free product. The result is interesting in itself and it will be used to easily obtain ucp radial multipliers on the vertex-reduced amalgamated free product.

**Proposition 2.16.** *Let  $A_1, A_2, A_3$  be unital  $C^*$ -algebras with a common unital  $C^*$ -subalgebra  $B$  and conditional expectations  $E_k : A_k \rightarrow B$ . After identification of  $A_1$  with a  $C^*$ -subalgebra of both  $A_1 \overset{1}{*}_B A_2$  and  $A_1 \overset{1}{*}_B A_3$ , the canonical GNS-faithful ucp maps  $A_1 \overset{1}{*}_B A_2 \rightarrow A_1$  and  $A_1 \overset{1}{*}_B A_3 \rightarrow A_1$  become conditional expectations and, with respect to these GNS-faithful conditional expectations, we have canonical isomorphisms*

- $\left( A_1 \overset{1}{*}_B A_2 \right) \overset{e}{*}_{A_1} \left( A_1 \overset{1}{*}_B A_3 \right) \simeq A_1 \overset{1}{*}_B \left( A_2 \overset{e}{*}_B A_3 \right)$ .
- $\left( A_1 \overset{2}{*}_B A_2 \right) \overset{e}{*}_{A_2} \left( A_3 \overset{2}{*}_B A_2 \right) \simeq \left( A_1 \overset{e}{*}_B A_3 \right) \overset{2}{*}_B A_2$ .

*Proof.* We prove the first point. The proof of the second point is similar. We write  $\tilde{A} = A_1 \overset{1}{*}_B \left( A_2 \overset{e}{*}_B A_3 \right)$ . Let  $\rho : A_2 \overset{e}{*}_B A_3 \rightarrow A_2 \overset{e}{*}_B A_3$  and  $\tilde{\pi} : A_1 \overset{1}{*}_B \left( A_2 \overset{e}{*}_B A_3 \right) \rightarrow \tilde{A}$  be the canonical surjections and  $\tilde{\mathbb{E}} : \tilde{A} \rightarrow A_1$  the canonical GNS-faithful ucp map. Define, for  $k = 1, 2$ ,  $\nu_k : A_k \rightarrow \tilde{A}$  by  $\nu_1 = \tilde{\pi}|_{A_1}$  and  $\nu_2 = \tilde{\pi} \circ \rho|_{A_2}$ . By definition,  $\nu_1(b) = \nu_2(b)$  for all  $b \in B$  and  $\nu_1$  is faithful. Let  $C$  be the  $C^*$ -subalgebra of  $\tilde{A}$  generated by  $\nu_1(A_1) \cup \nu_2(A_2)$ . We claim that there exists a (unique) unital faithful  $*$ -homomorphism  $\nu : A_1 \overset{1}{*}_B A_2 \rightarrow \tilde{A}$  such that  $\nu \circ \pi_1|_{A_k} = \nu_k$  for  $k = 1, 2$ , where  $\pi_1 : A_1 \overset{1}{*}_B A_2 \rightarrow A_1 \overset{1}{*}_B A_2$  is the canonical surjection. By the universal property of the 1-vertex-reduced amalgamated free product, it suffices to show the following claim, where  $E = \tilde{\mathbb{E}}|_C : C \rightarrow A_1$ .

**Claim.** *The ucp map  $E$  is GNS-faithful and satisfies  $E \circ \nu_1 = id_{A_1}$  and, for all  $a = a_1 \dots a_n \in A_f$  reduced with  $a_k \in A_{i_k}^\circ$ ,  $E(\nu_{i_1}(a_1) \dots \nu_{i_n}(a_n)) = 0$  whenever  $n \geq 2$  or  $n = 1$  and  $a = a_1 \in A_2^\circ$ .*

*Proof of the Claim.* The fact the  $E$  vanishes on the reduced operators (not in  $A_1^\circ$ ) is obvious, since  $\tilde{\mathbb{E}}$  satisfies the same property. The only non-trivial property to check is the fact that  $E$  is GNS-faithful: indeed, it is not true, in general, that the restriction of a GNS-faithful ucp map to a subalgebra is again GNS-faithful. So suppose that there exists  $x \in C$  such that  $E(y^*x^*xy) = 0$  for all  $y \in C$  and let us show that  $x$  is equal to zero. Since  $\tilde{\mathbb{E}} : \tilde{A} \rightarrow A_1$  is GNS-faithful, it suffices to show that  $\tilde{\mathbb{E}}(y^*x^*xy) = 0$  for all  $y \in \tilde{A}$ . By hypothesis, we know that it is true for all  $y \in C$ . Since  $\tilde{A}$  is the closed linear span of  $\tilde{\pi}(A_1)$  and  $\tilde{\pi}(z)$ , for  $z \in A_1 \overset{1}{*}_B \left( A_2 \overset{e}{*}_B A_3 \right)$  a reduced operator not in  $A_1^\circ$  and since  $\tilde{\pi}(A_1) \cup \tilde{\pi} \circ \rho(A_2) \subset C$ , it suffices to show that  $\tilde{\mathbb{E}}(y^*x^*xy) = 0$  for  $y = \tilde{\pi}(z)$  and  $z = z_1 \dots z_n \in A_1 \overset{1}{*}_B \left( A_2 \overset{e}{*}_B A_3 \right)$  a reduced operator with letters  $z_k$  alternating from  $A_1^\circ$ ,  $\rho(A_2^\circ)$  and  $\rho(A_3^\circ)$  and containing at least one letter in  $\rho(A_3^\circ)$ . Since one of the  $z_k$  is in  $\rho(A_3^\circ)$  and  $x \in C$  we have, by the property of  $\tilde{\mathbb{E}}$ ,  $\tilde{\mathbb{E}}(y^*(x^*x - \tilde{\mathbb{E}}(x^*x))y) = 0$ . Hence,  $\tilde{\mathbb{E}}(y^*x^*xy) = \tilde{\mathbb{E}}(y^*\tilde{\mathbb{E}}(x^*x)y) = \tilde{\mathbb{E}}(y^*E(x^*x)y) = 0$ , since  $E(x^*x) = 0$ .

*End of the proof of the Proposition.* Define, for  $k = 1, 3$ , the unital  $*$ -homomorphism  $\eta_k : A_k \rightarrow \tilde{A}$  by  $\eta_1 = \tilde{\pi}|_{A_1} = \nu_1$  and  $\eta_3 = \tilde{\pi} \circ \rho|_{A_3}$ . Using the universal property of the 1-vertex-reduced amalgamated free product one can show, using exactly the same arguments we used to construct the homomorphism  $\nu$ , that there exists a (necessarily unique) unital faithful  $*$ -homomorphism

$\eta : A_1 \underset{B}{*} A_3 \rightarrow \tilde{A}$  such that  $\eta \circ \pi'_1|_{A_k} = \eta_k$  for  $k = 1, 3$ , where  $\pi'_1 : A_1 \underset{B}{*} A_3 \rightarrow A_1 \underset{B}{*} A_3$  is the canonical surjection. Note that  $\nu(a) = \eta(a)$  for all  $a \in A_1$  and  $\tilde{A}$  is generated, as a  $C^*$ -algebra, by  $\nu(A_1 \underset{B}{*} A_2) \cup \eta(A_1 \underset{B}{*} A_3)$ . Since the GNS-faithful ucp map  $\tilde{\mathbb{E}} : \tilde{A} \rightarrow A_1$  obviously satisfies the condition on the reduced operators we may use the universal property of the edge-reduced amalgamated free product to conclude that there exists a canonical  $*$ -isomorphism

$$\left( A_1 \underset{B}{*} A_2 \right) \underset{A_1}{*} \left( A_1 \underset{B}{*} A_3 \right) \rightarrow \tilde{A}.$$

□

Using the previous identifications one can prove the following result about completely positive radial multipliers. For  $\underline{i} = (i_1, \dots, i_n) \in I$  and  $l \in \{1, 2\}$  we define the number

$$n(\underline{i}, l) = |\{s \in \{1, \dots, n\} : i_s = l\}|.$$

**Proposition 2.17.** *For all  $k, l \in \{1, 2\}$  and all  $0 < r \leq 1$  there exists a unique ucp map  $\varphi_r : A_{v,k} \rightarrow A_{v,k}$  such that  $\varphi_r(\pi_k(b)) = \pi_k(b)$  for all  $b \in B$  and,*

$$\varphi_r(\pi_k(a_1 \dots a_n)) = r^{n(\underline{i}, l)} \pi_k(a_1 \dots a_n) \text{ for all } a_1 \dots a_n \in A_f \text{ reduced with } a_k \in A_{i_k}^\circ \text{ and } \underline{i} = (i_1, \dots, i_n).$$

*Proof.* We first prove the proposition for  $k = 1$ . We separate the proof in two cases.

**Case 1:**  $l = 2$ . Since  $\pi_1$  is faithful on  $A_1$ , we may and will view  $A_1 \subset A_{v,1}$ . After this identification, the canonical GNS-faithful ucp map  $\mathbb{E}_1 : A_{v,1} \rightarrow A_1$  becomes a conditional expectation. Consider the conditional expectation  $\tau \otimes \text{id} : C([0, 1]) \otimes B \rightarrow B$ , where  $\tau$  is the integral with respect to the normalized Lebesgue measure on  $[0, 1]$ . We will also view  $A_1 \subset A_1 \underset{B}{*} (C([0, 1]) \otimes B)$  so that the canonical GNS-faithful ucp map  $\tilde{\mathbb{E}}_1 : A_1 \underset{B}{*} (C([0, 1]) \otimes B) \rightarrow A_1$  is a conditional expectation. Define  $\tilde{A} = A_{v,1} \underset{A_1}{*} \left( A_1 \underset{B}{*} (C([0, 1]) \otimes B) \right)$  with respect to the conditional expectations  $\mathbb{E}_1$  and  $\tilde{\mathbb{E}}_1$ . Since  $\mathbb{E}_1$  and  $\tilde{\mathbb{E}}_1$  are GNS-faithful, the edge-reduced and the  $k$ -vertex-reduced amalgamated free products coincide for  $k = 1, 2$ . Hence, we may and will view  $A_{v,1} \subset \tilde{A}$  and we have a canonical GNS-faithful conditional expectation  $\tilde{\mathbb{E}} : \tilde{A} \rightarrow A_{v,1}$ . Also, by the first assertion of Proposition 2.16 we have a canonical identification  $\tilde{A} = A_1 \underset{B}{*} \tilde{A}_2$ , where  $\tilde{A}_2 = A_2 \underset{B}{*} (C([0, 1]) \otimes B)$ . Let  $\tilde{\rho}_2 : A_2 \underset{B}{*} C([0, 1]) \otimes B \rightarrow \tilde{A}_2$  be the canonical surjection from the full to the edge-reduced amalgamated free product and  $\tilde{\pi} : A_1 \underset{B}{*} \tilde{A}_2 \rightarrow A_1 \underset{B}{*} \tilde{A}_2 = \tilde{A}$  be the canonical surjection from the full to the vertex-reduced amalgamated free product. Fix  $t \in \mathbb{R}$  and define the unitary  $v_t \in C([0, 1])$  by  $v_t(x) = e^{2i\pi tx}$ . Let  $\rho_t = |\tau(v_t)|^2$  and  $u_t = \tilde{\pi} \circ \tilde{\rho}_2(v_t \otimes 1_B) \in \tilde{A}$ . Define the unital  $*$ -homomorphisms  $\nu_1 = \tilde{\pi}|_{A_1} : A_1 \rightarrow \tilde{A}$  and  $\nu_2 : \tilde{A}_2 \rightarrow \tilde{A}$  by  $\nu_2(x) = u_t \tilde{\pi}(x) u_t^*$ . Note that  $\nu_1$  is faithful. To simplify the notations we put  $\tilde{A}_1 := A_1$ .

**Claim.** *For all  $x = x_1 \dots x_n \in A_1 \underset{B}{*} \tilde{A}_2$  reduced with  $x_k \in \tilde{A}_{i_k}^\circ$  and  $\underline{i} = (i_1, \dots, i_n) \in I$  one has:*

$$\tilde{\mathbb{E}}(\nu_{i_1}(x_1) \dots \nu_{i_n}(x_n)) = \begin{cases} \rho_t^{n(\underline{i}, l)} \tilde{\pi}(x_1 \dots x_n) & \text{if } \tilde{\pi}(x) \in A_{v,1}, \\ 0 & \text{if } \tilde{\mathbb{E}}(\tilde{\pi}(x)) = 0. \end{cases}$$

