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Emmanuel Germain, Pierre Ueber

► **To cite this version:**

Emmanuel Germain, Pierre Ueber. KK-theory of  $A$ -valued semi-circular systems. *Comptes Rendus. Mathématique*, 2016, 354 (1), pp.87 - 90. 10.1016/j.crma.2015.10.013 . hal-01876541

**HAL Id: hal-01876541**

**<https://normandie-univ.hal.science/hal-01876541>**

Submitted on 21 May 2024

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Functional analysis

## KK-theory of A-valued semi-circular systems



## KK-théorie des systèmes semi-circulaires A-valués

Emmanuel Germain<sup>a</sup>, Pierre Ueber<sup>b</sup><sup>a</sup> LMNO UMR 6139, Université de Caen et CNRS, France<sup>b</sup> ENS Lyon, France

## ARTICLE INFO

## Article history:

Received 10 July 2015

Accepted after revision 21 October 2015

Available online 7 December 2015

Presented by Alain Connes

## ABSTRACT

We compute in this article the KK-theory of A-valued semi-circular systems thanks to tools developed by Pimsner (see [1]) to study generalized Toeplitz algebras.

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## R É S U M É

On calcule dans cet article la KK-théorie de systèmes semi-circulaires A-valués à l'aide d'outils développés par Pimsner (voir [1]) pour étudier les algèbres de Toeplitz généralisées.

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To begin with, we will need a result in Hilbert module theory.

**Proposition 0.1.** *Let  $A, C$  be  $C^*$ -algebras,  $B$  a sub- $C^*$ -algebra of  $C$ ,  $E$  a Hilbert module over  $A$ , and  $\phi : A \rightarrow B$  a  $*$ -morphism. Let  $j : B \rightarrow C$  be the inclusion,  $\psi \stackrel{\text{def}}{=} j \circ \phi$  and  $E_0 \stackrel{\text{def}}{=} \{ \sum x_i \otimes b_i, x_i \in E, b_i \in B \} \subset E \otimes_\psi C$ . Then  $E_0$  is naturally endowed with a structure of Hilbert module over  $B$  and  $E_0 \simeq E \otimes_\phi B$ .*

Indeed, let  $x_i, x'_k \in E, b_i, b'_k \in B$ . We have:

$$\langle \sum x_i \otimes b_i, \sum x'_k \otimes b'_k \rangle = \sum_{i,k} b_i^* \phi(\langle x_i, x'_k \rangle) b'_k \in B.$$

As  $B$  is closed in  $C$ , we have:  $\forall x, y \in E_0, \langle x, y \rangle \in B$ . As a result,  $E_0$  is naturally endowed with a structure of pre-Hilbert module over  $B$ , which is complete because  $E_0$  is a closed subspace of the Hilbert module  $E \otimes_\psi C$ .

For the second part of the proposition, let  $\pi : (x, b) \in E \times B \mapsto x \otimes b \in E_0$ . If  $a \in E$ , then  $\pi(x \cdot a, b) = x \cdot a \otimes b = x \otimes j \circ \phi(a)b = x \otimes \phi(a)b = \pi(x, a \cdot b)$ . Then  $\pi$  induces  $\tilde{\pi} : E \otimes_{\text{alg}} B \rightarrow E_0$ . We clearly have:

$$\langle \tilde{\pi}(\sum x_i \otimes b_i), \tilde{\pi}(\sum x_i \otimes b_i) \rangle = \langle \sum x_i \otimes b_i, \sum x_i \otimes b_i \rangle,$$

E-mail addresses: [emmanuel.germain@unicaen.fr](mailto:emmanuel.germain@unicaen.fr) (E. Germain), [pierre.umber@ens-lyon.fr](mailto:pierre.umber@ens-lyon.fr) (P. Ueber).

so  $\tilde{\pi}$  is an isometry. As  $E_0$  is complete,  $\tilde{\pi}$  extends to an isometry  $\widehat{\pi}$  on  $E \otimes_\phi B$ . As  $\widehat{\pi}$  is an isometry,  $Im(\widehat{\pi})$  is closed in  $E_0$ , but  $Im(\widehat{\pi})$  contains a dense subspace of  $E_0$ , so  $\widehat{\pi}$  is an isomorphism and  $E_0 \simeq E \otimes_\phi B$ .