*Proof of the Claim.* To prove the formula of the claim, we may and will assume that each letter  $x_k$  of  $x$  coming from  $\tilde{A}_2^\circ$  is a reduced word, in the edge reduced amalgamated free product  $\tilde{A}_2 = A_2 \underset{B}{*} (C([0, 1]) \otimes B)$  with letters alternating from  $\tilde{\rho}_2(A_2^\circ)$  and  $\tilde{\rho}_2((B \otimes C([0, 1]))^\circ)$ . Now for such  $x$ , by the property of the conditional expectation  $\tilde{\mathbb{E}}$  and the canonical identification  $A_1 \underset{B}{*} \tilde{A}_2 = A_{v,1} \underset{A_1}{*} \left( A_1 \underset{B}{*} (C([0, 1]) \otimes B) \right)$ , we have  $\tilde{\pi}(x) \in A_{v,1}$  if and only if all the letters  $x_k$  of  $x$  for which  $x_k \in \tilde{A}_2^\circ$  we actually have  $x_k \in \tilde{\rho}_2(A_2^\circ)$ . Note that we also have  $\tilde{\mathbb{E}}(\tilde{\pi}(x)) = 0$  if and only if there is at least one letter  $x_k$  of  $x$  coming from  $\tilde{A}_2^\circ$  which, itself viewed as a reduced word in  $\tilde{A}_2 = A_2 \underset{B}{*} (C([0, 1]) \otimes B)$ , contains a letter which comes from  $\tilde{\rho}_2((C([0, 1]) \otimes B)^\circ)$ . We prove the formula by induction on  $n$ . If  $n = 1$  we have either  $x \in A_1^\circ$  in that case  $\tilde{\mathbb{E}}(\nu_1(x)) = \tilde{\mathbb{E}}(\tilde{\pi}(x)) = \tilde{\pi}(x)$  or  $x \in \tilde{\rho}_2(\tilde{A}_2^\circ)$  and

$$\begin{aligned} \tilde{\mathbb{E}}(\nu_2(x)) &= \tilde{\mathbb{E}}(u_t \tilde{\pi}(x) u_t^*) \\ &= \tilde{\mathbb{E}}((u_t - \tau(v_t)) \tilde{\pi}(x) (u_t^* - \overline{\tau(v_t)})) + \tau(v_t) \tilde{\mathbb{E}}(\tilde{\pi}(x) (u_t^* - \overline{\tau(v_t)})) \\ &\quad + \overline{\tau(v_t)} \tilde{\mathbb{E}}((u_t - \tau(v_t)) \tilde{\pi}(x)) + |\tau(v_t)|^2 \tilde{\mathbb{E}}(\tilde{\pi}(x)) \\ &= |\tau(v_t)|^2 \tilde{\mathbb{E}}(\tilde{\pi}(x)) = \rho_t \tilde{\mathbb{E}}(\tilde{\pi}(x)). \end{aligned}$$

$$\text{Hence, } \tilde{\mathbb{E}}(\nu_2(x)) = \begin{cases} \rho_t \tilde{\pi}(x) & \text{if } \tilde{\pi}(x) \in A_{v,1}, \\ 0 & \text{if } \tilde{\mathbb{E}}(\tilde{\pi}(x)) = 0. \end{cases}$$

This proves the formula for  $n = 1$ . Suppose that the formula holds for a given  $n \geq 1$ . Let  $x = x_1 \dots x_{n+1}$  be reduced with  $x_k \in \tilde{A}_{i_k}^\circ$  and define  $x' = x_1 \dots x_n$  and  $z = \nu_{i_1}(x_1) \dots \nu_{i_n}(x_n)$ . Let  $\underline{i} = (i_1, \dots, i_{n+1})$  and  $\underline{i}' = (i_1, \dots, i_n)$ .

Suppose that  $x_{n+1} \in A_1^\circ$ . Then  $n(\underline{i}, 2) = n(\underline{i}', 2)$  and,

$$\tilde{\mathbb{E}}(\nu_{i_1}(x_1) \dots \nu_{i_n}(x_n) \nu_{i_{n+1}}(x_{n+1})) = \tilde{\mathbb{E}}(\nu_{i_1}(x_1) \dots \nu_{i_n}(x_n) \tilde{\pi}(x_{n+1})) = \tilde{\mathbb{E}}(z) \tilde{\pi}(x_{n+1}).$$

Hence, if  $\tilde{\pi}(x) \in A_{v,1}$  then also  $\tilde{\pi}(x') \in A_{v,1}$  and we have, by the induction hypothesis,

$$\tilde{\mathbb{E}}(\nu_{i_1}(x_1) \dots \nu_{i_n}(x_n) \nu_{i_{n+1}}(x_{n+1})) = \rho_t^{n(\underline{i}', 2)} \tilde{\pi}(x') \tilde{\pi}(x_{n+1}) = \rho_t^{n(\underline{i}, 2)} \tilde{\pi}(x).$$

If  $\tilde{\mathbb{E}}(\tilde{\pi}(x)) = 0$  then also  $\tilde{\mathbb{E}}(\tilde{\pi}(x')) = 0$  and we have, by the induction hypothesis,  $\tilde{\mathbb{E}}(z) = 0$  so  $\tilde{\mathbb{E}}(\nu_{i_1}(x_1) \dots \nu_{i_n}(x_n) \nu_{i_{n+1}}(x_{n+1})) = 0$ .

Suppose now that  $x_{n+1} \in \tilde{A}_2^\circ$  then  $x_n \in A_1^\circ$  and we have,

$$\begin{aligned} \tilde{\mathbb{E}}(z \nu_{i_{n+1}}(x_{n+1})) &= \tilde{\mathbb{E}}(z u_t \tilde{\pi}(x_{n+1}) u_t^*) \\ &= \tilde{\mathbb{E}}(z (u_t - \tau(v_t)) \tilde{\pi}(x_{n+1}) (u_t^* - \overline{\tau(v_t)})) + \tau(v_t) \tilde{\mathbb{E}}(z \tilde{\pi}(x_{n+1}) (u_t^* - \overline{\tau(v_t)})) \\ &\quad + \overline{\tau(v_t)} \tilde{\mathbb{E}}(z (u_t - \tau(v_t)) \tilde{\pi}(x_{n+1})) + |\tau(v_t)|^2 \tilde{\mathbb{E}}(z \tilde{\pi}(x_{n+1})) \\ &= |\tau(v_t)|^2 \tilde{\mathbb{E}}(z \tilde{\pi}(x_{n+1})) = \rho_t \tilde{\mathbb{E}}(z \tilde{\pi}(x_{n+1})). \end{aligned}$$

Hence, if  $\tilde{\pi}(x) \in A_{v,1}$  then also  $\tilde{\pi}(x') \in A_{v,1}$  and  $x_{n+1} \in \tilde{A}_2^\circ$  so  $\tilde{\pi}(x_{n+1}) \in A_{v,1}$  and  $n(\underline{i}, 2) = n(\underline{i}', 2) + 1$ . By the preceding computation and the induction hypothesis we find:

$$\tilde{\mathbb{E}}(z \nu_{i_{n+1}}(x_{n+1})) = \rho_t \tilde{\mathbb{E}}(z \tilde{\pi}(x_{n+1})) = \rho_t \tilde{\mathbb{E}}(z) \tilde{\pi}(x_{n+1}) = \rho_t \rho_t^{n(\underline{i}', 2)} \tilde{\pi}(x') \tilde{\pi}(x_{n+1}) = \rho_t^{n(\underline{i}, 2)} \tilde{\pi}(x).$$

Finally, if  $\tilde{\mathbb{E}}(\tilde{\pi}(x)) = 0$ , we need to prove that  $\tilde{\mathbb{E}}(z\tilde{\pi}(x_{n+1})) = 0$ . Note that, since  $x_n \in A_1^\circ$ , we have  $z = \nu_{i_1}(x_1) \dots \nu_{i_{n-1}}(x_{n-1})x_n$ . Hence, if  $\tilde{\mathbb{E}}(\tilde{\pi}(x')) = 0$  so by the induction hypothesis we have  $\tilde{\mathbb{E}}(z) = 0$ ,  $z$  may be written as a sum of reduced operators, containing at least one letter from  $\tilde{\rho}_2((C([0, 1]) \otimes B)^\circ)$  and ending with a letter from  $A_1^\circ$ . It follows that  $z\tilde{\pi}(x_{n+1})$  may be written as a sum of reduced operators, containing at least one letter from  $\tilde{\rho}_2((C([0, 1]) \otimes B)^\circ)$ . Hence,  $\tilde{\mathbb{E}}(z\tilde{\pi}(x_{n+1})) = 0$ . If  $\tilde{\mathbb{E}}(\tilde{\pi}(x)) = 0$  and  $\tilde{\mathbb{E}}(\tilde{\pi}(x')) \in A_{v,1}$  then,  $x_1, \dots, x_n \in A_1^\circ \cup A_2^\circ$  but  $\tilde{\mathbb{E}}(\tilde{\pi}(x_{n+1})) = 0$ . It follows that  $z = \nu_{i_1}(x_1) \dots \nu_{i_{n-1}}(x_{n-1})x_n$  may be written as a sum of reduced operators ending with a letter from  $A_1^\circ$ . Hence,  $z\tilde{\pi}(x_{n+1})$  be written as a sum of reduced operators containing at least one letter from  $\tilde{\rho}_2((C([0, 1]) \otimes B)^\circ)$ . Hence,  $\tilde{\mathbb{E}}(z\tilde{\pi}(x_{n+1})) = 0$ .

*End of the proof of the Proposition.* By the Claim,  $\mathbb{E}_1 \circ \tilde{\mathbb{E}}(\nu_{i_1}(x_1) \dots \nu_{i_n}(x_n)) = 0$  for all reduced operators  $x = x_1 \dots x_n \in A_1 *_B \tilde{A}_2$  which are not in  $A_1$  and, we obviously have,  $\mathbb{E}_1 \circ \tilde{\mathbb{E}} \circ \nu_1 = \text{id}_{A_1}$ .

Viewing  $\tilde{A} = A_1 *_B \tilde{A}_2$  and using the universal property of the vertex-reduced amalgamated free product, there exists, for all  $t \in \mathbb{R}$ , a unique unital  $*$ -isomorphism  $\alpha_t : \tilde{A} \rightarrow \tilde{A}$  such that  $\alpha_t(\tilde{\pi}(a)) = \tilde{\pi}(a)$  if  $a \in A_1$  and  $\alpha_t(\tilde{\pi}(x)) = u_t \tilde{\pi}(x) u_t^*$  if  $x \in A_2 *_B (C([0, 1]) \otimes B)$ . In particular, it follows from the Claim that  $\tilde{\mathbb{E}} \circ \alpha_t|_{A_{v,1}} : A_{v,1} \rightarrow A_{v,1}$ , which is a ucp map, satisfies the properties of the map  $\varphi_r$  described in the statement of the proposition, with  $r = \rho_t = \left| \frac{\sin(\pi t)}{\pi t} \right|^2$ . This concludes the proof.

**Case 2:**  $l = 1$ . The proof is similar. This time, the automorphism  $\alpha_t : \tilde{A} \rightarrow \tilde{A}$  is defined, by the universal property, starting with the maps  $\nu_1 : A_1 \rightarrow \tilde{A}$  and  $\nu_2 : \tilde{A}_2 \rightarrow \tilde{A}$  defined by  $\nu_1(a) = u_t \tilde{\pi}(a) u_t^*$  and  $\nu_2(x) = \tilde{\pi}(x)$ . The remainder of the proof is the same.

The proof for  $k = 2$  is the same, using the second assertion of Proposition 2.16.  $\square$

### 3. $K$ -EQUIVALENCE BETWEEN THE FULL AND REDUCED AMALGAMATED FREE PRODUCTS

Let  $A_1, A_2$  be two unital  $C^*$ -algebras with a common  $C^*$ -subalgebra  $B \subset A_k$ ,  $k = 1, 2$  and denote by  $A_f$  the full amalgamated free product.

Let  $A := A_1 *_B^v A_2$  be the vertex-reduced amalgamated free product. For  $k = 1, 2$ , let  $E_{A_k}$  (resp.  $E_B$ ) be the canonical conditional expectation from  $A$  to  $A_k$  (resp. from  $A$  to  $B$ ). We will denote by the same symbol  $\mathcal{A}$  the set of reduced operators viewed in  $A$  or in  $A_f$ . Recall that the linear span of  $\mathcal{A}$  and  $B$  is a dense unital  $*$ -subalgebra of  $A$  (resp.  $A_f$ ).

We denote by  $\lambda : A_f \rightarrow A$  the canonical surjective unital  $*$ -homomorphism which is the identity on  $\mathcal{A}$ . In this section we prove the following result.

**Theorem 3.1.**  $[\lambda] \in \text{KK}(A_f, A)$  is invertible.

The following lemma is well known (see [Ve04, Lemma 3.1]). We include a proof for the convenience of the reader.

**Lemma 3.2.** Let  $n \geq 1$ ,  $a_k \in A_{l_k}^\circ$  for  $1 \leq k \leq n$ , and  $a = a_1 \dots a_n \in A$  a reduced word. For  $i = 1$  or  $2$ , one has

$$E_{A_i}(a^* a) = E_B(a^* a) \quad \text{whenever } l_n \neq i.$$

*Proof.* We prove it for  $i = 1$  by induction on  $n$ . The proof for  $i = 2$  is the same.

It's obvious for  $n = 1$ . Suppose that  $n \geq 2$ , define  $b = E_B(a_1^* a_1)^{\frac{1}{2}}$ ,  $x = (ba_2) \dots a_n$ . One has:

$$E_{A_1}(a^* a) = E_{A_1}(a_n^* \dots a_1^* a_1 \dots a_n) = E_{A_1}(a_n^* \dots a_2^* E_B(a_1^* a_1) a_2 \dots a_n) = E_{A_1}(x^* x) = E_B(x^* x),$$

where we applied the induction hypothesis to get the last equality. Since the same computation gives  $E_B(a^* a) = E_B(x^* x)$ , this concludes the proof.  $\square$

We denote by  $(H_k, \pi_k, \xi_k)$  (resp.  $(K, \rho, \eta)$ ) the GNS construction of  $E_{A_k}$  (resp.  $E_B$ ). We may and will assume that  $A \subset \mathcal{L}_{A_k}(H_k)$  and  $\pi_k = \text{id}$ .

Observe that the Hilbert  $A_k$ -module  $\xi_k \cdot A_k \subset H_k$  is orthogonally complemented i.e.  $H_k = \xi_k \cdot A_k \oplus H_k^\circ$ , as Hilbert  $A_k$ -modules, where  $H_k^\circ$  is the closure of  $\{a \xi_k : a \in A, E_{A_k}(a) = 0\}$ .