Let's turn now to our main result. Let  $A$  be a  $C^*$ -algebra with unit, and  $E$  be a Hilbert module over  $A$  with an isometric  $*$ -morphism  $\phi : A \rightarrow L_A(E)$  which endowed  $E$  with a left action. The algebra  $A$  is supposed to be separable and  $E$  countably generated. We will denote by  $\mathcal{F}(E)$  the Fock space associated to  $E$ , which is  $\mathcal{F}(E) = \bigoplus_{n \geq 0} E^{\otimes n}$  (where  $E^{\otimes 0} = A$ ). Each  $E^{\otimes n}$  is

a left  $A$ -module, thus  $\mathcal{F}(E)$  is endowed with a diagonal left action over  $A$ .

Let  $\xi \in E$  and  $T_\xi$  be the left creation operator  $\eta \mapsto \xi \otimes \eta$ . Then  $T_\xi \in L_A(\mathcal{F}(E))$  and the annihilation operator is given by  $T_\xi^*(\eta_1 \otimes \dots \otimes \eta_n) = \langle \xi, \eta_1 \rangle \eta_2 \otimes \dots \otimes \eta_n$ .

We denote by  $\mathcal{T}_E$  the associated Toeplitz algebra, which is the  $C^*$ -algebra generated by  $A$  and the operators  $T_\xi$ .

If  $E$  is also endowed with an anti-linear involution  $\xi \mapsto \xi^*$  then there is a natural  $*$ -subalgebra of  $\mathcal{T}_E$ , that we denote by  $\mathcal{S}_E$ , and is generated by  $A$  and elements  $T_\xi + T_{\xi^*}$ . This algebra is mainly studied in a Von Neumann algebra context (see for example [2] and [3]). We will here compute its KK-theory as a particular case of the following theorem.

**Theorem 0.2.** *Let  $S$  be any sub- $C^*$ -algebra of  $\mathcal{T}_E$  which contains  $A$  and is generated by linear combinations of creation and annihilation operators. Then  $S$  is KK-equivalent to  $A$*

According to Pimsner (see Proposition 3.3 in [1]), Toeplitz algebras satisfy the following universal property:

**Proposition 0.3.** *Let  $B$  be a  $C^*$ -algebra and  $\sigma : A \rightarrow B$  a  $*$ -morphism. We suppose that there is a family  $(t_\xi)_{\xi \in E}$  in  $B$  such that:*

- 1)  $\xi \mapsto t_\xi$  is  $\mathbb{C}$ -linear
- 2)  $t_\xi \sigma(a) = t_{\xi a}$  and  $\sigma(a) t_\xi = t_{\phi(a)\xi}$
- 3)  $t_\xi^* t_\zeta = \sigma(\langle \xi, \zeta \rangle)$

Then  $\sigma$  extends to a unique morphism on  $\mathcal{T}_E$  such that  $\sigma(T_\xi) = t_\xi$ .

We denote by  $i_A$  the inclusion of  $A$  in  $S$ ,  $i_S$  the inclusion of  $S$  in  $\mathcal{T}_E$  and  $j \stackrel{\text{def}}{=} i_S \circ i_A$ . Let  $P$  be the projection in  $\mathcal{F}(E)$  onto  $E^{\otimes 0} = A$  and  $Q \stackrel{\text{def}}{=} 1 - P$ . Let  $\pi_0 : A \rightarrow L_A(\mathcal{F}(E))$  given by the diagonal left action of  $\mathcal{F}(E)$ , and  $\tilde{\pi}_1 \stackrel{\text{def}}{=} Q \pi_0 = \pi_0 Q$ . We also define  $\tilde{T}_\xi \stackrel{\text{def}}{=} Q T_\xi Q$ . Then  $(\tilde{\pi}_1, \tilde{T}_\xi)$  satisfies conditions in Proposition 0.3, so  $\tilde{\pi}_1$  extends to a representation  $\pi_1$  of  $\mathcal{T}_E$ .

Let  $\beta \stackrel{\text{def}}{=} (\mathcal{F}(E) \oplus \mathcal{F}(E), (\pi_0, \pi_1), F)$  where  $F : \mathcal{F}(E) \oplus \mathcal{F}(E) \rightarrow \mathcal{F}(E) \oplus \mathcal{F}(E)$  is defined by  $F(\xi \oplus \zeta) = \zeta \oplus \xi$ . Then  $\beta$  is an element of  $KK(\mathcal{T}_E, A)$  (see Lemma 4.2 and Definition 4.3 in [1]).