We now define a partial isometry  $F_k \in \mathcal{L}_{A_k}(H_k, K \otimes_B A_k)$  in the following way. First we put  $F_k(\xi_k \cdot a) = 0$  for all  $a \in A_k$ . Then, it follows from Lemma 3.2 that we can define an isometry  $F_k : H_k^\circ \rightarrow K \otimes_B A_k$  by the following formula:

$$F_k(a_1 \dots a_n \xi_k) = \begin{cases} \rho(a_1 \dots a_n) \eta \otimes_B 1 & \text{if } l_n \neq k \\ \rho(a_1 \dots a_{n-1}) \eta \otimes_B a_n & \text{if } l_n = k \end{cases} \quad \text{for all } a_1 \dots a_n \in A \text{ a reduced operator.}$$

Let  $q_k \in \mathcal{L}_B(K)$  be the orthogonal projection onto words which do not end with  $k$  i.e. onto the complemented  $B$  submodule  $\bigoplus_{i=(i_1, \dots, i_n) \in I, i_n \neq k} K_i^\circ$  and note that  $F_k$  defines a bounded linear map from  $H_k$  to  $K \otimes_B A_k$  with image the complemented sub  $A_k$ -module  $(q_k \otimes 1) K \otimes_B A_k$ . Hence,  $F_k \in \mathcal{L}_{A_k}(H_k, K \otimes_B A_k)$  is a well defined partial isometry such that  $1 - F_k^* F_k$  is the orthogonal projection onto  $\xi_k \cdot A_k$ , and  $F_k F_k^* = q_k \otimes_B 1$ . Note also that the image of  $1 - F_k F_k^*$  is

$$((1 - q_k) \otimes 1) K \otimes_B A_k = (\eta \otimes 1) \cdot A_k \oplus \overline{\text{Span}\{\rho(a_1 \dots a_n) \eta \otimes 1 : a = a_1 \dots a_n \in A \text{ reduced with } l_n = k\}} \cdot A_k.$$

We will denote in the sequel  $q_0$  the orthogonal projection of  $K$  onto  $\eta \cdot B$ . It is clear that  $1 = q_1 + q_2 + q_0$  and that these projections are pairwise orthogonal. Define also  $\bar{F}_k = F_k + \theta_{\eta \otimes_B 1, \xi_k}$ . It is again clear that  $\bar{F}_k$  is an isometry and  $\bar{F}_k \bar{F}_k^* = q_k + q_0 = 1 - q_l$  for  $k \neq l$ .

**Lemma 3.3.** *For  $k = 1, 2$  the following facts hold.*

- (1)  $\rho(a) F_k = F_k a \in \mathcal{L}_{A_k}(H_k, K \otimes_B A_k)$  for all  $a \in A_k$ .
- (2)  $\rho(a) \bar{F}_k = \bar{F}_k a \forall a \in A_l$  with  $l \neq k$ .
- (3) For all  $x \in A$ ,  $\rho(x) \bar{F}_k - \bar{F}_k x \in \mathcal{K}_{A_k}(H_k, K \otimes_B A_k)$  and  $\rho(x) F_k - F_k x \in \mathcal{K}_{A_k}(H_k, K \otimes_B A_k)$ .

*Proof.* We prove the lemma for  $k = 1$ . The proof for  $k = 2$  is the same.

(1). When  $a \in B$  the commutation is obvious hence we may and will assume that  $a \in A_1^\circ$ . One has  $F_1 a \xi_1 = 0 = \rho(a) F_1 \xi_1$ . Let now  $n \geq 1$  and  $x = a_1 \dots a_n \in A$ ,  $a_k \in A_{l_k}^\circ$ , be a reduced operator with  $E_{A_1}(x) = 0$ . It suffices to show that  $F_1 a x \xi_1 = \rho(a) F_1 x \xi_1$ . If  $n = 1$  we must have  $x \in A_2^\circ$  and  $F_1 a x \xi_1 = \rho(ax) \eta \otimes 1 = \rho(a) F_1 x \xi_1$ . Suppose that  $n \geq 2$ . If  $l_1 = 2$  then  $ax$  is reduced and ends with a letter from  $A_{l_n}^\circ$ . It follows that  $F_1 a x \xi_2 = \rho(a) F_1 x \xi_2$ . If  $l_1 = 1$  then we can write  $ax = (aa_1)^\circ a_2 \dots a_n + E_B(aa_1) a_2 \dots a_n$ . Since  $a_2 \dots a_n$  is reduced and ends with  $l_n$  we find again that  $F_1 a x \xi_1 = \rho(a) F_1 x \xi_1$ .

(2). Let  $a \in A_2^\circ$ . Clearly  $\overline{F}_1 a \xi_1 = F_1 a \xi_1 = \rho(a) \eta \otimes 1$  and  $\rho(a) \overline{F}_1 \xi_1 = \rho(a) \eta \otimes 1$ . Let now  $n \geq 1$  and  $x = a_1 \dots a_n \in A$ ,  $a_k \in A_{i_k}^\circ$ , be a reduced operator with  $E_{A_1}(x) = 0$ . If  $n = 1$  we must have  $x \in A_2^\circ$ . It follows that  $\overline{F}_1 a x \xi_1 = F_1 (a x)^\circ \xi_1 + \theta_{\eta \otimes 1, \xi_1} E_B(a x) \xi_1 = \rho((a x)^\circ) \eta \otimes 1 + E_B(a x) \eta \otimes 1$  and  $\rho(a) \overline{F}_1 x \xi_1 = \rho(a) F_1 x \xi_1 = \rho(a x) \eta \otimes 1$ . If  $n \geq 2$ , arguing as in the proof of (1), we see that  $\overline{F}_1 a x \xi_1 = F_1 a x \xi_1 = \rho(a) F_1 x \xi_1 = \rho(a) \overline{F}_1 x \xi_1$ .

(3). We only prove the first statement of (3), the proof of the second statement is the same. By statement (2), it suffices to prove that  $\rho(x) \overline{F}_1 - \overline{F}_1 x \in \mathcal{K}_{A_1}(H_k, K \otimes_B A_k)$  for  $x \in A_1$ . Note that  $\overline{F}_1$  is a compact perturbation of  $F_1$  and denote by  $\theta$  the compact operator  $\theta := \overline{F}_1 - F_1$  then, using statement (1) we get, for  $x \in A_1$ ,  $\rho(x) \overline{F}_1 - \overline{F}_1 x = \rho(x)(F_1 + \theta) - (F_1 + \theta)x = \rho(x)\theta - \theta x$ , which is compact since  $\theta$  is.  $\square$

We define the following Hilbert  $A_f$ -modules:

$$H_m = H_1 \otimes_{A_1} A_f \oplus H_2 \otimes_{A_2} A_f \quad \text{and} \quad K_m = K \otimes_B A_f = \left( K \otimes_B A_k \right) \otimes_{A_k} A_f,$$

with the canonical representations  $\pi : A \rightarrow \mathcal{L}_{A_f}(H_m)$ ,  $\pi(x) = x \otimes_{A_1} 1_{A_f} \oplus x \otimes_{A_2} 1_{A_f}$  and  $\bar{\rho} : A \rightarrow \mathcal{L}_{A_f}(K_m)$ ,  $\bar{\rho}(x) = \rho(x) \otimes_B 1_{A_f}$ . We consider, for  $k = 1, 2$ , the partial isometry

$$F_k \otimes_{A_k} 1_{A_f} \in \mathcal{L}_{A_f}(H_k \otimes_{A_k} A_f, (K \otimes_B A_k) \otimes_{A_k} A_f).$$

Observe that  $F_1 \otimes_{A_1} 1_{A_f}$  and  $F_2 \otimes_{A_2} 1_{A_f}$  have orthogonal images. Indeed, the image of  $F_k \otimes_{A_k} 1_{A_f}$  is the closed linear span of  $\{\rho(a_1 \dots a_n) \eta \otimes_B y : y \in A_f \text{ and } a_1 \dots a_n \in A \text{ reduced with } a_n \notin A_k^\circ\}$ . Hence the operator  $F \in \mathcal{L}_{A_f}(H_m, K_m)$  defined by  $F = F_1 \otimes_{A_1} 1_{A_f} \oplus F_2 \otimes_{A_2} 1_{A_f}$  is a partial isometry such that  $1 - FF^*$  is the orthogonal projection onto  $(\eta \otimes_B 1_{A_f}) \cdot A_f$  and  $1 - F^*F$  is the orthogonal projection onto  $(\xi_1 \otimes_{A_1} 1_{A_f}) \cdot A_f \oplus (\xi_2 \otimes_{A_2} 1_{A_f}) \cdot A_f$ . In particular  $1 - F^*F$  and  $1 - FF^*$  belongs to  $\mathcal{K}_{A_f}(H_m)$  and  $\mathcal{K}_{A_f}(K_m)$  respectively. Moreover, it follows from lemma 3.3 that  $F\pi(x) - \bar{\rho}(x)F \in \mathcal{K}_{A_f}(H_m, K_m)$  for all  $x \in A$ . Hence, we get an element  $\alpha = [(H_m \oplus K_m, \pi \oplus \bar{\rho}, F)] \in \text{KK}(A, A_f)$ .

To prove Theorem 3.1 it suffices to prove that  $\alpha \otimes_{A_f} [\lambda] = [\text{id}_A]$  in  $\text{KK}(A, A)$  and  $[\lambda] \otimes_A \alpha = [\text{id}_{A_f}]$  in  $\text{KK}(A_f, A_f)$ . We prove the easy part in the next proposition.

**Proposition 3.4.** *One has  $[\lambda] \otimes_A \alpha = [\text{id}_{A_f}]$  in  $\text{KK}(A_f, A_f)$ .*

*Proof.* Observe that  $[\lambda] \otimes_A \alpha = [(H_m \oplus K_m, \pi_m \oplus \rho_m, F)]$  where  $\pi_m = \pi \circ \lambda : A_f \rightarrow \mathcal{L}_{A_f}(H_m)$  and  $\rho_m = \bar{\rho} \circ \lambda : A_f \rightarrow \mathcal{L}_{A_f}(K_m)$ . Hence, by compact perturbation,  $[\lambda] \otimes_A \alpha - [\text{id}_{A_f}]$  is represented by the Kasparov triple  $(H_m \oplus \tilde{K}_m, \pi_m \oplus \tilde{\rho}_m, \tilde{F})$ , where  $\tilde{K}_m = K_m \oplus A_f$  and  $\tilde{\rho}_m(x) = \rho_m(x) \oplus x$ , where we view  $A_f = \mathcal{L}_{A_f}(A_f)$  by left multiplication. Finally,  $\tilde{F} \in \mathcal{L}_{A_f}(H_m, \tilde{K}_m)$  is the unitary defined by

$$\begin{aligned} \tilde{F}(\xi_1 \otimes_{A_1} 1_{A_f}) &= \eta \otimes_B 1_{A_f}, & \tilde{F}(\xi_2 \otimes_{A_2} 1_{A_f}) &= 1_{A_f} \quad \text{and,} \\ \tilde{F}(\xi) &= F(\xi) \text{ for all } \xi \in H_m \ominus \left( (\xi_1 \otimes_{A_1} 1_{A_f}) \cdot A_f \oplus (\xi_2 \otimes_{A_2} 1_{A_f}) \cdot A_f \right). \end{aligned}$$



We collect some computations in the following claim.

**Claim.** Let  $v \in \mathcal{L}_{A_f}(H_m)$  be the self-adjoint unitary defined by the identity on  $H_m \ominus ((\xi_1 \otimes_{A_1} 1_{A_f}) \cdot A_f \oplus (\xi_2 \otimes_{A_2} 1_{A_f}) \cdot A_f)$  and  $v(\xi_1 \otimes_{A_1} 1_{A_f}) = \xi_2 \otimes_{A_2} 1_{A_f}$ ,  $v(\xi_2 \otimes_{A_2} 1_{A_f}) = \xi_1 \otimes_{A_1} 1_{A_f}$ . One has:

- (1)  $\tilde{F}^* \tilde{\rho}_m(b) \tilde{F} = \pi_m(b)$  and  $v^* \pi_m(b) v = \pi_m(b)$  for all  $b \in B$ .
- (2)  $\tilde{F}^* \tilde{\rho}_m(a) \tilde{F} = v^* \pi_m(a) v$  for all  $a \in A_1$ .
- (3)  $\tilde{F}^* \tilde{\rho}_m(a) \tilde{F} = \pi_m(a)$  for all  $a \in A_2$ .

*Proof of the claim.* The proof of (1) is obvious and we leave it to the reader.

(2). By (1), it suffices to prove (2) for  $a \in A_1^\circ$ . Let  $a \in A_1^\circ$ . On the one hand:

$$\tilde{F}^* \tilde{\rho}_m(a) \tilde{F} \xi_1 \otimes_{A_1} 1_{A_f} = \tilde{F}^* (\rho(a) \eta \otimes_B 1_{A_f}) = a \xi_2 \otimes_{A_2} 1_{A_f} \quad \text{and} \quad \tilde{F}^* \tilde{\rho}_m(a) \tilde{F} \xi_2 \otimes_{A_2} 1_{A_f} = \tilde{F}^*(a) = \xi_2 \otimes_{A_2} a.$$

On the other hand:

$$v^* \pi_m(a) v \xi_1 \otimes_{A_1} 1_{A_f} = v^* (a \xi_2 \otimes_{A_2} 1_{A_f}) = a \xi_2 \otimes_{A_2} 1_{A_f} \quad \text{and} \quad v^* \pi_m(a) v \xi_2 \otimes_{A_2} 1_{A_f} = v^* (a \xi_1 \otimes_{A_1} 1_{A_f}) = \xi_2 \otimes_{A_2} a.$$

Let now  $x = a_1 \dots a_n \in A$  be a reduced operator with  $a_k \in A_{l_k}^\circ$ . We prove by induction on  $n$  that  $\tilde{F}^* \tilde{\rho}_m(a) \tilde{F} x \xi_k \otimes_{A_k} 1_{A_f} = v^* \pi_m(a) v x \xi_k \otimes_{A_k} 1_{A_f}$  for all  $k \in \{1, 2\}$ . Suppose that  $n = 1$  so  $x \in A_1^\circ \cup A_2^\circ$  and let  $k \in \{1, 2\}$  such that  $x \notin A_k^\circ$  (the case  $x \in A_k^\circ$  has been done before). We have:

$$\tilde{F}^* \tilde{\rho}_m(a) \tilde{F} x \xi_k \otimes_{A_k} 1_{A_f} = \tilde{F}^* (\rho(ax) \eta \otimes_B 1_{A_f}) = \begin{cases} (ax)^\circ \xi_2 \otimes_{A_2} 1_{A_f} + \xi_1 \otimes_{A_1} E_B(ax) & \text{if } x \in A_1^\circ, \\ ax \xi_1 \otimes_{A_1} 1_{A_f} & \text{if } x \in A_2^\circ. \end{cases}$$

On the other hand we have:

$$v^* \pi_m(a) v x \xi_k \otimes_{A_k} 1_{A_f} = v^* (ax \xi_k \otimes_{A_k} 1_{A_f}) = \begin{cases} (ax)^\circ \xi_2 \otimes_{A_2} 1_{A_f} + \xi_1 \otimes_{A_1} E_B(ax) & \text{if } x \in A_1^\circ (k = 2), \\ ax \xi_1 \otimes_{A_1} 1_{A_f} & \text{if } x \in A_2^\circ (k = 1). \end{cases}$$

Finally, suppose that  $n \geq 2$  and the formula holds for  $n - 1$ . Write  $ax = y + z$ , where, if  $l_1 = 1$ ,  $y = (aa_1)^\circ a_2 \dots a_n$  and  $z = E_B(aa_1) a_2 \dots a_n$  and, if  $l_1 = 2$ ,  $y = ax$  and  $z = 0$ . Observe that, in both cases,  $y$  is a reduced operator ending with a letter from  $A_{l_n}^\circ$  and  $z$  is either 0 or a reduced operator ending with a letter from  $A_{l_n}^\circ$ . By the induction hypothesis, we may and will assume that  $k \neq l_n$ . We have:

$$\begin{aligned} \tilde{F}^* \tilde{\rho}_m(a) \tilde{F} x \xi_k \otimes_{A_k} 1_{A_f} &= \tilde{F}^* (\rho(ax) \eta \otimes_B 1_{A_f}) = \tilde{F}^* (\rho(y) \eta \otimes_B 1_{A_f}) + \tilde{F}^* (\rho(z) \eta \otimes_B 1_{A_f}) \\ &= y \xi_k \otimes_{A_k} 1_{A_f} + z \xi_k \otimes_{A_k} 1_{A_f} = ax \xi_k \otimes_{A_k} 1_{A_f}. \end{aligned}$$

Moreover,

$$\begin{aligned} v^* \pi_m(a) v x \xi_k \otimes_{A_k} 1_{A_f} &= v^* (ax \xi_k \otimes_{A_k} 1_{A_f}) = v^* (y \xi_k \otimes_{A_k} 1_{A_f}) + v^* (z \xi_k \otimes_{A_k} 1_{A_f}) \\ &= y \xi_k \otimes_{A_k} 1_{A_f} + z \xi_k \otimes_{A_k} 1_{A_f} = ax \xi_k \otimes_{A_k} 1_{A_f}. \end{aligned}$$

The proof of (3) is similar. □

*End of the proof of Proposition 3.4.* Let  $t \in \mathbb{R}$  and define  $v_t = \cos(t) + iv \sin(t) \in \mathcal{L}_{A_f}(H_m)$ . Since  $v = v^*$  is unitary,  $v_t$  is a unitary for all  $t \in \mathbb{R}$ . Moreover, assertion (1) of the Claim implies that  $v_t \pi_m(b) v_t^* = \pi_m(b)$  for all  $b \in B$ . It follows from the universal property of  $A_f$  that there exists a unique unital  $*$ -homomorphism  $\pi_t : A_f \rightarrow \mathcal{L}_{A_f}(H_m)$  such that:

$$\pi_t(a) = \begin{cases} v_t^* \pi_m(a) v_t & \text{if } a \in A_1, \\ \pi_m(a) & \text{if } a \in A_2. \end{cases}$$

Then the triple  $\alpha_t = (H_m \oplus \tilde{K}_m, \pi_t \oplus \tilde{\rho}_m, \tilde{F})$  gives a homotopy between  $\alpha_0$  which represents  $[\lambda] \otimes \alpha - [\text{id}_{A_f}]$  and  $\alpha_{\frac{\pi}{2}}$  which is degenerated by the claim.  $\square$

We finish the proof of Theorem 3.1 in the next proposition.

**Proposition 3.5.** *One has  $\alpha \otimes [\lambda] = [\text{id}_A]$  in  $\text{KK}(A, A)$ .*

*Proof.* Observe that  $\alpha \otimes [\lambda] = [(H_r \oplus K_r, \pi_r \oplus \rho_r, F_r)]$  where

$$H_r = H_m \otimes_{\lambda} A = H_1 \otimes_{A_1} A \oplus H_2 \otimes_{A_2} A \quad \text{and} \quad K_r = K_m \otimes_{\lambda} A = K \otimes_B A = \left( K \otimes_B A_k \right) \otimes_{A_k} A,$$

with the canonical representations  $\pi_r : A \rightarrow \mathcal{L}_A(H_r)$ ,  $\pi_r(x) = \pi(x) \otimes 1 = x \otimes 1_{A_1} \oplus x \otimes 1_{A_2}$  and  $\rho_r : A \rightarrow \mathcal{L}_A(K_r)$ ,  $\rho_r(x) = \bar{\rho}(x) \otimes 1 = \rho(x) \otimes 1_A$  and with the operator  $F_r = F \otimes 1 \in \mathcal{L}_A(H_r, K_r)$ .

Hence,  $\alpha \otimes [\lambda] - [\text{id}_A]$  is represented by the Kasparov triple  $(H_r \oplus \tilde{K}_r, \pi_r \oplus \tilde{\rho}_r, \tilde{F}_r)$ , where  $\tilde{K}_r = K_r \oplus A$  and  $\tilde{\rho}_r(x) = \rho_r(x) \oplus x$ , where we view  $A = \mathcal{L}_A(A)$  by left multiplication. Finally,  $\tilde{F}_r \in \mathcal{L}_A(H_r, \tilde{K}_r)$  is the unitary defined by

$$\tilde{F}_r(\xi_1 \otimes 1_A) = \eta \otimes 1_A, \quad \tilde{F}_r(\xi_2 \otimes 1_A) = 1_A \quad \text{and,}$$

$$\tilde{F}(\xi) = F(\xi) \text{ for all } \xi \in H_r \ominus \left( (\xi_1 \otimes 1_A).A \oplus (\xi_2 \otimes 1_A).A \right).$$

The claim in the proof of Proposition 3.4 implies the following claim.

**Claim.** *Let  $u \in \mathcal{L}_A(H_r)$  be the self-adjoint unitary defined by the identity on  $H_r \ominus ((\xi_1 \otimes 1_A).A \oplus (\xi_2 \otimes 1_A).A)$  and  $u(\xi_1 \otimes 1_A) = \xi_2 \otimes 1_A$ ,  $u(\xi_2 \otimes 1_A) = \xi_1 \otimes 1_A$ . One has:*

- (1)  $\tilde{F}^* \tilde{\rho}_r(b) \tilde{F} = \pi_r(b)$  and  $u^* \pi_r(b) u = \pi_r(b)$  for all  $b \in B$ .
- (2)  $\tilde{F}^* \tilde{\rho}_r(a) \tilde{F} = u^* \pi_r(a) u$  for all  $a \in A_1$ .
- (3)  $\tilde{F}^* \tilde{\rho}_r(a) \tilde{F} = \pi_r(a)$  for all  $a \in A_2$ .

Let  $t \in \mathbb{R}$  and define the unitary  $u_t = \cos(t) + iu \sin(t) \in \mathcal{L}_A(H_r)$ . Assertion (1) of the Claim implies that  $u_t^* \pi_r(b) u_t = \pi_r(b)$  for all  $b \in B$ . By the universal property of full amalgamated free products, for all  $t \in \mathbb{R}$ , there exists a unique unital  $*$ -homomorphism  $\pi_t : A_f \rightarrow \mathcal{L}_A(H_r)$  such that:

$$\pi_t(a) = \begin{cases} u_t^* \pi_r(a) u_t & \text{if } a \in A_1, \\ \pi_r(a) & \text{if } a \in A_2. \end{cases}$$

Arguing as in the end of the proof of Proposition 3.4, we see that it suffices to show that, for all  $t \in [0, \frac{\pi}{2}]$ ,  $\pi_t$  factorizes through  $A$  i.e.  $\ker(\lambda) \subset \ker(\pi_t)$ . Since it is obvious for  $t = 0$ , we only need to show that  $\ker(\lambda) \subset \ker(\pi_t)$  for all  $t \in ]0, \frac{\pi}{2}]$ . To do that, we need the following claim.

**Claim.** For all  $t \in \mathbb{R}$  and all  $a = a_1 \dots a_n \in \mathcal{A}$  a reduced operator with  $a_k \in A_{l_k}^\circ$  one has

- (1)  $\pi_t(a)u_t^*(\xi_2 \otimes_{A_2} 1_A) = e^{-it}(a\xi_2 \otimes_{A_2} 1_A)$  if  $l_n = 1$  and  $\pi_t(a)(\xi_1 \otimes_{A_1} 1_A) = a\xi_1 \otimes_{A_1} 1_A$  if  $l_n = 2$ .
- (2)  $\langle u_t^*(\xi_1 \otimes_{A_1} 1_A), \pi_t(a)u_t^*(\xi_1 \otimes_{A_1} 1_A) \rangle = \sin^{2k}(t)a$  where  $k = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd and } l_n = 1, \\ \frac{n+1}{2} & \text{if } n \text{ is odd and } l_n = 2. \end{cases}$
- (3)  $\langle \xi_2 \otimes_{A_2} 1_A, \pi_t(a)\xi_2 \otimes_{A_2} 1_A \rangle = \sin^{2k}(t)a$  where  $k = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd and } l_n = 1, \\ \frac{n-1}{2} & \text{if } n \text{ is odd and } l_n = 2. \end{cases}$

*Proof of the Claim.* (1) is obvious by induction on  $n$  once observed that  $u_t\xi = e^{it}\xi$  (and  $u_t^*\xi = e^{-it}\xi$ ) for all  $\xi \in H_r \ominus (\xi_1 \otimes_{A_1} 1_A \oplus \xi_2 \otimes_{A_2} 1_A)$ .

(2). Define, for  $a_1 \dots a_n \in \mathcal{A}$ ,  $F(a_1, \dots, a_n) = \langle u_t^*(\xi_1 \otimes_{A_1} 1_A), \pi_t(a)u_t^*(\xi_1 \otimes_{A_1} 1_A) \rangle$ . First suppose that  $a \in A_1^\circ$  then  $F(a) = \langle u_t^*(\xi_1 \otimes_{A_1} 1_A), u_t^*\pi_r(a)(\xi_1 \otimes_{A_1} 1_A) \rangle = \langle \xi_1 \otimes_{A_1} 1_A, \xi_1 \otimes_{A_1} a \rangle = a$ . Now, let  $a = a_1 \dots a_n \in \mathcal{A}$  with  $n \geq 2$  and  $l_n = 1$ . We have:

$$F(a_1, \dots, a_n) = \langle u_t^*(\xi_1 \otimes_{A_1} 1_A), \pi_t(a_1 \dots a_{n-1})u_t^*(\xi_1 \otimes_{A_1} a_n) \rangle = F(a_1, \dots, a_{n-1})a_n.$$

Hence, it suffices to show the formula for  $l_n = 2$ . Suppose  $a \in A_2^\circ$ , we have:

$$\begin{aligned} F(a) &= \langle u_t^*(\xi_1 \otimes_{A_1} 1_A), \pi_r(a)u_t^*(\xi_1 \otimes_{A_1} 1_A) \rangle \\ &= \langle \cos(t)\xi_1 \otimes_{A_1} 1_A - i \sin(t)\xi_2 \otimes_{A_2} 1_A, \cos(t)a\xi_1 \otimes_{A_1} 1_A - i \sin(t)\xi_2 \otimes_{A_2} a \rangle = \sin^2(t)a. \end{aligned}$$

Now suppose  $a_1 a_2 \in \mathcal{A}$ , with  $l_2 = 2, l_1 = 1$ . We have:

$$\begin{aligned} F(a_1, a_2) &= \langle \xi_1 \otimes_{A_1} 1_A, \pi_r(a_1)u_t\pi_r(a_2)u_t^*(\xi_1 \otimes_{A_1} 1_A) \rangle \\ &= \langle \xi_1 \otimes_{A_1} 1_A, \pi_r(a_1)u_t(\cos(t)a_2\xi_1 \otimes_{A_1} 1_A - i \sin(t)\xi_2 \otimes_{A_2} a_2) \rangle \\ &= \langle \xi_1 \otimes_{A_1} 1_A, \cos(t)e^{it}a_1 a_2 \xi_1 \otimes_{A_1} 1_A - i \cos(t) \sin(t)a_1 \xi_2 \otimes_{A_2} a_2 + \sin^2(t)\xi_1 \otimes_{A_1} a_1 a_2 \rangle \\ &= \sin^2(t)a_1 a_2. \end{aligned}$$