We have the relations  $j \otimes_{\mathcal{T}_E} \beta = 1_A$  and  $\beta \otimes_A j = 1_{\mathcal{T}_E}$ , where  $1_C$ , for every  $C^*$ -algebra  $C$ , is the multiplicative unit in the ring  $KK(C, C)$  (see Theorem 4.4 in [1]). We consider  $\alpha \stackrel{\text{def}}{=} i_S \otimes_{\mathcal{T}_E} \beta \in KK(S, A)$ .

**Proposition 0.4.** *We have the relations  $i_A \otimes_S \alpha = 1_A$  and  $\alpha \otimes_A i_A = 1_S$ .*

Indeed, for the first one we have  $i_A \otimes_S \alpha = i_A \otimes_S i_S \otimes_{\mathcal{T}_E} \beta = j \otimes_{\mathcal{T}_E} \beta = 1_A$ .

For the second one, we first recall all the tools which are introduced in Pimsner's article in the proof of Theorem 4.4. Let  $\tau_1 : \mathcal{T}_E \rightarrow L_{\mathcal{T}_E}(\mathcal{F}(E) \otimes_j \mathcal{T}_E)$  be the operator such that, for  $T \in \mathcal{T}_E$ ,  $\tau_1(T)$  acts on  $A \otimes_j \mathcal{T}_E \simeq \mathcal{T}_E$  by  $\tau_1(T)(S) \stackrel{\text{def}}{=} TS$  and is equal to zero on  $\bigoplus_{n \geq 1} E^{\otimes n} \otimes_j \mathcal{T}_E$  (note that  $\tau_1$  is a  $*$ -morphism). Let  $\tau_0 : \mathcal{T}_E \rightarrow L_{\mathcal{T}_E}(\mathcal{F}(E) \otimes_j \mathcal{T}_E)$  be the operator such

that, for  $T_\xi \in \mathcal{T}_E$ ,  $\tau_0(T_\xi)$  acts on  $A \otimes_j \mathcal{T}_E \simeq \mathcal{T}_E$  by  $\tau_0(T_\xi)(S) \stackrel{\text{def}}{=} \xi \otimes S$  and is equal to zero on  $\bigoplus_{n \geq 1} E^{\otimes n} \otimes_j \mathcal{T}_E$ . Note that

$(\tau_0(T_\xi))^*(\eta \otimes S) = \langle \xi, \eta \rangle S$  on  $E \otimes_j \mathcal{T}_E$  and is equal to zero on  $A \otimes_j \mathcal{T}_E$  and  $\bigoplus_{n \geq 2} E^{\otimes n} \otimes_j \mathcal{T}_E$ .

**Lemma 0.5.** *Consider  $T_\xi \in \mathcal{T}_E$  and  $t \in [0, 1]$ . We define*

$$\tilde{T}_{\xi,t} \stackrel{\text{def}}{=} \cos\left(\frac{\pi}{2}t\right)\tau_0(T_\xi) + \sin\left(\frac{\pi}{2}t\right)\tau_1(T_\xi) + \pi_1(T_\xi) \otimes 1_{\mathcal{T}_E}.$$

The couple  $(\pi_0 \otimes 1_{\mathcal{T}_E}, \tilde{T}_{\xi,t})$  satisfies the conditions in Proposition 0.3, and thus  $\pi_0 \otimes 1_{\mathcal{T}_E}$  extends to a representation  $\pi_t : \mathcal{T}_E \rightarrow L_{\mathcal{T}_E}(\mathcal{F}(E) \otimes_j \mathcal{T}_E)$ .

Conditions 1) and 2) are easy to check.

As regards condition 3), we have:  $\tilde{T}_{\xi,t}^* \tilde{T}_{\zeta,t} = I + J + K$  where

$$\begin{aligned}
 I &= \cos^2\left(\frac{\pi}{2}t\right)(\tau_0(T_\xi))^* \tau_0(T_\zeta) + \sin\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right)(\tau_0(T_\xi))^* \tau_1(T_\zeta) \\
 &\quad + \cos\left(\frac{\pi}{2}t\right)(\tau_0(T_\xi))^* \pi_1(T_\zeta) \otimes 1_{\mathcal{T}_E}, \\
 J &= \cos\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}t\right)(\tau_1(T_\xi))^* \tau_0(T_\zeta) + \sin^2\left(\frac{\pi}{2}t\right)(\tau_1(T_\xi))^* \tau_1(T_\zeta) \\
 &\quad + \sin\left(\frac{\pi}{2}t\right)(\tau_1(T_\xi))^* \pi_1(T_\zeta) \otimes 1_{\mathcal{T}_E}, \\
 K &= \cos\left(\frac{\pi}{2}t\right)((\pi_1(T_\xi))^* \otimes 1_{\mathcal{T}_E})\tau_0(T_\zeta) + \sin\left(\frac{\pi}{2}t\right)((\pi_1(T_\xi))^* \otimes 1_{\mathcal{T}_E})\tau_1(T_\zeta) \\
 &\quad + ((\pi_1(T_\xi))^* \pi_1(T_\zeta)) \otimes 1_{\mathcal{T}_E}.
 \end{aligned}$$

Then we compute each term on the subspace where it doesn't vanish. Remark that the subspaces  $E^{\otimes n} \otimes_j \mathcal{T}_E$  of  $\mathcal{F}(E) \otimes_j \mathcal{T}_E$  are stable for  $\pi_1 \otimes 1_{\mathcal{T}_E}$ . Let  $T \in A \otimes_j \mathcal{T}_E \simeq \mathcal{T}_E$ ,  $\eta \in E$ . We have:

$$\begin{aligned}
 (\tau_0(T_\xi))^* \tau_0(T_\zeta)(T) &= (\tau_0(T_\xi))^*(\zeta \otimes T) = \langle \xi, \zeta \rangle T; \\
 (\tau_0(T_\xi))^* \tau_1(T_\zeta)(T) &= 0; \\
 (\tau_0(T_\xi))^* (\pi_1(T_\zeta) \otimes 1_{\mathcal{T}_E})(\eta \otimes T) &= (\tau_0(T_\xi))^*(Q T_\zeta Q \eta \otimes T) = 0; \\
 (\tau_1(T_\xi))^* \tau_0(T_\zeta)(T) &= 0; \\
 (\tau_1(T_\xi))^* \tau_1(T_\zeta)(T) &= T_\xi^* T_\zeta T = \langle \xi, \zeta \rangle T; \\
 (\tau_1(T_\xi))^* (\pi_1(T_\zeta) \otimes 1_{\mathcal{T}_E}) &= 0; \\
 ((\pi_1(T_\xi))^* \otimes 1_{\mathcal{T}_E})\tau_0(T_\zeta)(T) &= (\pi_1(T_\xi^*)\zeta) \otimes T = 0; \\
 ((\pi_1(T_\xi))^* \otimes 1_{\mathcal{T}_E})\tau_1(T_\zeta)(T) &= (\pi_1(T_\xi^*)1) \otimes T_\zeta T = 0.
 \end{aligned}$$

For the last two statements, we use the fact that  $\pi_1(T_\xi^*)$  vanishes on the subspaces  $A = E^{\otimes 0}$  and  $E = E^{\otimes 1}$  of  $\mathcal{F}(E)$ . As regards the last term, let  $\eta \in \mathcal{F}(E)$ . We have:  $(\pi_1(T_\xi^*)\pi_1(T_\zeta))\eta \otimes T = \langle \xi, \eta \rangle (Q\eta) \otimes T$ . Finally:

$$\begin{aligned}
 \tilde{T}_{\xi,t}^* \tilde{T}_{\zeta,t}(\eta \otimes T) &= (\cos^2\left(\frac{\pi}{2}t\right) + \sin^2\left(\frac{\pi}{2}t\right)) \langle \xi, \eta \rangle (P\eta) \otimes T + \langle \xi, \eta \rangle (Q\eta) \otimes T \\
 &= \langle \xi, \zeta \rangle \eta \otimes T
 \end{aligned}$$

so  $\tilde{T}_{\xi,t}^* \tilde{T}_{\zeta,t} = (\pi_0 \otimes 1_{\mathcal{T}_E})(\langle \xi, \zeta \rangle)$ .