Finally, suppose that  $n \geq 3$  and  $a_1 \dots a_n \in \mathcal{A}$  with  $l_n = 2$ . Define  $x = a_1 \dots a_{n-2}$ . We have

$$\begin{aligned} F(a_1, \dots, a_n) &= \langle u_t^*(\xi_1 \otimes_{A_1} 1_A), \pi_t(x)u_t^*\pi_r(a_{n-1})u_t\pi_r(a_n)u_t^*(\xi_1 \otimes_{A_1} 1_A) \rangle \\ &= \langle u_t^*(\xi_1 \otimes_{A_1} 1_A), \pi_t(x)u_t^*\pi_r(a_{n-1})u_t(\cos(t)a_n\xi_1 \otimes_{A_1} 1_A - i \sin(t)\xi_2 \otimes_{A_2} a_n) \rangle \\ &= \langle u_t^*(\xi_1 \otimes_{A_1} 1_A), \pi_t(x)u_t^*(\cos(t)e^{it}a_{n-1}a_n\xi_1 \otimes_{A_1} 1_A - i \cos(t) \sin(t)a_{n-1}\xi_2 \otimes_{A_2} a_n + \sin^2(t)\xi_1 \otimes_{A_1} a_{n-1}a_n) \rangle \\ &= \langle u_t^*(\xi_1 \otimes_{A_1} 1_A), \cos(t)a_1 \dots a_n \xi_1 \otimes_{A_1} 1_A - ie^{-it} \cos(t) \sin(t)a_1 \dots a_{n-1} \xi_2 \otimes_{A_2} a_n \rangle \\ &\quad + \langle u_t^*(\xi_1 \otimes_{A_1} 1_A), \sin^2(t)\pi_t(x)u_t^*\xi_1 \otimes_{A_1} a_{n-1}a_n \rangle. \end{aligned}$$

Hence we find:

$$F(a_1, \dots, a_n) = \sin^2(t) \langle u_t^*(\xi_1 \otimes_{A_1} 1_A), \pi_t(x) u_t^* \xi_1 \otimes_{A_1} a_{n-1} a_n \rangle = \sin^2(t) F(a_1, \dots, a_{n-2}) a_{n-1} a_n.$$

The result now follows by an obvious induction on  $n$ . The proof of (3) is similar.  $\square$

*End of the proof of Proposition 3.5.* Fix  $t \in ]0, \frac{\pi}{2}]$  and let  $A_t$  be the  $C^*$ -subalgebra of  $\mathcal{L}_A(H_r)$  generated by  $\pi_t(A_1) \cup \pi_t(A_2)$ . Hence,  $\pi_t : A_f \rightarrow A_t$  is surjective. Consider the ucp map  $\varphi_t : A_t \rightarrow A$  defined by  $\varphi_t(x) = \frac{1}{2} \left( \langle u_t^*(\xi_1 \otimes_{A_1} 1_A), x u_t^*(\xi_1 \otimes_{A_1} 1_A) \rangle + \langle \xi_2 \otimes_{A_2} 1_A, x \xi_2 \otimes_{A_2} 1_A \rangle \right)$  and note that  $\varphi_t$  is GNS faithful. Indeed, let  $x \in A_t$  such that  $\varphi_t(y^* x^* x y) = 0$  for all  $y \in A_t$ . Then  $L \subset \ker(x)$  where,

$$\begin{aligned} L &= \overline{\text{Span}} \left( A_t u_t^*(\xi_1 \otimes_{A_1} 1_A) \cdot A \cup A_t (\xi_2 \otimes_{A_2} 1_A) \cdot A \right) = \overline{\text{Span}} \left( A_t (\xi_1 \otimes_{A_1} 1_A) \cdot A \cup A_t (\xi_2 \otimes_{A_2} 1_A) \cdot A \right) \\ &= \overline{\text{Span}} \left( A_t (\xi_1 \otimes_{A_1} 1_A) \cdot A \cup A_t u_t^*(\xi_2 \otimes_{A_2} 1_A) \cdot A \right) = H_r, \end{aligned}$$

where we used Assertion (1) of the Claim for the last equality. Hence  $x = 0$ . Let  $A_{v,k}$  for  $k = 1, 2$  be the  $k$ -vertex-reduced free product and call  $i_k$  the natural inclusion of  $A$  in  $A_{v,k}$  and  $\pi_k = i_k \circ \lambda$  the natural map from  $A_f$  to  $A_{v,k}$ . Clearly  $\|x\|_A = \max(\|i_1(x)\|, \|i_2(x)\|)$  for any  $x$  in the vertex-reduced free product  $A$ . From the Assertions (2) and (3) of the Claim and Proposition 2.17 with  $r = \sin^2(t) > 0$  we deduced that for any  $k = 1, 2$  there exists two ucp maps  $\psi_1^k$  and  $\psi_2^k$  from  $A_{v,k}$  to itself such that  $i_k(\varphi_t(\pi_t(a))) = \frac{1}{2}(\psi_1^k(\pi_k(a)) + \psi_2^k(\pi_k(a)))$  for all  $a \in A_f$ . Therefore  $\|\varphi_t(\pi_t(a))\|_A \leq \max(\|\pi_1(a)\|, \|\pi_2(a)\|) = \|\lambda(a)\|$  for all  $a \in A_f$ . Let us show that  $\ker(\lambda) \subset \ker(\pi_t)$ . Let  $x \in \ker(\lambda)$ . Then, for all  $y \in A_f$  we have  $\lambda(y^* x^* x y) = 0$ . Therefore  $\varphi_t \circ \pi_t(y^* x^* x y) = 0$  for all  $y \in A_f$ . Since  $\pi_t$  is surjective we deduce that  $\varphi_t(y^* \pi_t(x)^* \pi_t(x) y) = 0$  for all  $y \in A_t$ . Using that  $\varphi_t$  is GNS faithful we deduce that  $\pi_t(x) = 0$ .  $\square$

We obtain the following obvious Corollary of Theorem 3.1 and Corollary 2.9.

**Corollary 3.6** ([Cu82]). *If we have conditional expectations  $E_k : A_k \rightarrow B$  which are also unital  $*$ -homomorphism, then the canonical surjection  $A_1 *_{A_2} A_2 \rightarrow A_1 \oplus_B A_2$  is  $K$ -invertible.*

#### 4. A SIX TERM EXACT SEQUENCE IN $KK$ -THEORY FOR FULL AMALGAMATED FREE PRODUCTS

Let  $A_1$  and  $A_2$  two unital  $C^*$ -algebras with a common unital  $C^*$ -subalgebra  $B$ . We will denote by  $i_l$  the inclusion of  $B$  in  $A_l$  for  $l = 1, 2$ . The algebra  $A_f$  is the full amalgamated free product. To simplify notation we will denote by  $S$  the algebra  $C_0(]-1, 1[)$ .

Let  $D$  be the subalgebra of  $S \otimes A_f$  consisting of functions  $f$  such that  $f(]-1, 0]) \subset A_1$ ,  $f(]0, 1[) \subset A_2$  and  $f(0) \in B$ . This algebra is of course isomorphic to the cone of  $i_1 \oplus i_2$  from  $B$  to  $A_1 \oplus A_2$ . We call  $j$  the inclusion of  $D$  in the suspension of  $A_f$ .

**Theorem 4.1.** *Suppose that there exist unital conditional expectations from  $A_l$  to  $B$  for  $l = 1, 2$ , then the map  $j$ , seen as an element  $[j]$  of  $KK^0(D, S \otimes A_f)$ , is invertible.*

The proof of this result will be done in several steps. We will start with the construction of an element  $x$  of  $KK^1(A_f, D)$ . As  $KK^1(A_f, D)$  is isomorphic to  $KK^0(S \otimes A_f, D)$  this will produce a candidate  $y$  for the inverse of  $j$ . The proof that  $y \otimes_D [j]$  is the identity of the suspension of  $A_f$  in

$KK^0(A_f, A_f)$  will use 3.4. Finally the proof that  $[j] \otimes_{S \otimes A_f} y$  is the identity of  $D$  in  $KK^0(D, D)$  will be done indirectly by using a short exact sequence for  $D$ .

**4.1. An inverse in KK-theory.** In order to present the inverse, we need some additional notations and preliminaries. Let  $\kappa_1$  be the inclusion of  $C_0(\ ] - 1, 0[; A_1)$  in  $D$  and  $\kappa_2$  the inclusion of  $C_0(\ ]0, 1[; A_2)$  in  $D$ . There is also  $\kappa_0$  the obvious map from  $S \otimes B$  in  $D$ . As  $K$  of the preceding section is a  $B$ -module, we can define

$$K_0 = (K \otimes S) \otimes_{\kappa_0} D, \quad K_1 = (K \otimes_{i_1} A_1 \otimes C_0(\ ] - 1, 0[)) \otimes_{\kappa_1} D \text{ and } K_2 = (K \otimes_{i_2} A_2 \otimes C_0(\ ]0, 1[)) \otimes_{\kappa_2} D.$$

If one defines  $I_l$  as the images of  $\kappa_l$  in  $D$  for  $l = 1, 2$ , it is clear that these are ideals in  $D$ .

**Lemma 4.2.**  *$K_l$  is canonically isomorphic to  $\overline{K_0 \cdot I_l}$  for  $l = 1, 2$  as  $D$  Hilbert module.*

*Proof.* We will show the statement for  $l = 1$ . Indeed as  $I_1 = \overline{C_0(\ ] - 1, 0[) \cdot I_1}$  because an approximate unit for  $C_0(\ ] - 1, 0[)$  is also one for  $I_1$ , it is easy to see that  $\overline{K_0 \cdot I_1}$  is isomorphic to  $\overline{(K \otimes S) \cdot C_0(\ ] - 1, 0[)} \otimes_{\kappa_0} D \cdot I_1$ , i.e.  $(K \otimes C_0(\ ] - 1, 0[)) \otimes_{\kappa_0} D \cdot I_1$ . Considering that  $C_0(\ ] - 1, 0[; A_1) \otimes_{\kappa_1} D$  is  $D \cdot I_1$ , one gets that  $\overline{K_0 \cdot I_1}$  is nothing but  $(K \otimes C_0(\ ] - 1, 0[)) \otimes_{\tilde{\kappa}_0} C_0(\ ] - 1, 0[; A_1) \otimes_{\kappa_1} D$  where  $\tilde{\kappa}_0$  is the natural inclusion of  $C_0(\ ] - 1, 0[; B)$  in  $C_0(\ ] - 1, 0[; A_1)$ , i.e.  $i_1 \otimes Id_{C_0(\ ] - 1, 0[)}$ . Therefore  $(K \otimes_{i_1} A_1) \otimes C_0(\ ] - 1, 0[)$  is  $(K \otimes C_0(\ ] - 1, 0[)) \otimes_{\tilde{\kappa}_0} C_0(\ ] - 1, 0[; A_1)$  and  $\overline{K_0 \cdot I_1}$  is  $K_1$ .  $\square$

We will also need the following lemmas :

- Lemma 4.3.** (1) *If  $f \in C(\ ] - 1, 1[; \mathbb{R})$ , then  $f$  is a self-adjoint element in  $Z(M(D))$  and more generally for any  $D$ -Hilbert module  $\mathcal{E}$  the right multiplication by  $f$  induces an element  $\hat{f} \in Z(\mathcal{L}_D(\mathcal{E}))$  such that the map  $f \mapsto \hat{f}$  is an algebra homomorphism.*  
 (2) *Let  $f$  in  $C_0(\ ] - 1, 0[; \mathbb{R})$ . Then  $f \in I_1 \cap Z(D)$  and the right multiplication by  $f$  induces a morphism  $\hat{f}$  of  $\mathcal{L}_D(K_0, K_1)$  such that  $\hat{f}^* \hat{f} = \hat{f}^2$  in  $\mathcal{L}_D(K_0)$  and  $\hat{f} \hat{f}^* = \hat{f}^2$  in  $\mathcal{L}_D(K_1)$ .*  
 (3) *Let  $f$  in  $C_0(\ ]0, 1[; \mathbb{R})$ . Then  $f \in I_2 \cap Z(D)$  and the right multiplication by  $f$  induces a morphism  $\hat{f}$  of  $\mathcal{L}_D(K_0, K_2)$  such that  $\hat{f}^* \hat{f} = \hat{f}^2$  in  $\mathcal{L}_D(K_0)$  and  $\hat{f} \hat{f}^* = \hat{f}^2$  in  $\mathcal{L}_D(K_2)$ .*

The first point is pretty obvious and (2) and (3) are also clear in view of Lemma 4.2.

- Lemma 4.4.** (1) *If  $f \in C_0(\ ] - 1, 1[; \mathbb{R})$  then for any  $B$ -module  $\mathcal{E}$  and  $F \in \mathcal{K}_B(\mathcal{E})$ , we have  $((F \otimes 1_S) \otimes_{\kappa_0} 1_D) \hat{f}$  is a compact operator of  $(\mathcal{E} \otimes S) \otimes_{\kappa_0} D$ .*  
 (2) *If  $f \in C_0(\ ] - 1, 0[; \mathbb{R})$  then for any  $A_1$ -module  $\mathcal{E}$  and  $F \in \mathcal{K}_{A_1}(\mathcal{E})$ , we have  $(F \otimes 1_{C_0(\ ] - 1, 0[)} \otimes_{\kappa_1} 1_D) \hat{f}$  is a compact operator of  $(\mathcal{E} \otimes C_0(\ ] - 1, 0[)) \otimes_{\kappa_1} D$ .*  
 (3) *Similarly for  $f \in C_0(\ ]0, 1[; \mathbb{R})$  and  $A_2$ -modules.*

*Proof.* Point (2) and (3) are similar to (1). To show (1), let  $F$  be the rank one operator  $\theta_{\xi, \eta}$  for  $\xi$  and  $\eta$  vectors in  $\mathcal{E}$  which is defined as  $\theta_{\xi, \eta}(x) = \xi \langle \eta, x \rangle$  for all  $x$  in  $\mathcal{E}$ . Then  $(F \otimes 1_S) \otimes_{\kappa_0} 1_D \hat{f}$  is  $\theta_{\xi \otimes f_2 \otimes f_2, \eta \otimes f_2 \otimes f_2} \hat{f}$  and therefore compact for any function  $f = f_1 f_2^4$  with  $f_1$  and  $f_2$  in  $C_0(\ ] - 1, 1[; \mathbb{R})$ . As any function can be written like that, for example by polar decomposition, we get our result.  $\square$

Define now two functions in  $C(\ ] - 1, 1[; \mathbb{R})$  :  $C^+(t)$  is  $\cos(\pi t)$  if  $t \geq 0$  and 1 if  $t \leq 0$ , the function  $C^-(t)$  is  $\cos(\pi t)$  if  $t \leq 0$  and 1 if  $t \geq 0$ . Similarly, we have two functions in  $S$  ;  $S^+$  is  $\sin(\pi t)$  if  $t \geq 0$  and 0 if  $t \leq 0$ , the function  $S^-(t)$  is  $\sin(\pi t)$  if  $t \leq 0$  and 0 if  $t \geq 0$ . And finally  $T$  is the identity function of  $C(\ ] - 1, 1[; \mathbb{R})$ .