We now focus on  $\alpha \otimes_A i_A$ . Likewise, we can define  $\tau_1^S : S \rightarrow L_S(\mathcal{F}(E) \otimes_{i_A} S)$  and  $\tau_0^S : S \rightarrow L_S(\mathcal{F}(E) \otimes_{i_A} S)$ . The element  $\alpha \otimes_A i_A$  is given by the Kasparov module

$$(\mathcal{F}(E) \otimes_{i_A} S \oplus \mathcal{F}(E) \otimes_{i_A} S, (\pi_0 \otimes 1_S \circ i_S) \oplus (\pi_1 \otimes 1_S \circ i_S), F \otimes 1_S).$$

Then the element  $\alpha \otimes_A i_A - 1_B$  can be represented by the Kasparov module  $\gamma \stackrel{\text{def}}{=} ((\mathcal{F}(E) \otimes_{i_A} S) \oplus (\mathcal{F}(E) \otimes_{i_A} S), \pi_0^S \oplus \pi_1^S, F \otimes 1_S)$  where  $\pi_1^S = \tau_1^S \oplus (\pi_1 \otimes 1_S \circ i_S)$  and  $\pi_0^S = \pi_0 \otimes 1_S \circ i_S$ .

We also have  $\pi_0 \otimes 1 \circ i_S = \tau_0^S \oplus (\pi_1 \otimes 1 \circ i_S)$ .

**Lemma 0.6.** Consider the  $\mathbb{C}$ -subspace  $\mathcal{F}(E) \otimes_{i_S} S$  of  $\mathcal{F}(E) \otimes_j \mathcal{T}_E$  (see Proposition 0.1) and let  $t \in [0, 1]$ . Then the representation  $\pi_t$  in Lemma 0.5 induces a representation  $\pi_t^S : S \rightarrow L_S(\mathcal{F}(E) \otimes_{i_A} S)$ .

Indeed, let  $g \stackrel{\text{def}}{=} \sum_{i=1}^n \lambda_i T_{\xi_i} + \sum_{i=1}^m \mu_i T_{\zeta_i}^*$  be a generator of the  $C^*$ -algebra  $S$ . We first show that  $\pi_t(g)$  stabilizes  $\mathcal{F}(E) \otimes_{i_A} S$ .

Let  $E_0 \stackrel{\text{def}}{=} \{\sum \xi_i \otimes b_i, x_i \in \mathcal{F}(E), b_i \in S\} \subset \mathcal{F}(E) \otimes_{i_A} S$ .

We have  $\pi_t(g) = L + M + N$  where

$$\begin{aligned}
 L &= \sum_{i=1}^n \lambda_i \tau_1(T_{\xi_i}) + \sum_{i=1}^m \mu_i (\tau_1(T_{\zeta_i}))^* \\
 M &= \sum_{i=1}^n \lambda_i \tau_0(T_{\xi_i}) + \sum_{i=1}^m \mu_i (\tau_0(T_{\zeta_i}))^* \\
 N &= \sum_{i=1}^n \lambda_i (\pi_0(T_{\xi_i})) \otimes 1_{\mathcal{T}_E} + \sum_{i=1}^m \mu_i (\pi_0(T_{\xi_i}))^* \otimes 1_{\mathcal{T}_E}
 \end{aligned}$$

As in the proof of Lemma 0.5 we only pay attention on the subspaces where terms do not vanish. Let  $b \in A \otimes_{i_A} S \simeq S$ ,  $\eta \in E$ . Then we have:

$$\begin{aligned} \left(\sum_{i=1}^n \lambda_i \tau_1(T_{\xi_i}) + \sum_{i=1}^m \mu_i (\tau_1(T_{\zeta_i}))^*\right) b &= \left(\sum_{i=1}^n \lambda_i T_{\xi_i} + \sum_{i=1}^m \mu_i T_{\zeta_i}^*\right) b \in A \otimes_{i_A} S \simeq S; \\ \left(\sum_{i=1}^n \lambda_i \tau_0(T_{\xi_i}) + \sum_{i=1}^m \mu_i (\tau_0(T_{\zeta_i}))^*\right) b &= \left(\sum_{i=1}^n \lambda_i \xi_i\right) \otimes b \in E \otimes S; \\ \left(\sum_{i=1}^n \lambda_i \tau_0(T_{\xi_i}) + \sum_{i=1}^m \mu_i (\tau_0(T_{\zeta_i}))^*\right) \eta \otimes b &= \left\langle \sum_{i=1}^m \mu_i \zeta_i, \eta \right\rangle b \in A \otimes_{i_A} S \simeq S. \end{aligned}$$

The last term clearly stabilizes  $E_0$ . By linearity,  $\pi_t(g)$  stabilizes  $E_0$ . As  $\pi_t(g)$  is continuous on  $E_0$  and  $(\pi_t(g))^* = \pi_t(g^*)$ ,  $\pi_t$  induces a  $*$ -morphism  $\pi_t^S$  on the involutive algebra generated by  $g$  valued in  $L_S(\mathcal{F}(E) \otimes_{i_A} S)$ . We now have to extend  $\pi_t^S$  to a morphism on  $S$ . We note that  $\|\pi_t^S(g)\| \leq \|\pi_t(g)\| \leq \|g\|$  because  $\pi_t$  is a  $*$ -morphism between  $C^*$ -algebras. Then  $\pi_t^S$  is continuous and extends to a (unique) morphism on  $S$ , still denoted by  $\pi_t^S$ . For  $t = 0$  or  $t = 1$ , we find the same  $\pi_0^S$  and  $\pi_1^S$  introduced before.