With the notation of the first part, we have a natural  $D$ -module

$$H = (H_1 \otimes C_0(]-1, 0[)) \otimes_{\kappa_1} D \oplus (H_2 \otimes C_0(]0, 1[)) \otimes_{\kappa_2} D \oplus (K \otimes S) \otimes_{\kappa_0} D.$$

It is also clear that  $H$  is endowed with a natural (left) action of  $A_f$  as  $H_1, H_2$  and  $K$  have it.

Let  $G$  be the operator of  $\mathcal{L}_D(H)$  defined in matrix form by

$$G = \begin{pmatrix} \widehat{C}^- & 0 & -((F_1 \otimes 1_{C_0(]-1, 0[)})^* \otimes_{\kappa_1} 1) \widehat{S}^- \\ 0 & -\widehat{C}^+ & ((F_2 \otimes 1_{C_0(]0, 1[)})^* \otimes_{\kappa_2} 1) \widehat{S}^+ \\ -\widehat{S}^-^* ((F_1 \otimes 1_{C_0(]-1, 0[)}) \otimes_{\kappa_1} 1) & \widehat{S}^+^* ((F_2 \otimes 1_{C_0(]0, 1[)}) \otimes_{\kappa_2} 1) & Z \end{pmatrix}$$

where  $Z = -\widehat{C}^-((q_1 \otimes 1_S) \otimes_{\kappa_0} 1) + \widehat{C}^+((q_2 \otimes 1_S) \otimes_{\kappa_0} 1) - \widehat{T}((q_0 \otimes 1_S) \otimes_{\kappa_0} 1)$ . Thanks to Lemma 4.3,  $G$  is well-defined. Moreover the following facts hold.

**Proposition 4.5.** *The operator  $G$  satisfies that  $G^2 - 1$  is a compact operator of  $H$  and  $G$  commutes modulo compact operators with the action of  $A_f$ .*

*Proof.* To make reading easier, we will note, in this proof only,  $\overline{F}_1$  for  $(F_1 \otimes 1_{C_0(]-1, 0[)}) \otimes_{\kappa_1} 1$  and  $\overline{F}_2$  for  $(F_2 \otimes 1_{C_0(]0, 1[)}) \otimes_{\kappa_2} 1$ . Computing  $G^2$  one gets as upper left  $2 \times 2$  corner :

$$\begin{pmatrix} \widehat{C}^-^2 + \overline{F}_1^* \widehat{S}^- \widehat{S}^-^* \overline{F}_1 & -\overline{F}_1^* \widehat{S}^- \widehat{S}^+^* \overline{F}_2 \\ -\overline{F}_2^* \widehat{S}^-^* \widehat{S}^+ \overline{F}_1 & \widehat{C}^+^2 + \overline{F}_2^* \widehat{S}^+ \widehat{S}^+^* \overline{F}_2 \end{pmatrix}$$

As  $F_1^* F_1$  is the identity modulo compact operators, using Lemma 4.4 ( the function  $(S^-)^2$  is in  $C_0(]-1, 1[)$  ) one has that  $\overline{F}_1^* \widehat{(S^-)^2} \overline{F}_1$  is  $\widehat{(S^-)^2}$  modulo compact operators. Recalling also that  $F_1^* F_2 = 0$ , one gets that this matrix is then the identity modulo compact operators.

Let's focus now on the last row of  $G^2$ . We get first  $-\widehat{C}^- \overline{F}_1^* \widehat{S}^- - \overline{F}_1^* \widehat{S}^- Z$ . As  $F_1^* q_1 \otimes_{i_1} 1 = F_1^*$  and  $F_1^* q_2 \otimes_{i_1} 1 = 0$  along with  $F_1^* q_0 \otimes_{i_1} 1 = 0$ ,  $\overline{F}_1^* \widehat{S}^- Z$  is  $-\overline{F}_1^* \widehat{S}^- \widehat{C}^-$ . The second component of that row is treated in the same way. Finally the last component is  $\widehat{S}^-^2 \overline{F}_1 \overline{F}_1^* + \widehat{S}^+^2 \overline{F}_2 \overline{F}_2^* + \widehat{C}^-^2 ((q_1 \otimes 1_S) \otimes_{\kappa_0} 1) + \widehat{C}^+^2 ((q_2 \otimes 1_S) \otimes_{\kappa_0} 1) + \widehat{T}^2 ((q_0 \otimes 1_S) \otimes_{\kappa_0} 1)$  as  $q_0, q_1, q_2$  are commuting projections. But  $F_1 F_1^*$  is  $q_1 \otimes_{i_1} 1$  so  $\widehat{S}^-^2 \overline{F}_1 \overline{F}_1^*$  is  $\widehat{S}^-^2 ((q_1 \otimes 1_S) \otimes_{\kappa_0} 1)$ . Hence, as  $q_1 + q_2 + q_0 = 1$ , the last component is  $1 + \widehat{T}^2 - 1 ((q_0 \otimes 1_S) \otimes_{\kappa_0} 1)$ . As  $T^2 - 1$  is in  $C_0(]-1, 1[)$  and  $q_0$  is compact, this component is then 1 modulo compact operators.

Addressing now the compact commutation with the left action of  $A_f$ , it is very obvious using Lemma 4.4 and Lemma 3.3 (3) for every component of  $G$  except  $Z$  as it contains multiplication with functions not in  $C_0(]-1, 1[)$ . So let  $a$  be in  $A_1$ . We need to compute  $[Z, \rho(a) \otimes_{\kappa_0} 1]$ . But we know that  $[q_1, \rho(a)] = 0$ . As  $q_2 = 1 - q_1 - q_0$  we get that  $[Z, \rho(a) \otimes_{\kappa_0} 1] = -(\widehat{C}^+ + T)(([q_0, \rho(a)]) \otimes_{\kappa_0} 1)$  which is compact as  $C^+ + T$  is a function that vanishes on  $-1$  and  $1$ . The case when  $a$  is in  $A_2$  is treated in a similar way, hence the compact commutation property is proved for all  $a$  in  $A_f$ .  $\square$

As a consequence, the couple  $(H, G)$  defines an element of  $KK^1(A_f, D)$  which we will call  $x$  in the sequel.

**4.2. K-equivalence.** In all the following proofs we will very often use the external tensor product of Kasparov elements. Instead of the traditional notation  $\tau_C(x)$  for the tensorisation with the algebra  $C$  of an element  $x$  in  $KK^*(A, B)$ , we will write  $1_C \otimes x$  for the element in  $KK^*(C \otimes A, C \otimes B)$  or  $x \otimes 1_C$  for the element in  $KK^*(A \otimes C, B \otimes C)$ . Of course  $B \otimes C$  is (non canonically) isomorphic to  $C \otimes B$ , but as we will perform several times this operation, the order will matter. Note that we do not specify the tensor norm as the algebra  $C$  we will be using is always nuclear. Also when  $\pi$  is a morphism between  $A$  and  $B$ , we will write  $[\pi]$  for the canonical element in  $KK^0(A, B)$ .

We will denote by  $b$  the element of  $KK^1(\mathbb{C}, S)$  which is defined on the  $S$ -Hilbert module  $S$  itself by the operator  $\widehat{T}$ . It is well known that  $b$  is invertible. Indeed let's describe its form as an extension. The projection associated to the orthogonal symmetry  $\widehat{T}$  is the multiplication by the function  $p(t) = (1+t)/2$  on  $C_b([-1, 1])/C_0([-1, 1])$ . Now in  $C_b([-1, 1])$ , the  $C^*$ -algebra generated by  $C_0([-1, 1])$  and  $p$  is obviously  $C_0([-1, 1])$ . So the extension we have to consider is given by the map from  $\mathbb{C}$  to  $C_0([-1, 1])/C_0([-1, 1])$  that sends  $\lambda$  to  $\lambda p$ . Using the evaluation at 1, one gets the standard extension

$$0 \rightarrow C_0([-1, 1]) \rightarrow C_0([-1, 1]) \rightarrow \mathbb{C} \rightarrow 0.$$

Using UCT for example, as all K-groups appearing here are torsion-free, we deduce that  $b$  is invertible. The interested reader can also check section 19.2 of [Bl86].

**Proposition 4.6.** *With the hypothesis of Theorem 4.1, one has in  $KK^1(A_f, A_f \otimes S)$  that  $x \otimes_D [j]$  is homotopic to  $1_{A_f} \otimes b$ .*

*Proof.* We will actually show that  $x \otimes_D [j]$  is homotopic to  $[Id_{A_f}] \otimes_{A_f} (1_{A_f} \otimes b)$ . To prove that, we will choose the representant of  $[Id_{A_f}]$  that appear in 3.4 and show that its Kasparov product with  $1_{A_f} \otimes b$  is homotopic to  $x \otimes_D [j]$ . Call  $j_l$  for  $l = 1, 2$  the inclusions of  $A_l$  in  $A_f$  and  $j_0 = j_1 \circ i_1 = j_2 \circ i_2$  the inclusion of  $B$  in  $A_f$ . First it is obvious that  $H \otimes_j (A_f \otimes S)$  is  $H_1 \otimes_{j_1} A_f \otimes C_0([-1, 0]) \oplus H_2 \otimes_{j_2} A_f \otimes C_0([0, 1]) \oplus K \otimes_{j_0} A_f \otimes S$  which is not quite the same as  $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f) \otimes S$ . So we will realize now a homotopy to fix that.

**Lemma 4.7.** *Consider the following two spaces :  $\Delta_1 = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq 1, -1 < t < s\}$  and  $\Delta_2 = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq 1, -s < t < 1\}$ . The Hilbert module  $\overline{H} = H_1 \otimes_{j_1} A_f \otimes C_0(\Delta_1) \oplus H_2 \otimes_{j_2} A_f \otimes C_0(\Delta_2) \oplus K \otimes_{j_0} A_f \otimes S \otimes C([0, 1])$  is endowed with a natural structure of  $A_f \otimes S \otimes C([0, 1])$  Hilbert module, as  $C_0(\Delta_1)$  and  $C_0(\Delta_2)$  naturally embed in  $C_0([-1, 1] \times [0, 1])$  (with inclusion maps  $d_1$  and  $d_2$ ), and  $A_f$  left action. Moreover the operator*

$$\overline{G} = \begin{pmatrix} \widehat{C}^- \otimes 1_{C([0,1])} & 0 & -(F_1 \otimes_{j_1} 1 \otimes d_1)^*(\widehat{S}^- \otimes 1_{C([0,1])}) \\ 0 & -\widehat{C}^+ \otimes 1_{C([0,1])} & (F_2 \otimes_{j_2} 1 \otimes d_2)^*(\widehat{S}^+ \otimes 1_{C([0,1])}) \\ -(\widehat{S}^-)^* \otimes 1_{C([0,1])} (F_1 \otimes_{j_1} 1 \otimes d_1) & (\widehat{S}^+)^* \otimes 1_{C([0,1])} (F_2 \otimes_{j_2} 1 \otimes d_2) & \overline{Z} \end{pmatrix}$$

with  $\overline{Z} = \widetilde{Z} \otimes 1_{C([0,1])}$  where  $\widetilde{Z} = -\widehat{C}^-(q_1 \otimes_{j_0} 1 \otimes 1_S) + \widehat{C}^+(q_2 \otimes_{j_0} 1 \otimes 1_S) - \widehat{T}(q_0 \otimes_{j_0} 1 \otimes 1_S)$  makes the pair  $(\overline{H}, \overline{G})$  into an element of  $KK^1(A_f, A_f \otimes S \otimes C([0, 1]))$  for which the evaluation at  $s = 0$  is  $x \otimes_D [j]$  and the evaluation at  $s = 1$  has  $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f) \otimes S$  as

$$\text{module and } \widetilde{G} = \begin{pmatrix} \widehat{C}^- & 0 & -(F_1^* \otimes_{j_1} 1 \otimes 1_S) \widehat{S}^- \\ 0 & -\widehat{C}^+ & (F_2^* \otimes_{j_2} 1 \otimes 1_S) \widehat{S}^+ \\ -\widehat{S}^-^* (F_1 \otimes_{j_1} 1 \otimes 1_S) & \widehat{S}^+^* (F_2 \otimes_{j_2} 1_S) & \widetilde{Z} \end{pmatrix} \text{ as operator.}$$

*Proof.* As it is a straightforward check, details will be omitted.  $\square$

Using Connes- Skandalis characterization of the Kasparov product, we will now establish that  $\tilde{G}$  is a representant of the Fredholm operator for the product  $[Id_{A_f}] \otimes_{A_f} (1_{A_f} \otimes b)$  by checking the connection and positivity properties (see [Bl86] Chap 18.4). But to do that we of course need to revert to the general presentation of  $KK^1$  elements as graded  $KK$  elements (see preliminaries). Let's denote  $e_0 = (1, 1)$  and  $e_1 = (1, -1)$  the basis of  $\mathbb{C}_1$ . The element of graded  $KK$ -theory that we have now for  $x \otimes_D [j]$  is given by the module  $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f) \otimes S \otimes \mathbb{C}_1$  and operator  $R$  such that if  $\xi$  is in  $H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f$  and  $f$  in  $S$ ,  $R(\xi \otimes f \otimes e_0) = \tilde{G}(\xi \otimes f) \otimes e_1$ . As  $R$  is  $\mathbb{C}_1$ -linear, that completely characterizes  $R$ . There is a similar statement for  $b$  as an element of  $KK(\mathbb{C}, S \otimes \mathbb{C}_1)$ . We will call  $T$  the 1-graded operator that appears.

Looking first at the module for  $[Id_{A_f}] \otimes_{A_f} (1_{A_f} \otimes b)$ , we obtain  $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f) \hat{\otimes} (S \otimes \mathbb{C}_1)$ . Note that we used the graded tensor product. Of course when one term is trivially graded the graded tensor product is the usual tensor product. At first look, it is the same as  $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f) \otimes S \otimes \mathbb{C}_1$  except that the grading is not the same. But of course there is a  $A_f \otimes S \otimes \mathbb{C}_1$ -isomorphism  $U$  that corrects that, sending  $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f) \otimes S \hat{\otimes} e_0 \oplus (K \otimes_{j_0} A_f) \otimes S \hat{\otimes} e_1$  to  $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f) \otimes S \otimes e_0$ . Through this isomorphism,  $R$  becomes  $\bar{R}$ .

Let's look now at the connection condition (see [Bl86] Definition 18.3.1 p 170). As  $\bar{R}$  and  $T$  are self-adjoint, there is only one condition to test. For  $\xi$  in  $H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f$ , one should look at the  $A_f \otimes S \otimes \mathbb{C}_1$ -linear map from  $A_f \otimes S \otimes \mathbb{C}_1$  to  $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f) \hat{\otimes} (S \otimes \mathbb{C}_1)$  defined for  $f$  in  $S$  and  $a$  in  $A_f$  as  $a \otimes f \otimes e_0 \mapsto (-1)^{0 \times 1} \bar{R}(\xi \cdot a \hat{\otimes} (f \otimes e_0)) - \xi \cdot a \hat{\otimes} T(f \otimes e_0)$  (as we do the Kasparov product with  $1_{A_f} \otimes b$ ) and prove that it is compact. Observe that the operator leaves in  $C([-1, 1]) \otimes \mathcal{L}(A_f \otimes \mathbb{C}_1, (H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f) \hat{\otimes} \mathbb{C}_1)$ . As  $A_f$  is unital and therefore  $Id_{A_f}$  compact, this is  $C([-1, 1])$  tensored by compact operators. Hence we simply need proving that the evaluation at  $-1$  and  $1$  of the operator is 0. On both ends  $\tilde{G}$  is diagonal,

equals to  $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  or the opposite matrix as  $q_1 + q_2 + q_0 = 1$ . So the evaluation at  $-1$  of  $(-1)^{0 \times 1} \bar{R}$  will send  $(\xi \hat{\otimes} e_0)$  to  $-\xi \hat{\otimes} e_1$  which is what the evaluation at  $-1$  of  $T$  does. For the evaluation at  $1$ , the two operators are also identical.

Similarly if  $\xi$  in  $K \otimes_{j_0} A_f$ , one looks at  $(-1)^{1 \times 1} \bar{R}(\xi \cdot a \hat{\otimes} (f \otimes e_0)) - \xi \cdot a \hat{\otimes} T(f \otimes e_0)$ . The evaluation at  $-1$  of  $(-1)^{1 \times 1} \bar{R}$  will send  $(\xi \hat{\otimes} e_0)$  to  $\xi \hat{\otimes} e_1$  which is again what the evaluation at  $-1$  of  $T$  does and similarly for the evaluation at  $1$ .

We now concentrate on the commutator condition (see [Bl86] Definition 18.4.1 p 172). One needs to compute the anti-commutator of  $\bar{R}$  with  $F \hat{\otimes} 1$ , using the operator  $F$  that appeared before Proposition 3.4. We will call  $G_0$  and  $G_1$  the diagonal and anti-diagonal part of  $\tilde{G}$ .

For  $\xi$  in  $H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f$  and  $f$  in  $S$ , one has that  $F \hat{\otimes} 1(\xi \otimes f \otimes e_0) = F(\xi) \hat{\otimes} (f \otimes e_0)$ . As  $F(\xi)$  is then of degree 1,  $\bar{R}(F(\xi) \hat{\otimes} (f \otimes e_0)) = R(F(\xi) \otimes f \otimes e_1) = U^* \tilde{G}(F(\xi) \otimes f) \otimes e_0 = G_0(F(\xi) \otimes f) \hat{\otimes} e_1 + G_1(F(\xi) \otimes f) \hat{\otimes} e_0$ . On the other hand  $(F \hat{\otimes} 1) \cdot \bar{R}(\xi \hat{\otimes} (f \otimes e_0)) = (F \otimes 1_S) \cdot G_0(\xi \otimes f) \hat{\otimes} e_1 + (F \otimes 1_S) \cdot G_1(\xi \otimes f) \hat{\otimes} e_0$ . As the same is true for  $\xi$  in  $K \otimes_{j_0} A_f$ , we will be done once the following Lemma is proved.



**Lemma 4.8.** *The anti-commutator of  $G_0$  and  $F \otimes 1_S$  is 0 modulo compact operators and the anti-commutator of  $G_1$  and  $F \otimes 1_S$  is positive modulo compact operators.*

*Proof.* It is clear that  $\begin{pmatrix} \widehat{C}^- & 0 & 0 \\ 0 & -\widehat{C}^+ & 0 \\ 0 & 0 & \widehat{Z} \end{pmatrix}$  and  $\begin{pmatrix} 0 & 0 & F_1^* \otimes_{i_1} 1 \otimes 1_S \\ 0 & 0 & F_2^* \otimes_{i_2} 1 \otimes 1_S \\ F_1 \otimes_{i_1} 1 \otimes 1_S & F_2 \otimes_{i_2} 1 \otimes 1_S & 0 \end{pmatrix}$

anti-commutes modulo compact operator as we have (modulo compact operator)  $q_1 F_1 = F_1$  and  $q_2 F_1 = q_0 F_1 = 0$ . On the other hand the anti-commutator with the anti-diagonal part is

$$\begin{pmatrix} -2((F_1^* F_1) \otimes_{j_1} 1 \otimes 1_S) \widehat{S}^- & 0 & 0 \\ 0 & 2((F_2^* F_2) \otimes_{j_2} 1 \otimes 1_S) \widehat{S}^+ & 0 \\ 0 & 0 & -2(q_1 \otimes_{j_0} 1 \otimes 1_S) \widehat{S}^- + 2(q_2 \otimes_{j_2} 1 \otimes 1_S) \widehat{S}^+ \end{pmatrix}$$

As  $-S^-$  and  $S^+$  are positive functions and  $q_1$  and  $q_2$  are orthogonal projections, the previous matrix is a diagonal matrix of positive operators hence positive.  $\square$

*End of the proof of Proposition 4.6.* Having checked the two conditions that characterize the Kasparov product we have our proposition. Note that as  $[Id_{A_f}]$  is a Kasparov cycle given by a homomorphism, we obviously have  $[Id_{A_f}] \otimes_{A_f} (1_{A_f} \otimes b) = (1_{A_f} \otimes b) \otimes_{A_f \otimes S} ([Id_{A_f}] \otimes 1_S)$ . Hence  $x \otimes_D [j]$  is also equal to  $(1_{A_f} \otimes b) \otimes_{A_f \otimes S} ([Id_{A_f}] \otimes 1_S)$ . This is the form we need in the final stage of our proof of the theorem.  $\square$

We need now the following two lemmas to get some information about  $[j] \otimes_{A_f \otimes S} (x \otimes 1_S)$  as an element of  $KK^1(D, D \otimes S)$ .

**Lemma 4.9.** *Call  $ev_0$  the morphism from  $D$  to  $B$  that evaluates a function at 0. Then we have in  $KK^1(D, B \otimes S)$  that  $[j] \otimes_{A_f \otimes S} ((x \otimes_D [ev_0]) \otimes 1_S) = -[ev_0] \otimes_B (1_B \otimes b)$ .*

*Proof.* Let's first describe the left hand side. The Hilbert module is  $K \otimes S$  as the module  $(H_1 \otimes C_0([-1, 0])) \otimes_{\kappa_1} D \otimes_{ev_0} B$  is 0. The left  $D$  action is given by  $(\rho \otimes 1_S) \circ j$  and the operator is just  $(-q_1 + q_2) \otimes 1_S$ . We can replace this operator with  $G_0 = (-q_1 + q_2) \otimes 1_S - \widehat{T}(q_0 \otimes 1_S)$  as for any  $f$  in  $D$ ,  $(\rho \otimes 1_S) \circ j(f) \widehat{T}(q_0 \otimes 1_S)$  is compact. Observe that  $G_0$  is an operator of  $C([-1, 1]) \otimes \mathcal{K}(K)$ . Note now that the evaluation at  $-1$  of  $G_0$  is  $-q_1 + q_2 + q_0 = (1 - 2q_1)$  and at  $1$  is  $-q_1 + q_2 - q_0 = 2q_2 - 1$  as  $q_1 + q_2 + q_0 = 1$ . It then enables us to do a homotopy. Consider the pair  $(K \otimes S \otimes C([0, 1]), G_0 \otimes 1_{C([0, 1])})$  where the left action of  $D$  is defined now for any  $f$  in  $D$  and  $k \in C([-1, 1] \times [0, 1]; K)$  as  $(f.k)(t, s) = \rho(f(t(1-s)))k(t, s)$ . This is still a Kasparov element as  $(G_0^2 - 1) \otimes 1_{C([0, 1])} = ((\widehat{T}^2 - 1)(q_0 \otimes 1_S)) \otimes 1_{C([0, 1])}$  hence compact. Also the commutator of the left action with the operator  $G_0 \otimes 1$  is compact. Indeed, as the  $q_i$  have compact commutators with the left action, any commutator of  $G_0 \otimes 1$  with a left-acting element lives in  $C([-1, 1]) \otimes \mathcal{K}(K)$ . It is then only necessary to check that the evaluation at  $-1$  or  $1$  of any commutator is 0. But this is true as  $[q_1, \rho(A_1)] = 0$  and  $[q_2, \rho(A_2)] = 0$ . Therefore  $[j] \otimes_{A_f \otimes S} ((x \otimes_D [ev_0]) \otimes 1_S)$  is homotopic to an element of  $KK^1(D, B \otimes S)$  which is described with the pair  $(K \otimes S, G_0)$  where  $D$  acts on  $K \otimes S$  as the constant morphism  $\rho \circ ev_0$ . So it is  $[ev_0] \otimes_B z$  with  $z$  an element of  $KK^1(B, B \otimes S)$  which is only non trivial on  $q_0 K \otimes S \simeq B \otimes S$  where  $G_0$  acts as  $-\widehat{T}$ . Thus  $z = -1_B \otimes b$ .  $\square$

Recall that for  $l = 1, 2$ ,  $\kappa_l$  is the inclusion of  $A_l \otimes C([-1, 0])$  in  $D$ . To be precise we will use  $\bar{\kappa}_l$  for the induced map from  $A_l \otimes S$  to  $D$  via the isomorphism of  $C([-1, 0])$  with  $S$ .

**Lemma 4.10.** *For all  $l = 1, 2$ , one has  $[j_l] \otimes_{A_f} x = ([Id_{A_l}] \otimes b) \otimes_{A_l \otimes S} [\bar{\kappa}_l] \in KK^1(A_l, D)$ .*

*Proof.* We will do the lemma for  $l = 1$ . The element  $[j_1] \otimes_{A_f} x$  has the same module and operator as  $x$ , the only change is that we only consider a left action of  $A_1$ . We first perform a compact perturbation of the operator  $G$ . With the operators  $\bar{F}_l$  defined before Lemma 3.3, consider

$$G_1 = \begin{pmatrix} \widehat{C}^- & 0 & -((F_1 \otimes 1_{C_0(]-1,0])})^* \otimes_{\kappa_1} 1) \widehat{S}^- \\ 0 & -\widehat{C}^+ & ((\bar{F}_2 \otimes 1_{C_0(]0,1])})^* \otimes_{\kappa_2} 1) \widehat{S}^+ \\ -\widehat{S}^- * ((F_1 \otimes 1_{C_0(]-1,0])}) \otimes_{\kappa_1} 1 & \widehat{S}^+ * ((\bar{F}_2 \otimes 1_{C_0(]0,1])}) \otimes_{\kappa_2} 1 & \bar{Z} \end{pmatrix},$$

where  $\bar{Z} = -\widehat{C}^-((q_1 \otimes 1_S) \otimes_{\kappa_0} 1) + \widehat{C}^+(((1 - q_1) \otimes 1_S) \otimes_{\kappa_0} 1)$ .

As  $F_2 - \bar{F}_2$  is compact (see Lemma 3.3) and  $\bar{Z} - Z = \widehat{C}^+ + T((q_0 \otimes 1_S) \otimes_{\kappa_0} 1)$  is compact as  $C^+ + T$  is in  $S$ , we get the same element of  $KK^1(A_1, D)$ . Observe now that  $G_1^2$  is the identity because  $\bar{F}_2$  is an isometry and  $\widehat{S}^-((F_1 \otimes 1_{C_0(]-1,0])}) \otimes_{\kappa_1} 1$  vanishes and that  $G_1$  commutes exactly with the left action of  $A_1$  as  $F_1$  and  $\bar{F}_2$  does.