To end the proof, we will show that the family  $\pi_t^S$  is a homotopy, and thus  $\gamma = 0$ . First we have to show that, for fixed  $b \in S$ ,  $t \rightarrow \pi_t^S$  is continuous. For that, as  $\|\pi_t^S(s)\| \leq \|s\|$ , we only have to see it on generators  $g \in S$ , which is obvious.

Besides, we need to show that, for  $b \in S$  and  $t \in [0, 1]$  fixed, we have:

$$\pi_t^S(b) - \pi_0^S(b) \in \mathcal{K}_S(\mathcal{F}(E) \otimes_{i_A} S).$$

We only need to check it for  $g \in S$  generator with  $g \stackrel{\text{def}}{=} \sum_{i=1}^n \lambda_i T_{\xi_i} + \sum_{i=1}^m \mu_i T_{\zeta_i}^*$ . The projection  $P$ , introduced at the beginning,

is clearly a compact operator of  $\mathcal{F}(E)$ , so  $P \otimes 1_S$  is a compact operator of  $\mathcal{F}(E) \otimes_{i_A} S$ . We can see that  $\pi_t^S(g) - \pi_0^S(g) = U + V \in \mathcal{K}_S(\mathcal{F}(E) \otimes_{i_A} S)$  where

$$\begin{aligned} U &= \sum_{i=1}^n \lambda_i (\pi_t^S(T_{\xi_i}) - \pi_0^S(T_{\xi_i})) (P \otimes 1_S) \\ V &= \sum_{i=1}^m \mu_i (P \otimes 1_S) (\pi_t^S(T_{\zeta_i}^*) - \pi_0^S(T_{\zeta_i}^*)). \end{aligned}$$

Thus we have the relation  $\alpha \otimes_A i_A = 1_S$ .

**Corollary 0.7.** *We have  $K_0(S) = K_0(A)$ . Particularly, we have:*

$$K_0(\mathcal{S}_E) = K_0(A)$$

And thus a different proof of the result of [4]:

**Corollary 0.8.** *Let  $\varphi : f \in \mathcal{C}([0, 1]) \mapsto \int_0^1 f(t) dt$ .  $\varphi$  is a state of the  $C^*$ -algebra  $\mathcal{C}([0, 1])$ .*

*We have  $K_0((\mathcal{C}([0, 1]), \varphi) *_r (\mathcal{C}([0, 1]), \varphi)) = K_0(\mathbb{C})$ .*

Indeed, for  $A = \mathbb{C}$  and  $E = \mathbb{C}^2$ , if  $S_1$  and  $S_2$  are the creation operators associated with the vectors  $(1, 0)$  and  $(0, 1)$ , we consider  $C^*(1, S_1 + S_1^*, S_2 + S_2^*)$ . It is well known that  $C^*(1, S_1 + S_1^*, S_2 + S_2^*) \simeq (\mathcal{C}([-2, 2]), \psi) *_r (\mathcal{C}([-2, 2]), \psi)$ , where

$\psi : f \in \mathcal{C}([-2, 2]) \mapsto \frac{1}{2\pi} \int_{-2}^2 f(t) \sqrt{4 - t^2} dt$  (see [5]), and that there is an homeomorphism on  $[-2, 2]$  onto  $[0, 1]$  that sends

the semi-circular measure to the Lebesgue one. That gives rise to an  $*$ -isomorphism:

$$(C([0, 1]), \varphi) *_r (C([0, 1]), \varphi) \simeq (C([-2, 2]), \psi) *_r (C([-2, 2]), \psi)$$

so  $K_0((C([0, 1]), \varphi) *_r (C([0, 1]), \varphi)) = K_0(C^*(1, S_1 + S_1^*, S_2 + S_2^*)) = K_0(\mathbb{C})$ .

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