We will now construct a homotopy to remove the  $[0, 1[$  part of our module. Consider the space  $\Delta_3 = \{(t, s) \in \mathbb{R} : 0 \leq s \leq 1, 0 < t < s\}$  and  $\Delta_4 = \{(t, s) \in \mathbb{R} : 0 \leq s \leq 1, -1 < t < s\}$  which are open in  $] -1, 1[ \times [0, 1]$ . Hence we also have a natural imbedding  $\delta_4$  of  $C_0(\Delta_4; B)$  in  $D \otimes C([0, 1])$  and  $\delta_3$  of  $C_0(\Delta_3; A_2)$  in  $D \otimes C([0, 1])$ . Then  $\tilde{H} = (H_1 \otimes C_0(]-1, 0]) \otimes_{\kappa_1} D \otimes C([0, 1]) \oplus (H_2 \otimes C_0(\Delta_3)) \otimes_{\delta_3} D \otimes C([0, 1]) \oplus (K \otimes C_0(\Delta_4)) \otimes_{\delta_4} D \otimes C([0, 1])$  is well defined and the pair  $(\tilde{H}, G_1 \otimes 1_{C([0, 1])})$  is a Kasparov element in  $KK^1(A_1, D \otimes C([0, 1]))$ . Indeed the only thing to check is whether  $G_1^2 \otimes 1_{C([0, 1])}$  is the identity modulo compact operator as  $G_1 \otimes 1_{C([0, 1])}$  has exact commutation with the action of  $A_1$ . But this is true by the previous observation.

Therefore  $[j_l] \otimes_{A_f} x$  can be represented by the evaluation at 0 of this Kasparov element. Let's describe it: the module part is  $(H_1 \oplus K \otimes_{i_1} A_1) \otimes C_0(]-1, 0]) \otimes_{\kappa_1} D$  with obvious left  $A_1$  action as  $(K \otimes C_0(]-1, 0]) \otimes_{\kappa_0} D$  is isomorphic to  $(K \otimes_{i_1} A_1) \otimes C_0(]-1, 0]) \otimes_{\kappa_1} D$ . With this identification, the operator is

$$E_1 = \begin{pmatrix} \widehat{C}^- & -((F_1^* \otimes 1_{C_0(]-1,0])}) \otimes_{\kappa_1} 1) \widehat{S}^- \\ -\widehat{S}^- * ((F_1 \otimes 1_{C_0(]-1,0])}) \otimes_{\kappa_1} 1 & Z_1 \end{pmatrix}$$

where  $Z_1 = -\widehat{C}^-((q_1 \otimes_{i_1} 1) \otimes 1_{C_0(]-1,0])}) \otimes_{\kappa_1} 1 + (((1 - q_1) \otimes_{i_1} 1) \otimes 1_{C_0(]-1,0])}) \otimes_{\kappa_1} 1$ .

It is then clear, after identifying  $C_0(]-1, 0])$  with  $S$ , that  $[j_1] \otimes_{A_f} x$  is  $z \otimes_{A_1 \otimes S} [\bar{\kappa}_1]$  with  $z$  in  $KK^1(A_1, A_1 \otimes S)$ . By recalling that  $1 - q_1$  commutes with the left action of  $A_1$ , it is obvious that  $z$

is represented by the pair  $((H_1 \oplus q_1 K \otimes_{i_1} A_1) \otimes S, \bar{E}_1)$  with  $\bar{E}_1 = \begin{pmatrix} \widehat{C}_1 & -(F_1^* \otimes 1_S) \widehat{S}_1 \\ -\widehat{S}_1^* (F_1 \otimes 1_S) & -\widehat{C}_1 (q_1 \otimes_{i_1} 1 \otimes 1_S) \end{pmatrix}$

where  $C_1$  is the function  $\cos(\pi(t/2 - 1/2))$  and  $S_1$  the function  $\sin(\pi(t/2 - 1/2))$ .

Following the proof of Proposition 4.6,  $z$  is obviously the product  $z' \otimes b$  where  $z'$  is the element of  $KK^0(A_1, A_1)$  given by the module  $H_1 \oplus q_1 K \otimes_{i_1} A_1$  with  $H_1$  positively graded and the obvious left action of  $A_1$  and the operator  $\begin{pmatrix} 0 & F_1^* \\ F_1 & 0 \end{pmatrix}$ . Now the action of  $A_1$  stabilizes  $H_1^\circ$  and commutes with  $F_1$  by 3.3 (1) and moreover  $F_1$  is a unitary between  $H_1^\circ$  and  $q_1 K \otimes_{i_1} A_1$ . Hence this part is degenerated and can be removed from the Kasparov element. What remains is the graded

module  $\xi_1.A_1 \oplus 0$  with left action of  $A_1$  by multiplication and 0 as operator. This is a description of  $[Id_{A_1}]$ . □

We are now ready to prove Theorem 4.1.

*Proof of Theorem 4.1.* Call  $a \in KK^1(S, \mathbb{C})$  the inverse of  $b$ . The element  $y = (1_{A_f} \otimes a) \otimes_{A_f} x$  is an element of  $KK^0(A_f \otimes S, D)$ . We claim that this is the inverse of  $[j]$ . Indeed thanks to 4.6 we have that

$$y \otimes_D [j] = (1_{A_f} \otimes a) \otimes_{A_f} x \otimes_D [j] = (1_{A_f} \otimes a) \otimes_{A_f} (1_{A_f} \otimes b) \otimes_{A_f \otimes S} ([Id_{A_f}] \otimes 1_S).$$

As  $a \otimes_{\mathbb{C}} b = [Id_S]$  we get that  $y \otimes_D [j] = (1_{A_f} \otimes [Id_S]) \otimes_{A_f \otimes S} ([Id_{A_f}] \otimes 1_S)$  is  $[Id_{A_f \otimes S}]$ . To prove the reverse equality, we will need a trick that can be found already in [Pi86]. Observe first that for any  $l = 1, 2$  and using Lemma 4.10,

$$\begin{aligned} [\bar{\kappa}_l] \otimes_D [j] \otimes_{A_f \otimes S} y &= [j \circ \bar{\kappa}_l] \otimes_{A_f \otimes S} y = ([j_l] \otimes 1_S) \otimes_{A_f \otimes S} (1_{A_f} \otimes a) \otimes_{A_f} x \\ &= (1_{A_l} \otimes a) \otimes_{A_l} [j_l] \otimes_{A_f} x \\ &= (1_{A_l} \otimes a) \otimes_{A_l} (1_{A_l} \otimes b) \otimes_{A_l} ([Id_{A_l}] \otimes 1_S) \otimes_{A_l \otimes S} [\bar{\kappa}_l] \\ &= [\bar{\kappa}_l]. \end{aligned}$$

Now we need to compute  $[j] \otimes_{A_f \otimes S} y \otimes_D [ev_0]$ . To do this we will use the following lemma.

**Lemma 4.11.** *In  $KK^1(D \otimes S, A_f \otimes S)$ , one has  $([j] \otimes_{A_f \otimes S} (1_{A_f} \otimes a)) \otimes 1_S = -(1_D \otimes a) \otimes_D [j]$ .*

*Proof.* Indeed,

$$(1_D \otimes b) \otimes_{D \otimes S} ([j] \otimes_{A_f \otimes S} (1_{A_f} \otimes a)) \otimes 1_S = [j] \otimes_{A_f \otimes S} (1_{A_f} \otimes (1_S \otimes b)) \otimes_{S \otimes S} (a \otimes 1_S).$$

If  $\Sigma$  is the flip automorphism of  $S \otimes S$  then clearly  $[\Sigma] = -[Id_{S \otimes S}]$  in  $KK^0(S \otimes S, S \otimes S)$ . As a consequence  $(1_S \otimes b) \otimes_{S \otimes S} (a \otimes 1_S) = -1_S \otimes (b \otimes_{\mathbb{C}} a) = -[Id_S]$ . Hence

$$(1_D \otimes b) \otimes_{D \otimes S} ([j] \otimes_{A_f \otimes S} (1_{A_f} \otimes a)) \otimes 1_S = -[j].$$

Multiplying both sides by  $1_D \otimes a$  gives the result. □

In view of Lemmas 4.11 and 4.9 one has:

$$\begin{aligned} ([j] \otimes_{A_f \otimes S} y \otimes_D [ev_0]) \otimes 1_S &= -(1_D \otimes a) \otimes_D ([j] \otimes_{A_f \otimes S} (x \otimes_D [ev_0]) \otimes 1_S) \\ &= +(1_D \otimes a) \otimes_D [ev_0] \otimes_B (1_B \otimes b) \\ &= (1_D \otimes a) \otimes_D (1_D \otimes b) \otimes_{D \otimes S} ([ev_0] \otimes 1_S) \\ &= [ev_0] \otimes 1_S \end{aligned}$$

As  $- \otimes 1_S$  from  $KK(B_1, B_2)$  to  $KK(B_1 \otimes S, B_2 \otimes S)$  is an isomorphism for any  $B_1$  and  $B_2$ , we get  $[j] \otimes_{A_f \otimes S} y \otimes_D [ev_0] = [ev_0]$ . Denote now  $q = [Id_D] - [j] \otimes_{A_f \otimes S} y$ . As  $y \otimes_D [j] = [Id_{A_f \otimes S}]$ ,  $q$  is an idempotent in the ring  $KK^0(D, D)$ . On the other hand,  $D$  fits into a short exact sequence

$$0 \rightarrow A_1 \otimes S \oplus A_2 \otimes S \xrightarrow{\bar{\kappa}_1 \oplus \bar{\kappa}_2} D \xrightarrow{ev_0} B \rightarrow 0.$$

The induced six term exact sequence for the functor  $KK^0(D, -)$  then shows that, as  $q \otimes_D [ev_0] = 0$ , there exist  $q_l$  in  $KK^0(D, A_l)$  for  $l = 1, 2$  such that  $q = (q_1 \oplus q_2) \otimes_{A_1 \oplus A_2} ([\bar{\kappa}_1] \oplus [\bar{\kappa}_2])$ . So  $q = q \otimes_D q = (q_1 \oplus q_2) \otimes_{A_1 \oplus A_2} ([\bar{\kappa}_1] \oplus [\bar{\kappa}_2]) \otimes_D q = 0$  because  $[\bar{\kappa}_l] \otimes_D q = 0$  for  $l = 1, 2$  as observed

before Lemma 4.11. Therefore  $[Id_D] = [j] \otimes_{A_f \otimes_S y}$  and the  $K$ -equivalence between  $A_f$  and  $D$  is established.  $\square$

We obtain the following immediate corollaries.

**Corollary 4.12.** *Let  $C$  be any separable  $C^*$ -algebra. Recall that  $i_l$  is the inclusion of  $B$  in  $A_l$  and  $j_l$  is the inclusion of  $A_l$  in  $A_1 *_B A_2$  for  $l = 1$  or  $2$ . Then we have the two 6-terms exact sequences,*

$$\begin{array}{ccccc} KK^0(C, B) & \xrightarrow{i_1^* \oplus i_2^*} & KK^0(C, A_1) \oplus KK^0(C, A_2) & \xrightarrow{j_1^* + j_2^*} & KK^0(C, A_1 *_B A_2) \\ \uparrow & & & & \downarrow \\ KK^1(C, A_1 *_B A_2) & \xleftarrow{j_1^* + j_2^*} & KK^1(C, A_1) \oplus KK^1(C, A_2) & \xleftarrow{i_1^* \oplus i_2^*} & KK^1(C, B) \end{array}$$

and

$$\begin{array}{ccccc} KK^0(B, C) & \xleftarrow{i_1^* + i_2^*} & KK^0(A_1, C) \oplus KK^0(A_2, C) & \xleftarrow{j_1^* \oplus j_2^*} & KK^0(A_1 *_B A_2, C) \\ \downarrow & & & & \uparrow \\ KK^1(A_1 *_B A_2, C) & \xrightarrow{j_1^* \oplus j_2^*} & KK^1(A_1, C) \oplus KK^1(A_2, C) & \xrightarrow{i_1^* + i_2^*} & KK^1(B, C) \end{array}$$

*Proof.* The proof can be found in [Ge97] or [Th03]. It is simply the application of the six-term exact sequence to the short exact sequence for  $D$  that has been used just above. Identification of the horizontal maps as well as the connecting maps can also be found there.  $\square$

The following is a generalization of a similar statement in [FF13].

**Corollary 4.13.** *Let  $G_1, G_2, H$  be compact quantum groups and suppose that  $\widehat{H}$  is a common discrete quantum subgroup of both  $\widehat{G}_1, \widehat{G}_2$  and  $\widehat{G}_k$  is  $K$ -amenable for  $k = 1, 2$ . Then the amalgamated free product of the two discrete quantum groups is  $K$ -amenable.*

*Proof.* Write, for  $k = 1, 2$ ,  $C_m(G_k), C_m(H)$  the full  $C^*$ -algebras and  $C(G_k), C(H)$  the reduced  $C^*$ -algebra and view  $C_m(H) \subset C_m(G_k), C(H) \subset C(G_k)$ , for  $k = 1, 2$ . Let  $\widehat{G}$  be the amalgamated free product discrete quantum group. One has  $C_m(G) = C_m(G_1) *_C C_m(G_2)$  and  $C(G) =$

$C(G_1) *_C C(G_2)$ , where the edge-reduced amalgamated free product is done with respect to the faithful Haar states on  $C(G_k)$ , for  $k = 1, 2$ . Let  $\lambda_{G_k} : C_m(G_k) \rightarrow C(G_k)$  be the canonical surjection. By assumption,  $\lambda_{G_k}$  is  $K$ -invertible for  $k = 1, 2$ . Observe that the canonical surjection  $\lambda_G : C_m(G) \rightarrow C(G)$  is given by  $\lambda_G = \pi \circ \lambda$ , where

$$\lambda : C_m(G_1) *_C C_m(G_2) \rightarrow C(G_1) *_C C(G_2)$$

is the free product of the maps  $\lambda_{G_1}$  and  $\lambda_{G_2}$  and  $\pi : C(G_1) *_C C(G_2) \rightarrow C(G_1) *_C C(G_2)$  is the canonical quotient map. By Theorem 3.1  $\pi$  is  $K$ -invertible and using the exact sequence of the full free product and the five Lemma,  $\lambda$  is  $K$ -invertible.  $\square$

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